ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 4, Fall 1994

OSCILLATORY AND ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF A CLASS OF OPERATOR-DIFFERENTIAL EQUATIONS

D.D. BAINOV, M.B. DIMITROVA AND A.D. MYSHKIS

ABSTRACT. In the present paper the oscillatory and asymptotic properties of the solutions of the operator-differential equation

$$[\tau_{n-1}(t)[\tau_{n-2}(t)[\dots[\tau_1(t)[\tau_0(t)x(t)]']'\dots]']'+\delta(Ax)(t)=0$$

are investigated, where A is a monotonic operator with certain properties.

Particular realizations of the operator A are given, for which the results obtained can be applied.

1. Introduction. In 1987 the book of Ladde, Lakshmikantham, Zhang [2] was published. In it for the first time in sufficient detail problems related to the oscillation and asymptotic theory of functional differential equations are considered. Parallel to the development of the oscillation theory of functional differential equations the development of the oscillation and asymptotic theory of various classes of ordinary differential equations began, such as differential equations with "maxima," impulsive differential equations, integro-differential equations, etc. We shall note that the results obtained for these equations are of isolated character and the traditional problems set in the oscillation theory are almost untouched for them.

In the present paper the oscillatory and asymptotic properties of the solutions of a class of homogeneous operator-differential equations are investigated, and, thus, by means of a single approach, the properties of the solutions of numerous little investigated classes of differential equations are studied. We shall note that an analogous approach was used in Mishev, Bainov [**3**].

Received by the editors on August 1, 1992.
The first and second authors were supported by the Bulgarian Ministry of Science and Higher Education under Grant MM-7.

2. Preliminary notes. Consider the operator-differential equation

$$(1)\qquad [\tau_{n-1}(t)[\tau_{n-2}(t)[\dots[\tau_1(t)[\tau_0(t)x(t)]']'\dots]']']' + \delta(Ax)(t) = 0$$

where $\delta = \pm 1$, $n \geq 1$; the number $t_0 \in R$ is fixed. A is an operator with certain properties: $\tau_i \in C([t_0,\infty),(0,\infty))$, $i = 0,\dots,n-1$. Here $C(M,N)$ is the set of all continuous functions $f : M \to N$.

Introduce the following notation:

$$(L_0x)(t) = \tau_0(t)x(t)$$

$$(L_ix)(t) = \tau_i(t)[(L_{i-1}x)(t)]', \qquad i = 1,\dots,n, \quad \tau_n(t) :\equiv 1.$$

$$R_0(t) = 1 \qquad R_1(t) = \int_{t_0}^{t} \frac{ds_1}{\tau_1(s_1)}$$

$$R_i(t) = \int_{t_0}^{t} \frac{ds_1}{\tau_1(s_1)} \left(\int_{t_0}^{s_1} \frac{ds_2}{\tau_2(s_2)} \left(\cdots \left(\int_{t_0}^{s_{i-1}} \frac{ds_i}{\tau_i(s_i)} \right) \cdots \right) \right),$$
$$i = 2,\dots,n-1,\ n > 2.$$

Denote by $\mathcal{D}_n$ the set of all functions $x \in C([T_x,\infty);R)$, $T_x \geq t_o$, such that the functions L_ix, $i = 0,1,\dots,n$, exist and are continuous for $[T_x,\infty)$.

Definition 1. The function $x : [T_x,\infty) \to R$ is said to be a *solution* of equation (1) if $x \in \mathcal{D}_n$ and x satisfies equation (1) for

$$t \geq \max\{T_x, TA_x\}, \qquad TA_x \geq t_0.$$

Definition 2. A given function $u : [t_0,\infty) \to R$ is said to *eventually* enjoy the property P if there exists a point $t_{p,u} \geq t_0$ such that for $t \geq t_{p,u}$ the property P is valid.

Definition 3. The function $x \in C([T_x,\infty);R)$ is said to be *eventually zero* if $x(t) = 0$ eventually, and *eventually nonzero* otherwise.

Definition 4. The function $x \in C([T_x,\infty);R)$ is said to *oscillate* if there exists a sequence of numbers $t_1 < t_2 < \cdots < t_n < \cdots$,

$\lim_{n\to\infty} t_n = \infty$ such that $x(t_i)x(t_{i+1}) < 0$. Otherwise the function is said to be *nonoscillating*. The function x is said to *weakly oscillate* if $\sup\{t : x(t) = 0\} = \infty$.

