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OSCILLATORY AND NONOSCILLATORY BEHAVIOR OF A SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATION

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Section 1. In this paper we are concerned with establishing criteria for the oscillatory behavior of the solutions of the second order nonlinear differential equation with deviating argument

$$(\mathrm{I}_1) \qquad [p(t)h(x)x'(t)]' + q(t)f(x(g(t))) = 0, \qquad ' = d/dt,$$

and of the corresponding ordinary differential equation

$$(\mathrm{I}_2) \qquad [p(t)h(x)x'(t)]' + q(t)f(x(t)) = 0,$$

where

H1) $q : [t_0, +\infty) \to \mathbf{R}$ is continuous and does not eventually vanish, i.e., there exists $\{t_k\}$, $t_k \to +\infty$, such that $q(t_k) \neq 0$;

H2) $p : [t_0, +\infty) \to \mathbf{R}$ is positive continuously differentiable;

H3) $g : [t_0, +\infty) \to \mathbf{R}$ is positive continuously differentiable such that $g'(t) \geq 0$ and $g(t) \to +\infty$ as $t \to +\infty$;

H4) $h : \mathbf{R} \to \mathbf{R}$ is continuously differentiable and $h(u) > 0$ for $u \neq 0$;

H5) $f : \mathbf{R} \to \mathbf{R}$ is continuously differentiable such that $uf(u) > 0$ for $u \neq 0$ and $df(u)/du \geq 0$.

Throughout by a *solution* of (I_1) $[(\mathrm{I}_2)]$ we shall mean a twice continuously differentiable function which exists on some half-line $[t_x, +\infty)$, satisfies (I_1) $[(\mathrm{I}_2)]$ and does not eventually vanish. For results concerning the continuability we refer the reader to [**3, 11, 12, 14, 20**] and references therein.

As usual, a solution of (I_1) $[(\mathrm{I}_2)]$ is said to be *oscillatory* or *nonoscillatory* according to whether it does or does not have arbitrarily large

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zeros. A nonoscillatory solution x of (I_1) $[(\mathrm{I}_2)]$ is said to be *weakly oscillatory* if x' changes sign for arbitrarily large values of t (see, for example, [**14, 15**]).

Many physical systems are modeled on equations (I_1) and (I_2). Such equations arise, for instance, in the study of celestial mechanics, of gas dynamics and fluid mechanics, of chemically reacting systems and also in many electromagnetic problems on atomic fields. An ample bibliography is contained in [**5, 16, 20**].

There are several results concerning oscillation criteria for the equations (I_1) or (I_2). Among the numerous papers dealing with this subject we refer in particular to [**5, 8, 14, 20, 22**] and to the references contained therein. The equation $x'' + q(t)|x(t)|^\gamma \operatorname{sgn} x(t) = 0$ has been especially deeply investigated. As contributions to the study of this problem from the point of view of the behavior of the integral average of the function q, we refer to the recent papers [**4, 9, 13, 19, 21**]. However, only a few results are known concerning the existence of weakly oscillatory solutions of (I_1) or (I_2) especially if the function q is allowed to change sign for arbitrarily large values of t.

When $q \leq 0$ for all large t, the equations (I_1) and (I_2) have been considered in [**5, 6**], respectively. The aim of this paper is to consider the cases "$q \geq 0$" and "q changes sign for all large t," to give sufficient conditions in order that every solution of (I_1) $[(\mathrm{I}_2)]$ is either oscillatory or weakly oscillatory and to study the asymptotic nature of nonoscillatory solutions of (I_1) $[(\mathrm{I}_2)]$.

When q is allowed to take on negative values for arbitrarily large t, equation (I_2) has been considered by many authors. In particular in [**9**] some criteria on the oscillatory behavior, which extend and improve previous results in [**2, 4, 8**] are established. Such criteria require, among other conditions, that f is strongly superlinear at infinity, i.e., such that

$$\int^{+\infty} \frac{1}{f(u)}\,du < +\infty, \qquad \int^{-\infty} \frac{1}{f(u)}\,du < +\infty.$$

This condition, which is also required by the majority of the quoted authors, is not assumed in our paper; results here obtained involve only, in certain cases, conditions on the integral of the function $h(u)/f(u)$ for u sufficiently small.