For any nonoscillating function $y \in C([T_y, \infty); R)$, $T_y \geq t_0$, we define the function

$$(\psi_{n-1}y)(t) = \int_t^\infty y(s)\,ds$$
$$(\psi_i y)(t) = \int_t^\infty \frac{(\psi_{i+1}y)(s)}{\tau_{i+1}(s)}\,ds,$$

$i = n-2, \dots, 0$; $n > 1$.

We shall say that conditions (H) are satisfied if the following conditions hold:

H1. $\tau_i \in C([t_0, \infty), (0, \infty))$, $i = 0, \dots, n-1$.

H2. $\int_{t_0}^\infty dt/\tau_i(t) = \infty$, $i = 1, \dots, n-1$.

H3. $A : \mathcal{D}_n \to C([T_{Ax}, \infty); R)$

H4. If the functions $x_1, x_2 \in \mathcal{D}_n$ and $x_1(t) \leq x_2(t)$ eventually, then $(Ax_1)(t) \leq (Ax_2)(t)$ eventually.

H5. If the function $x \in \mathcal{D}_n$ and $x(t) = 0$ eventually, then $(Ax)(t) = 0$ eventually.

H6. If the function $x \in \mathcal{D}_n$ and x is eventually nonzero and nonoscillating, then the function Ax is eventually nonzero.

Lemma 1. *Let the following conditions hold*:

1. *Conditions* H1 *and* H2 *hold.*
2. *The function* $x \in \mathcal{D}_n$
3. *The function* $L_n x$ *is eventually nonzero and nonoscillating.*

Then:

1. *Each function* $L_i x$, $i = 0, \dots, n-1$, *is eventually monotonic and nonzero.*

2. *If* $n > 1$ *and* $\lim_{t\to\infty}(L_i x)(t) \neq 0$ *for some* $i = 1, \dots, n-1$, *then* $\lim_{t\to\infty}(L_j x)(t) = \operatorname{sgn}(\lim_{t\to\infty}(L_i x)(t)) \cdot \infty$ *for any* $j = 0, \dots, i-1$.

3. *If* $\lim_{t\to\infty}(L_i x)(t) = 0$ *for some* $i = 0, \dots, n-1$, *then* $(L_j x)(t) \cdot (L_{j+1} x)(t) \le 0$ *eventually for* $j = i, \dots, n-1$.

4. *If* $(L_i x)(t) \cdot (L_{i+1} x)(t) \le 0$ *eventually for some* $i = 0, \dots, n-1$, *then* $\lim_{t\to\infty}(L_i x)(t) = c \in \mathbf{R}$.

5. *If* $\lim_{t\to\infty}(L_i x)(t) \in \mathbf{R}$ *for some* $i = 0, \dots, n-1$, *then*

$$\int_t^\infty \frac{|(L_{i+1}x)(s)|}{\tau_{i+1}(s)}\, ds < \infty$$

and

$$(L_i x)(t) = \lim_{t\to\infty}(L_i x)(t) + (-1)^{n-i}(\psi_i L_n x)(t) \tag{2}$$

6. *For* $n > 1$ *the following equality is valid*

$$\lim_{t\to\infty} \frac{(L_0 x)(t)}{R_i(t)} = \lim_{t\to\infty}(L_i x)(t), \qquad i = 1, \dots, n-1. \tag{3}$$

Remark 1. Lemma 1 is a corollary of the respective theorems proved in [**1, 5, 6, 7**].

3. Main results.

Theorem 1. *Let the following conditions hold*:

1. *Conditions* (H) *are met.*

2. *For any constant* $c \in \mathbf{R}\backslash\{0\}$ *there exists an integer* $i \in [0, n-1]$ *such that eventually the following relation is valid*

$$\left| \left(\psi_i A \frac{c}{\tau_0} \right)(t) \right| = \infty. \tag{4}$$

Then for the existence of an eventually nonzero nonoscillating solution x *of equation* (1) *for which* $L_0 x$ *is a bounded function it is necessary for* $\delta = 1$ ($\delta = -1$) *that* n *be an odd* (*even*) *number. For these solutions the following relations are valid*

$$\lim_{t\to\infty}(L_i x)(t) = 0$$

and

$$\operatorname{sgn}(L_i x)(t) = (-1)^i \operatorname{sgn} x(t), \qquad i = 0, \dots, n-1.$$