Further, by using averaging techniques, we shall be concerned with the existence of eventually monotone solutions of (I_1) and (I_2) and with their asymptotic behavior. As is well known, Sturm's theorem fails for equations (I_1) and (I_2) and so both oscillatory and monotone solutions may exist simultaneously. In addition, the relationship between the oscillatory behavior of the equations (I_1) and (I_2) is considered and the existence of bounded solutions is discussed.

Relationships and comparisons with known results will be made throughout the paper.

Section 2. Consider the equations (I_1) and (I_2) in the form

$$[p(t)h(x)x'(t)]' + q(t)f(x(g(t))) = 0. \tag{I}$$

Taking into account that q does not eventually vanish, any eventually constant function different from zero is not a solution of (I).

With respect to their asymptotic behavior, all the solutions of (I) may be *a priori* divided into the following classes:

$\mathbf{M}^+ = \{x = x(t)$ solution of (I) (not eventually vanishing): there exists $t_x \geq t_0 : x(t)x'(t) \geq 0$ for $t \geq t_x\}$

$\mathbf{M}^- = \{x = x(t)$ solution of (I) (not eventually vanishing): there exists $t_x \geq t_0 : x(t)x'(t) \leq 0$ for $t \geq t_x\}$

$\mathbf{O} = \{x = x(t)$ solution of (I) (not eventually vanishing): there exists $\{t_n\}, t_n \to +\infty : x(t_n) = 0\}$

$\mathbf{WO} = \{x = x(t)$ solution of (I): $x(t) \neq 0$ for t sufficiently large and for all $\alpha > 0$ there exists $t_{\alpha_1} > \alpha$, there exists $t_{\alpha_2} > \alpha : x'(t_{\alpha_1})x'(t_{\alpha_2}) < 0\}$.

With a very simple argument we can prove that $\mathbf{M}^+$, $\mathbf{M}^-$, $\mathbf{O}$, $\mathbf{WO}$ are mutually disjoint. By the above definitions, it turns out that solutions in the class $\mathbf{M}^+$ are eventually either positive nondecreasing or negative nonincreasing, solutions in the class $\mathbf{M}^-$ are eventually either positive nonincreasing or negative nondecreasing, solutions in the class $\mathbf{O}$ are oscillatory, and, finally, solutions in the class $\mathbf{WO}$ are weakly oscillatory.

Lemma 1. *If*

$$\limsup_{t\to+\infty}\int_{t_0}^{t} q(\tau)\,d\tau = +\infty, \tag{1}$$

then for equation (I) *we have* $\mathbf{M}^+ = \varnothing$.

Proof. Suppose that equation (I) has a solution $x \in \mathbf{M}^+$. There is no loss of generality in assuming that there exists t_1 such that $x(t) > 0$, $x'(t) \geq 0$, $x(g(t)) > 0$, $x'(g(t)) \geq 0$ for all $t \geq t_1$ since the proof is similar if $x(t) < 0$, $x(g(t)) < 0$ for all large t. From (I) we have $(t \geq t_1)$

$$\begin{aligned} \frac{p(t)h(x(t))x'(t)}{f(x(g(t)))} - \frac{p(t_1)h(x(t_1))x'(t_1)}{f(x(g(t_1)))} \\ + \int_{t_1}^{t} \frac{p(s)h(x(s))x'(s)f'(x(g(s)))x'(g(s))g'(s)}{f^2(x(g(s)))}\,ds \\ = -\int_{t_1}^{t} q(s)\,ds \end{aligned} \tag{2}$$

or

$$\frac{p(t)h(x(t))x'(t)}{f(x(g(t)))} - \frac{p(t_1)h(x(t_1))x'(t_1)}{f(x(g(t_1)))} \leq -\int_{t_1}^{t} q(s)\,ds.$$

From (1) we obtain

$$\liminf_{t\to+\infty}\frac{p(t)h(x(t))x'(t)}{f(x(g(t)))} = -\infty$$

which contradicts the assumption $x'(t) > 0$ for all large t. □

We point out that Lemma 1 does not require that the function f satisfies hypotheses on superlinearity and/or sublinearity. Furthermore, in Lemma 1, deviating type conditions are not assumed and so such a result may hold for ordinary, retarded, advanced and mixed type equations.

The following example shows that assumption (1) cannot be dropped without violating the validity of Lemma 1.

Example 1. Consider the ordinary differential equation

$$x'' + \cos t f(x) = 0 \tag{E$_1$}$$

where $f(u) = 1$ for $u \geq 1$. Let x be the solution of (E_1) such that $x(0) = 1$, $x'(0) = 2$. Assume that there exists $\tau > 0$ such that $x'(\tau) = 0$ and $x'(t) > 0$ for $0 < t < \tau$. Integrating (E_1) on $[0, \tau]$, we obtain the contradiction $-x'(0) + \int_0^\tau \cos r\, dr = -2 + \sin \tau = 0$. Consequently, $x'(t) > 0$ for all $t > 0$ and from (E_1) we get $x(t) = 2t + \cos t$. Thus, x is continuable for all $t > 0$ and so x belongs to the class $\mathbf{M}^+$.

Let us now examine the problem of the existence of solutions of (I) in the class $\mathbf{M}^-$. When the function $h(u)/f(u)$ is locally integrable for u sufficiently small, we have the following

Theorem 1. *Assume that $g(t) \leq t$. If the function $h(u)/f(u)$ is locally integrable on $(0, c)$ and $(-c, 0)$ for some $c > 0$, that is,*

$$\int_0^c \frac{h(u)}{f(u)}\, du < +\infty, \qquad \int_{-c}^0 \frac{h(u)}{f(u)}\, du > -\infty \tag{3}$$

and if

$$\limsup_{t\to+\infty} \int_T^t \frac{1}{p(s)} \int_T^s q(r)\, dr\, ds = +\infty \quad \text{for all } T \geq t_0, \tag{4}$$

then for equation (I) we have $\mathbf{M}^- = \varnothing$.

Proof. Suppose that equation (I) has a solution $x \in \mathbf{M}^-$. There is no loss of generality in assuming that there exists t_1 such that $x(t) > 0$, $x'(t) \leq 0$, $x(g(t)) > 0$, $x'(g(t)) \leq 0$ for all $t \geq t_1$ since the proof is similar if $x(t) < 0$, $x'(t) \geq 0$ for all large t. From (2) we obtain ($t \geq t_1$)

$$\begin{aligned}\frac{p(t)h(x(t))x'(t)}{f(x(g(t)))} &= \frac{p(t_1)h(x(t_1))x'(t_1)}{f(x(g(t_1)))}\\ &\quad - \int_{t_1}^t \frac{p(s)h(x(s))x'(s)f'(x(g(s)))x'(g(s))g'(s)}{f^2(x(g(s)))}\, ds\\ &\quad - \int_{t_1}^t q(s)\, ds \leq - \int_{t_1}^t q(s)\, ds\end{aligned}$$

or

$$\frac{h(x(t))x'(t)}{f(x(g(t)))} \le -\frac{1}{p(t)} \int_{t_1}^{t} q(s)\,ds. \tag{5}$$

Since x is a nonincreasing function for $t \ge t_1$ and $g(t) \le t$, we have $x(g(t)) \ge x(t)$ for every $t \ge t_1$ and so, by taking into account that f is a nondecreasing function, from (5) we get

$$\frac{h(x(t))x'(t)}{f(x(t))} \le \frac{h(x(t))x'(t)}{f(x(g(t)))} \le -\frac{1}{p(t)} \int_{t_1}^{t} q(s)\,ds.$$

Thus, we have

$$\int_{t_1}^{t} \frac{h(x(s))x'(s)}{f(x(s))}\,ds \le -\int_{t_1}^{t} \frac{1}{p(s)} \int_{t_1}^{s} q(r)\,dr\,ds$$

or

$$\int_{x(t)}^{x(t_1)} \frac{h(u)}{f(u)}\,du \ge \int_{t_1}^{t} \frac{1}{p(s)} \int_{t_1}^{s} q(r)\,dr\,ds$$

which, because of (4), implies

$$\limsup_{t\to+\infty} \int_{x(t)}^{x(t_1)} \frac{h(u)}{f(u)}\,du = +\infty. \tag{6}$$

This contradicts condition (3). The proof is now complete. $\square$

The following examples show that assumptions (3) and (4) cannot be dropped without violating the validity of Theorem 1.