Proof. Let $x \in \mathcal{D}_n$ be an eventually nonzero, nonoscillating solution of equation (1). Without loss of generality, we can assume that $x(t) \geq 0$ for $t \in [t_1, \infty)$, where $t_1 \geq t_0$. From condition H4 it follows that there exists a point $t_2 \geq t_1$ such that $(Ax)(t) \geq 0$ for $t \geq t_2$.

Then from equation (1) it follows that

$$\delta(L_n x)(t) = -(Ax)(t) \leq 0 \quad \text{for } t \geq t_2, \tag{5}$$

i.e., we can apply Lemma 1.

We shall prove that $\lim_{t\to\infty}(L_0 x)(t) = 0$. If we suppose that this is not true, then from assertion 1 of Lemma 1 it follows that $\tau_0(t)x(t) \geq c > 0$ eventually for some constant c.

From condition H4 and (1) it follows that eventually the following inequality holds

$$|(L_n x)(t)| = (Ax)(t) \geq \left(A\frac{c}{\tau_0}\right)(t). \tag{6}$$

Choose i, $i = 0, \dots, n-1$, which corresponds to the constant c (see condition 2 of Theorem 1). From assertion 2 of Lemma 1 it follows that $\lim_{t\to\infty}(L_i x)(t)$ is a finite number. Then from equality (2) it follows that

$$|(\psi_i L_n x)(t)| < \infty. \tag{7}$$

From inequalities (6) and (7) we obtain that

$$\left|\left(\psi_i A\frac{c}{\tau_0}\right)(t)\right| < \infty,$$

which contradicts condition (4). Hence

$$\lim_{t\to\infty}(L_0 x)(t) = 0. \tag{8}$$

From (8) and assertions 2 and 3 of Lemma 1 it follows that

$$(9)\qquad \begin{aligned}&(L_j x)(t)\cdot(L_{j+1}x)(t)\le 0, \qquad j=0,\dots,n-1\\ &\lim_{t\to\infty}(L_j x)(t)=0.\end{aligned}$$

From inequalities (9) and (5) it follows that for $\delta = 1$ ($\delta = -1$) the number n is odd (even). □

Theorem 2. *Let the following conditions hold*:

1. *Conditions* (H) *are met.*

2. *Condition* 2 *of Theorem* 1 *holds.*

3. $\delta = 1$.

4. *For $n > 1$ for any integer $i \in [0, n-2]$ which is odd or even just as n is, and for any constant $c \in \mathbf{R}\backslash\{0\}$ the following relation is eventually valid*

$$(10)\qquad \left|\left(\psi_{i+1}A\frac{cR_i}{\tau_0}\right)(t)\right| = \infty.$$

Then the assertion of Theorem 1 *is valid without the requirement for boundedness of the function L_0x.*

Proof. Let x be an eventually nonzero nonoscillating solution of equation (1). Without loss of generality assume that $x(t) \ge 0$ eventually. Then from conditions H4, H5, condition 3 of Theorem 1 and equation (1) it follows that $(L_n x)(t) \le 0$ eventually and we can apply Lemma 1. Let i be the greatest integer for which eventually the inequalities $(L_i x)(t) > 0$ and $(L_{i+1}x)(t) > 0$ are valid (if there exists no i with this property, then the boundedness of the function L_0x follows immediately from Lemma 1). From Lemma 1 it follows that $n - i \equiv 0 \pmod 2$ and $\lim_{t\to\infty}(L_i x)(t) > 0$. From the last inequality and from (3) we derive that eventually the following inequality is valid

$$\frac{(L_0x)(t)}{R_i(t)} \ge c > 0, \qquad \text{i.e., } x(t) \ge \frac{cR_i(t)}{\tau_0(t)}.$$