Example 2. Consider the ordinary differential equation

$$(t^3 x'(t))' + t x(t) = 0. \tag{E$_2$}$$

Since the function $x(t) = 1/t$ is a solution of (E$_2$) for $t \ge 1$, we have $\mathbf{M}^- \ne \varnothing$. Moreover, assumption (4) holds since ($T \ge 1$)

$$\lim_{t\to+\infty}\int_T^t \frac{1}{p(s)}\int_T^s q(r)\,dr\,ds$$
$$= \lim_{t\to+\infty} \frac{1}{4t^2}+\frac{1}{2}\ln t-\frac{1}{4T^2}-\frac{1}{2}\ln T=+\infty,$$

while condition (3) is not satisfied.

Example 3. Consider the ordinary differential equation

$$(t^4x^2(t)x'(t))' + 3tx(t)/4 = 0. \tag{E$_3$}$$

Since the function $x(t) = t^{-1/2}$ is a solution of (E_3) for $t \geq 1$, we have $\mathbf{M}^- \neq \varnothing$. For this equation the assumption (4) does not hold since ($T \geq 1$)

$$\limsup_{t\to+\infty}\int_T^t \frac{1}{p(s)}\int_T^s q(r)\,dr\,ds = \lim_{t\to+\infty} -\frac{3}{8t}+\frac{T^2}{8t^3}+\frac{1}{4T}=\frac{1}{4T},$$

while condition (3) is satisfied.

When $h \equiv 1$, assumption (3) implies that the function f is sublinear for u sufficiently small. In [**9, 19**] the equation (I_2) with $h \equiv 1$ and f strongly superlinear at infinity is considered and sufficient conditions are given in order that (I_2) is oscillatory. Thus, in this special case, $\mathbf{M}^+ = \mathbf{M}^- = \mathbf{WO} = \varnothing$. From Lemma 1 and Theorem 1 we get the following result which completes the ones quoted in [**9, 19**].

Such a result also extends to equations with deviating argument, an earlier oscillation criterion given in [**17**].

Corollary 1. *Assume that $g(t) \leq t$. If the assumptions* (1), (3) *and* (4) *are satisfied, then every solution of* (I) *is either oscillatory or weakly oscillatory. In addition, if $g(t) \equiv t$, then every solution of* (I) *is oscillatory.*

Proof. The first assertion follows immediately from Lemma 1 and Theorem 1; the last one was proved in [**17**]. $\square$

Lemma 1, Theorem 1 and Corollary 1 are also related to recent results in [**4, 21**] which involve different integral averages of the function q.

Section 3. In this section we examine the existence of weakly oscillatory solutions of (I). Under the assumptions of Corollary 1, (I_2) does not have weakly oscillatory solutions. In general this assertion cannot be made for the corresponding functional differential equation (I_1) since the deviating argument may generate weakly oscillatory solutions. The following example shows that, under assumptions (1), (3), (4), there exist equations of type (I) (with $g(t) \neq t$) with weakly oscillatory solutions.

Example 4. Consider the differential equation with delay

$$x''(t) + \frac{\sin t}{2 - \sin t} f(x(g(t))) = 0 \tag{E_4}$$

where $g(t) = t - \pi$ and f is a continuously differentiable function such that $f(u) = u$ for $u \geq 1$ and $f(u) = |u|^\alpha \operatorname{sgn} u$, $0 < \alpha < 1$, for $-1/2 < u < 1/2$.

The function $x(t) = 2 + \sin t$ is a weakly oscillatory solution of (E_4). Furthermore, all the hypotheses of Corollary 1 are satisfied, as can be seen by standard calculations. Clearly, assumption (3) holds. Now, by noting that

$$\int_0^{2\pi} \frac{\sin t}{2 - \sin t}\, dt = k > 0,$$

we obtain

$$\lim_{t\to+\infty} \int_0^t q(\tau)\, d\tau = +\infty \tag{7}$$

and so assumption (1) is satisfied. Finally, let us show that assumption (4) is also satisfied. By (7), for any $\tau \geq t_0$ there exists $T_\tau > \tau$ such that for all $t > T_\tau$,

$$\int_\tau^t q(s)\, ds \geq k_1 > 0.$$

Hence,

$$\begin{aligned}\int_\tau^t \left(\int_\tau^s q(r)\,dr\right) ds &= \int_\tau^{T_\tau} \left(\int_\tau^s q(r)\,dr\right) ds + \int_{T_\tau}^t \left(\int_\tau^s q(r)\,dr\right) ds \\ &\geq \int_\tau^{T_\tau} \left(\int_\tau^s q(r)\,dr\right) ds + k_1(t - T_\tau)\end{aligned}$$

and so assumption (4) is verified.

The above example demonstrates that equations with deviating argument create some new problems concerning the existence of weakly oscillatory solutions. We shall establish various sets of conditions under which equation (I) has no weakly oscillatory solutions. The following holds:

Theorem 2. a) *If the assumption*

$$\lim_{t\to+\infty} \int_{t_0}^t q(s)\,ds = +\infty \tag{8}$$

is satisfied, then for equation (I_2) *we have* $\mathbf{WO} = \varnothing$.

b) *If* $q(t) \geq 0$ *for all large* t*, then for equation* (I) *we have* $\mathbf{WO} = \varnothing$.

c) *If the following assumptions*

(c_1) $\quad p'(t)q(t)$ *does not change sign,*

(c_2) $\quad p'^2(t) + q^2(t) > 0$

hold for all t *sufficiently large, then for equation* (I) *with* $h \equiv 1$ *we have* $\mathbf{WO} = \varnothing$.

Proof. Claim a). Suppose that equation (I_2) has a solution $x \in \mathbf{WO}$. There is no loss of generality in assuming that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for all $t \geq t_1$ since the proof is similar if $x(t) < 0$ for all large t. From (2) we have $(t \geq t_1)$

$$\begin{aligned}\frac{p(t)h(x(t))x'(t)}{f(x(t))} &= \frac{p(t_1)h(x(t_1))x'(t_1)}{f(x(t_1))} \\ &\quad - \int_{t_1}^t \frac{p(s)h(x(s))f'(x(s))x'^2(s)}{f^2(x(s))}\,ds - \int_{t_1}^t q(s)\,ds \\ &\leq \frac{p(t_1)h(x(t_1))x'(t_1)}{f(x(t_1))} - \int_{t_1}^t q(s)\,ds\end{aligned}$$

and so, for all large t, $x'(t) < 0$, which gives a contradiction since x is weakly oscillatory.

Claim b). Let x be a weakly oscillatory solution of (I). There is no loss of generality in assuming that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(g(t)) > 0$ for all $t \geq t_1$ since the proof is similar if $x(t) < 0$, $x(g(t)) < 0$ for all large t. Consider the function F given by $F(t) = p(t)h(x(t))x'(t)$; we have for $t \geq t_1$, $F'(t) = -q(t)f(x(g(t))) \leq 0$ and so F is nonincreasing, which gives a contradiction since F is an oscillatory function.

Claim c). Let x be a weakly oscillatory solution of (I). Choose a large $t_1 \geq t_0$ such that assumptions (c_1) and (c_2) are satisfied for any $t \geq t_1$. Assume that $x(t) > 0$, $x(g(t)) > 0$ for all $t \geq t_1$. Let $\{t_n\}$ be a sequence of points of positive maximum for x', $t_n > t_1$, and let $\{\tau_n\}$ be a sequence of points of negative minimum for x', $\tau_n > t_1$.

Hence,

$$x'(t_n) > 0, \qquad x''(t_n) = 0, \qquad x'(\tau_n) < 0, \qquad x''(\tau_n) = 0. \tag{9}$$

Taking into account that $h \equiv 1$, from (I) we obtain

$$\begin{aligned} p'(t_n)x'(t_n) &= -q(t_n)f(x(g(t_n))) \\ p'(\tau_n)x'(\tau_n) &= -q(\tau_n)f(x(g(\tau_n))). \end{aligned}$$

Since the functions p' and q do not simultaneously vanish, we get

$$p'(t_n) \neq 0; \quad q(t_n) \neq 0, \quad p'(\tau_n) \neq 0, \quad q(\tau_n) \neq 0$$

and so (9) yields

$$p'(t_n)q(t_n) < 0, \qquad p'(\tau_n)q(\tau_n) > 0$$

which contradicts assumption (c_1). $\square$

We point out that if $q(t) \leq 0$ for all large t, then for equation (I) we have $\mathbf{WO} = \varnothing$ too. Such an assertion follows easily by the same argument as given in the above proof.