We apply Lemma 1 and obtain that $\lim_{t\to\infty}(L_{i+1}x)(t) < \infty$. Then from (2) it follows that

$$(11)\qquad |(\psi_{i+1}L_n x)(t)| < \infty.$$

From the fact that eventually

$$|(L_n x)(t)| = (Ax)(t) \geq \left(A\frac{cR_i}{\tau_0}\right)(t) \tag{12}$$

and from inequalities (11) and (12) it follows that

$$\left|\left(\psi_{i+1} A\frac{cR_i}{\tau_0}\right)(t)\right| < \infty$$

which contradicts condition (10). □

Corollary 1. *Let the conditions of Theorem* 2 *hold and let* n *be even. Then all solutions of equation* (1) *oscillate.*

Theorem 3. *Let the following conditions hold*:

1. *Conditions* (H) *are met.*
2. *Condition* 2 *of Theorem* 1 *holds.*
3. $\delta = -1$.
4. *For any integer* $i \in [0, n-1]$ *which is odd if* n *is even and vice versa, and for any constant* $c \in \mathbf{R}\backslash\{0\}$ *the following relation is eventually valid*

$$\left|\left(\psi_{\min\{i+1,n-1\}} A\frac{ck_i}{\tau_0}\right)(t)\right| = \infty. \tag{13}$$

Then for any nonoscillating solution x *of equation* (1) *just one of the following assertions is valid*:

1. $\lim_{t\to\infty}(L_j x)(t) = 0$, $j = 0, \dots, n-1$ *and then* n *is even.*
2. $|\lim_{t\to\infty}(L_j x)(t)| = \infty$, $j = 0, \dots, n-1$ *and then*

$$\frac{R_{n-1}(t)}{\tau_0(t)} = o(x(t)) \quad \textit{as } t \to \infty.$$

Proof. Let x be an eventually nonzero nonoscillating solution of equation (1) for $\delta = -1$. Without loss of generality we can assume that $x(t) \geq 0$. Then $(L_n x)(t) \geq 0$ eventually. Denote by i ($i = 0, \dots, n-1$) the least integer such that the inequalities $(L_i x)(t) \geq 0$ and $(L_{i+1} x)(t) \geq 0$ are eventually valid (if there exists no such integer $i \in [0, n-1]$), then from Lemma 1 it follows that the function $L_0 x$ is bounded. Then assertion 1 of Theorem 3 follows from Theorem 1). Then $n - i \equiv 1 \pmod 2$.

If $i = n-1$, then from Lemma 1 we obtain that $\lim_{t\to\infty}(L_{n-1}x)(t) > 0$. Suppose that $\lim_{t\to\infty}(L_{n-1}x)(t) < \infty$. Then eventually

$$(\psi_{n-1} A x)(t) = (\psi_{n-1} L_n x)(t) < \infty. \tag{14}$$

But from assertion 6 of Lemma 1 it follows that eventually

$$\frac{(L_0 x)(t)}{R_{n-1}(t)} \geq c > 0, \qquad \text{i.e., } x(t) \geq \frac{c R_{n-1}(t)}{\tau_0(t)}. \tag{15}$$

From (14) and (15) we obtain a contradiction with condition (13). Hence $\lim_{t\to\infty}(L_{n-1}x)(t) = \infty$. Then $\lim_{t\to\infty}(L_j x)(t) = \infty$ for $j = 0, \dots, n-1$. The relation $R_{n-1}(t)/\tau_0(t) = o(x(t))$ as $t \to \infty$ follows from (3) for $i = n-1$. For $i \leq n-3$ we get to a contradiction by arguments analogous to those in the proof of Theorem 2. □

Remark 2. In Theorems 1–3 let condition H6 be replaced by the weaker condition H7:

H7. If $x \in \mathcal{D}_n$ and $\sup\{t : x(t) = 0\} < \infty$, then the function Ax is eventually nonzero.

Then the assertions of Theorems 1–3 will be valid if we replaced in them *nonoscillating solution* by *nonweakly-oscillating solution.*

4. Some particular realizations of the operator.

Theorem 4. *Let the following conditions hold*:

1.

$$\begin{aligned}(Ax)(t) = &\max_{p_1(t)\leq s\leq q_1(t)} F_1(t, g_1(d_1(s)x(h_1(s)))) \\ &+ \min_{p_2(t)\leq s\leq q_2(t)} F_2(t, g_2(d_2(s)x(h_2(s))))\end{aligned} \tag{16}$$

where for each $i = 1, 2,$

$$p_i, q_i, d_i, h_i \in C([t_0, \infty); R)$$
$$\lim_{t\to\infty} p_i(t) = \lim_{t\to\infty} h_i(t) = \infty, \qquad d_i(t) > 0$$

$p_i(t) \le q_i(t)$, $g_i \in C(R, R)$ *are nondecreasing functions and* $\operatorname{sgn} g_i(u) = \operatorname{sgn} u$, $F_i \in C([t_0, \infty) \times R, R)$

$F_i(t, u)$ *are nondecreasing functions with respect to* u.