We make now some additional remarks related to Theorem 2: the claim a) fails for equations with deviating argument, as example (E_4) shows; in claims b), c) hypotheses on the deviating argument g and monotonicity assumptions on the function f are not required; in claim c), if (c_1) and (c_2) are eventually satisfied and p is strongly monotone, then q cannot eventually change sign.

Finally, we note that if the function p is constant and q changes sign, then assumption (c_2) is not fulfilled and Theorem 2-c) fails, as example (E_4) shows.

When no integrability requirement is assumed on the function $h(u)/f(u)$ for u sufficiently small, sufficient conditions in order that every solution of equation (I) is either oscillatory or weakly oscillatory are given in the following result. Such a criterion extends to the functional differential equation (I), a well-known oscillation result of Leighton and Wintner (see, e.g., [**7**] for the linear case and Bhatia [**1**] for the equation (I_2) with $h \equiv 1$.)

Theorem 3. *If the assumptions*

$$\lim_{t\to+\infty} \int_{t_0}^{t} q(s)\,ds = +\infty \tag{8}$$

and

$$\lim_{t\to+\infty} \int_{t_0}^{t} \frac{1}{p(s)}\,ds = +\infty \tag{10}$$

are satisfied, then every solution of (I_2) *is oscillatory and every solution of* (I) *is either oscillatory or weakly oscillatory.*

Proof. From Lemma 1 and Theorem 2 it follows that for equation (I_2), $\mathbf{M}^+ = \mathbf{WO} = \varnothing$ and for equation (I) $\mathbf{M}^+ = \varnothing$. Then in order to complete the proof it suffices to show that for equation (I) $\mathbf{M}^- = \varnothing$. Let x be a solution of class $\mathbf{M}^-$ of equation (I). There is no loss of generality in assuming that there exists t_1 such that $x(t) > 0$, $x'(t) \le 0$, $x(g(t)) > 0$, $x'(g(t)) \le 0$ for all $t \ge t_1$ since the proof is similar if

$x(t) < 0$, $x'(t) \geq 0$ for all large t. Proceeding as in the proof of Lemma 1 from (2), we obtain ($t \geq t_1$):

$$\begin{aligned}\omega(t) &= \omega(t_1) + \int_{t_1}^{t} \omega(s)\bigg(-\frac{f'(x(g(s)))x'(g(s))g'(s)}{f(x(g(s)))}\bigg)\,ds - \int_{t_1}^{t} q(s)\,ds\\ &\leq \omega(t_1) + \int_{t_1}^{t} \omega(s)\bigg(-\frac{f'(x(g(s)))x'(g(s))g'(s)}{f(x(g(s)))}\bigg)\,ds\end{aligned}$$

where $\omega(t) = p(t)h(x(t))x'(t)/f(x(g(t)))$ and $\omega(t_1) < 0$. By using Gronwall's inequality, we get $\omega(t) \leq \omega(t_1)f(x(g(t_1)))/f(x(g(t)))$ or $p(t)h(x(t))x'(t) \leq \omega(t_1)f(x(g(t_1))) = k$ ($k < 0$). Thus, for all large t, we have

$$\int_{x(t_1)}^{x(t)} h(u)\,du \leq k\int_{t_1}^{t} \frac{1}{p(s)}\,ds$$

which, because of (10), implies

$$\lim_{t\to+\infty} \int_{x(t)}^{x(t_1)} h(u)\,du = +\infty,$$

and so a contradiction since $\lim_{t\to+\infty} x(t)$ exists and is finite and h is continuous. The proof is now complete. □

Assumptions (8) and (10) guarantee that (I_2) does not have weakly oscillatory solutions. In general, this does not occur for the corresponding functional differential equation (I_1) since the deviating argument may generate weakly oscillatory solutions as Example 4 shows.