2. *Conditions* H1–H2 *hold.*

3. *Condition* 2 *of Theorem* 1 *holds.*

Then for the existence of an eventually nonzero nonoscillating solution x *of the equation*

$$(17) \qquad \begin{aligned}(L_n x)(t) + \delta\Big(&\max_{p_1(t)\le s\le q_1(t)} F_1(t, g_1(d_1(s)x(h_1(s))))\\ &+ \min_{p_2(t)\le s\le q_2(t)} F_2(t, g_2(d_2(s)x(h_2(s))))\Big) = 0\end{aligned}$$

for which $L_0 x$ *is a bounded function it is necessary for* $\delta = 1$ $(\delta = -1)$ *that* n *be an odd* (*even*) *number. For these solutions the following relations are valid*

$$\lim_{t\to\infty} (L_i x)(t) = 0$$

and

$$\operatorname{sgn}(L_i x)(t) = (-1)^i \operatorname{sgn} x(t), \qquad i = 0, \dots, n-1.$$

Theorem 5. *Let the following conditions hold*:

1. *Condition* 1 *of Theorem* 4 *holds.*

2. *Conditions* H1–H2 *hold.*

3. *Conditions* 2, 3 *and* 4 *of Theorem* 2 *are met.*

Then the assertion of Theorem 4 *is valid without the requirement for boundedness of the function* $L_0 x$.

Theorem 6. *Let the following conditions hold*:

1. *Condition* 1 *of Theorem* 4 *holds.*

2. *Conditions* H1–H2 *hold.*

3. *Conditions* 2, 3 *and* 4 *of Theorem* 3 *are met.*

Then, for any nonoscillating solution of equation (17) *just one of the following assertions is valid*:

1. $\lim_{t\to\infty}(L_j x)(t) = 0$, $j = 0, \dots, n-1$ *and then* n *is even.*

2. $|\lim_{t\to\infty}(L_j x)(t)| = \infty$, $j = 0, \dots, n-1$ *and then*

$$\frac{R_{n-1}(t)}{\tau_0(t)} = o(x(t)) \quad \text{as } t \to \infty.$$

Theorems 4, 5, and 6 are particular cases of Theorems 1, 2 and 3, respectively.

Remark 3. If only one of the two addends enter the right-hand side of (16), then the operator A satisfies conditions H3–H5 and H7.

Theorem 7. *Let the following conditions hold*:

$$\text{(18)} \qquad 1.\ (Ax)(t) = F\left(t, \int_{p(t)}^{q(t)} k(t, s, x(t), x(s))\, d_s\tau(t, s)\right)$$

where

a) *The function* F *satisfies condition* 1 *of Theorem* 4.

b) $p, q \in C([t_0, \infty); R)$, $\lim_{t\to\infty} p(t) = \infty$, $p(t) < q(t)$

c) $k \in C([t_0, \infty)^2 \times R^2, R)$. *The function* $k(t, s, u, v)$ *is nondecreasing with respect to* u *and* v.

d) $\operatorname{sgn} k(t, s, u, 0) = \operatorname{sgn} u$, $\operatorname{sgn} k(t, s, 0, v) = \operatorname{sgn} v$

e) *For any* $t \in [t_0, \infty)$ *the function* $s \to \tau(t, s)$ *is increasing.*

f) *The functions* $t \to \tau(t, p(t))$ *and* $t \to \tau(t, q(t))$ *are continuous and for* $t \in [t_0, \infty)$ *the following relation is valid*

$$\lim_{t'\to t} \int_{\max\{p(t), p(t')\}}^{\min\{q(t), q(t')\}} |\tau(t', s) - \tau(t, s)|\, ds = 0$$