Making use of the same argument given in the proof of Theorem 3, we can obtain the following

Corollary 2. *Assume that*

$$\lim_{t\to+\infty} \int_{t_0}^{t} \frac{1}{p(s)}\,ds = +\infty. \tag{10}$$

If either (8) *or*

$$\begin{cases} \lim_{t\to+\infty}\int_{t_0}^{t} q(s)\,ds & \textit{exists and is finite}\\ \int_t^{+\infty} q(s)\,ds \geq 0 & \textit{for all large } t\end{cases} \tag{11}$$

is satisfied, then for (I) *we have* $\mathbf{M}^- = \varnothing$.

Proof. The assertion follows by noting that condition (11) implies that

$$\liminf_{t\to+\infty} \int_T^t q(s)\,ds \geq 0 \quad \text{for all large } T \tag{12}$$

and reasoning as in the proof of Theorem 3. □

We remark that the role of (8) and (11) in the study of oscillation criteria is considered in [**13**]; in that paper it is also shown that (12) is equivalent to either (8) or (11).

Notice further that Theorem 3 and Corollary 2 do not assume conditions concerning the type of the deviating argument g and so such results cover the ordinary case as well as the delay, advanced, and mixed cases.

Section 4. We consider now the asymptotic behavior of the eventually monotone solutions of (I). As we have just stated, for (I_1) both oscillatory and monotone solutions may exist simultaneously as the following example shows.

Example 5. Consider the functional differential equation

$$x'' + 8\sin t\, f(x(t/2 + \pi/4)) = 0, \qquad t \geq 0, \tag{E_5}$$

where f is a continuously differentiable function such that $f(u) = 1$ for $u \geq 2$ and $f(u) = u$ for $-1 \leq u \leq 1$. The functions $x_1(t) = 10 + 8\sin t$ and $x_2(t) = \sin 2t$ are obviously solutions of (E_5) and $x_1 \in \mathbf{WO}$ and $x_2 \in \mathbf{O}$. Furthermore, by the same argument as that given in Example 1, it is easy to show that $x_3(t) = 8\sin t + 16t + 2$ is a solution of (E_5) in the class $\mathbf{M}^+$.

In a recent paper [**5**], assuming the function q to be negative, the authors gave sufficient conditions for the existence of solutions in the class $\mathbf{M}^-$ which approach zero as $t \to +\infty$. If the function q changes sign or is positive, then an interesting result on the asymptotic behavior of the solution in the class $\mathbf{M}^-$, is suggested by Theorem 1. The following holds:

Theorem 4. *Assume that $g(t) \leq t$. If assumption* (4) *is satisfied, then for every solution $x \in \mathbf{M}^-$ we have*

$$\lim_{t\to+\infty} x(t) = 0.$$

Proof. The assertion follows by the same argument as given in the proof of Theorem 1, taking into account that (6) implies $\lim_{t\to+\infty} x(t) = 0$. $\square$

Such a result completes recent ones obtained in [**10**]. Notice also that if we suppose, in addition to the assumptions of Theorem 4, that condition (1) is verified, then every solution x of (I) is either weakly oscillatory or such that $\liminf_{t\to+\infty} |x(t)| = 0$. This result is closely related to those recently obtained in [**9**] for an equation of the type $[p(t)x'(t)]' + q(t)f(x(t)) = e(t)$, under additional assumptions concerning the nonlinearity and the external force acting on the system.

Finally, we examine the asymptotic behavior of the solutions in the class $\mathbf{M}^+$. The following holds:

Theorem 5. *If the assumption*

$$\limsup_{t\to+\infty} \int_T^t q(s) \int_T^s \frac{1}{p(r)}\,dr\,ds = +\infty \quad \textit{for all } T \geq t_0 \tag{13}$$

is satisfied, then every solution in the class $\mathbf{M}^+$ is unbounded.