2. *Conditions* H1–H2 *are met.*
3. *Condition* 2 *of Theorem* 1 *holds.*

Then for the existence of an eventually nonzero nonoscillating solution x of the equation

$$(L_n x)(t) + \delta F\left(t, \int_{p(t)}^{q(t)} k(t,s,x(t),x(s))\, d_s\tau(t,s)\right) = 0 \tag{19}$$

for which $L_0 x$ is a bounded function it is necessary for $\delta = 1$ ($\delta = -1$) that n be an odd (*even*) *number. For these solutions the following relations are valid*

$$\lim_{t\to\infty} (L_i x)(t) = 0$$

and

$$\operatorname{sgn} (L_i x)(t) = (-1)^i \operatorname{sgn} x(t), \qquad i = 0, \dots, n-1.$$

Theorem 8. *Let the following conditions hold*:

1. *Condition* 1 *of Theorem* 7 *holds.*
2. *Conditions* H1–H2 *hold.*
3. *Conditions* 2, 3 *and* 4 *of Theorem* 2 *are met.*

Then the assertion of Theorem 7 *is valid without the requirement for boundedness of the function $L_0 x$.*

Theorem 9. *Let the following conditions hold*:

1. *Condition* 1 *of Theorem* 7 *holds.*
2. *Conditions* H1–H2 *hold.*
3. *Conditions* $2, 3$ *and* 4 *of Theorem* 3 *are met.*

Then for any nonoscillating solution x of equation (19) *just one of the following assertions is valid*:

1. $\lim_{t\to\infty} (L_j x)(t) = 0$, $j = 0, \dots, n-1$ *and then n is even.*
2. $|\lim_{t\to\infty} (L_j x)(t)| = \infty$, $j = 0, \dots, n-1$ *and then*

$$\frac{R_{n-1}(t)}{\tau_0(t)} = o(x(t)) \quad \text{as } t \to \infty.$$

Theorems 7, 8 and 9 are particular cases of Theorems 1, 2 and 3, respectively.

Remark 4. If the operator A defined by (18) satisfies condition 1 of Theorem 7 but conditions 1d and 1e are replaced by the conditions

$$k(t,s,u,v) > 0 \quad \text{for} \quad u > 0, v > 0$$
$$k(t,s,u,v) < 0 \quad \text{for} \quad u < 0, v < 0$$

and the function $s \to \tau(t,s)$ is nondecreasing and nonconstant in the interval $[p(t), q(t)]$, respectively. Then the operator A satisfies conditions H3–H5 and H7.

REFERENCES

1. I.T. Kiguradze, *Some singular boundary value problems for ordinary differential equations*, Izd. Tbilisi State Univ., Tbilisi, 1975 (in Russian).

2. G.S. Ladde, V. Lakshmikantham and B.G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Pure Appl. Math. **110** (1987), 305 pp.

3. D.P. Mishev and D.D. Bainov, *Some properties of the nonoscillating solutions of functional differential equations of n-th order*, Rend. Circ. Mat. Palermo **35** (1986), 233–243.

4. A.D. Myshkis, *Linear differential equations with retarded argument*, Nareka, Moscow, 1972 (in Russian).

5. Ch.G. Philos, *Oscillatory and asymptotic behavior of the bounded solutions of differential equations with deviating arguments*, Hiroshima Math. J. **8** (1978), 31–48.

6. ———, *Oscillatory and asymptotic behavior of all solutions of differential equations with deviating arguments*, Proc. Roy. Soc. Edinburgh Ser. A **81** (1978), 195–210.

7. ———, *On the asymptotic behavior of the nonoscillatory solutions of differential equations with deviating arguments*, Math. Nachr. **113** (1983), 107–128.

8. Ch. Philos and V.A. Staikos, *A basic asymptotic criterion for differential equations with deviating arguments and its applications to the nonoscillation of linear ordinary equations*, Nonlinear Anal. Theory Appl. **6** (1982), 1095–1113.

9. Ch.G. Philos, Y.G. Sficas and V.A. Staikos, *Some results on the asymptotic behavior of nonoscillatory solutions of differential equations with deviating arguments*, J. Austral. Math. Soc. **32** (1982), 295–317.

South-West University, Neophyte Rilski, Blagoevgrad, Bulgaria

Technical University, Silven, Bulgaria

Railway Transport Engineering Institute, Moscow, USSR