Proof. Let x be a solution of (I), $x \in \mathbf{M}^+$. There is no loss of generality in assuming that there exists t_1 such that $x(t) > 0$, $x'(t) \geq 0$, $x(g(t)) > 0$, $x'(g(t)) \geq 0$ for all $t \geq t_1$ since the proof is similar if $x(t) < 0$, $x'(t) \leq 0$ for all large t. Consider the function

$$w(t) = \frac{-p(t)h(x(t))x'(t)}{f(x(g(t)))} \int_{t_1}^t \frac{1}{p(s)}\,ds.$$

We have, for $t \geq t_1$,

$$\begin{aligned} w'(t) &= \frac{-(p(t)h(x(t))x'(t))'}{f(x(g(t)))} \int_{t_1}^{t} \frac{1}{p(s)}\,ds - \frac{h(x(t))x'(t)}{f(x(g(t)))} \\ &\quad + \frac{p(t)h(x(t))x'(t)x'(g(t))g'(t)f'(x(g(t)))}{f^2(x(g(t)))} \int_{t_1}^{t} \frac{1}{p(s)}\,ds \\ &\geq \frac{-(p(t)h(x(t))x'(t))'}{f(x(g(t)))} \int_{t_1}^{t} \frac{1}{p(s)}\,ds - \frac{h(x(t))x'(t)}{f(x(g(t)))} \\ &= q(t) \int_{t_1}^{t} \frac{1}{p(s)}\,ds - \frac{h(x(t))x'(t)}{f(x(g(t)))}. \end{aligned}$$

Hence,

$$w(t) \geq \int_{t_1}^{t} q(s) \int_{t_1}^{s} \frac{1}{p(r)}\,dr\,ds - \int_{t_1}^{t} \frac{h(x(s))x'(s)}{f(x(g(s)))}\,ds. \tag{14}$$

As the function $h(x(t))x'(t)/f(x(g(t)))$ is positive for $t \geq t_1$, then the limit

$$\lim_{t\to+\infty} \int_{t_1}^{t} \frac{h(x(s))x'(s)}{f(x(g(s)))}\,ds$$

exists. We claim that it is infinity. Assume that

$$\lim_{t\to+\infty} \int_{t_1}^{t} \frac{h(x(s))x'(s)}{f(x(g(s)))}\,ds = K < +\infty.$$

Taking into account (13), from (14) we get $\limsup_{t\to+\infty} w(t) = +\infty$ which gives a contradiction since w is negative for all values of $t \geq t_1$.

Thus

$$\lim_{t\to+\infty} \int_{t_1}^{t} \frac{h(x(s))x'(s)}{f(x(g(s)))}\,ds = +\infty. \tag{15}$$

Now for all values of $t \geq t_1$, we have $f(x(g(t))) \geq f(x(g(t_1))) = c$ and, consequently,

$$\int_{t_1}^{t} \frac{h(x(s))x'(s)}{f(x(g(s)))}\,ds \leq \frac{1}{c}\int_{t_1}^{t} h(x(s))x'(s)\,ds = \frac{1}{c}\int_{x(t_1)}^{x(t)} h(u)\,du. \tag{16}$$

From (15), by taking into account the continuity of the function h we get $\lim_{t\to+\infty} x(t) = +\infty$; the proof is now complete. □

A simple example of an equation with unbounded solutions of class $\mathbf{M}^+$ is provided by (E_1). We point out that the condition imposed in Theorem 5 does not involve conditions on the deviating argument g and so such results cover the ordinary case as well as the delay, advanced, and mixed cases. Furthermore, the proof just completed suggests the following stronger result:

Corollary 3. *If the assumptions*

$$\limsup_{t\to+\infty} \int_T^t q(s) \int_T^s \frac{1}{p(r)}\, dr\, ds = +\infty \quad \textit{for all } T \geq t_0 \tag{13}$$

$$\int_{-\infty}^{+\infty} h(u)\, du < +\infty \tag{17}$$

are satisfied, then for equation (I) *we have*

$$\mathbf{M}^+ = \varnothing.$$

Proof. Let x be a solution of (I), $x \in \mathbf{M}^+$. The same argument given in the proof of Theorem 5 shows that (16) holds and so (15) yields a contradiction. □

Theorems 1 and 5 also suggest the following result which is closely related to those recently obtained in [**14**, Chapter 4.3, **18**]:

Corollary 4. *Assume that $g(t) \leq t$. If the assumptions* (3), (4) *and* (13) *are satisfied, then every bounded solution of* (I) *is either oscillatory or weakly oscillatory.*

Proof. The assertion follows easily from Theorems 1 and 5. □

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