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u-INDEPENDENCE AND QUADRATIC u-INDEPENDENCE IN THE CONSTRUCTION OF INDECOMPOSABLE FINITELY GENERATED MODULES

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ABSTRACT. Let R be a valuation domain having an ideal I such that a maximal immediate extension S of R contains four units u-independent over I. We construct a 4-generated indecomposable R-module M with Goldie dimension $g(M) = 2$. We thus supplement a result by Lunsford who constructed indecomposable finitely generated R-modules making use of sets of quadratically u-independent elements of S.

1. Introduction. Let R be a valuation domain, and let S be a fixed maximal immediate extension of R. There is a somewhat standard way to define finitely generated R-modules M by generators and relations, relating M to a set of units $u_1, \dots, u_n$ of S. Starting with [**5**] and [**8**], an extensive use of this idea was made. See also the books by Fuchs and Salce [**2**, Chapter 9] and [**3**, Chapter 5]. The notion of u-independence of units $u_1, \dots, u_n$ of S over an ideal I of R was introduced in [**8**] and investigated further in [**9**]. It was used to show the existence of indecomposable finitely generated R-modules M (related with $u_1, \dots, u_n$) with minimal number of generators $l(M) = n+1$ and Goldie dimension $g(M) = n$. This solved the problem of finding indecomposable finitely generated R-modules with Goldie dimension greater than one. However, it is worth noting that the argument developed in [**8**] worked only in the case when $l(M) = g(M) + 1$.

Lunsford [**4**] in 1995 gave a natural generalization of u-independence, defining *quadratic u-independence* of units $u_1, \dots, u_n$ of S over an ideal I. Starting with a sufficiently large set of units of S, for any pair of positive integers h, k he defined by generators and relations an R-module M with $l(M) = h+k$ and $g(M) = h$. Actually this type of module had already been introduced in 1987 by Salce and Zanardo [**7**]. Using quadratic u-independence, Lunsford was able to prove that such

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an M is indecomposable. In fact, he proved much more, namely that $\mathrm{End}_R(M)$ is a local ring. Let us recall that the endomorphism ring of indecomposable R-modules is, in general, not local (see [**8**], [**6**]).

The above-mentioned constructions are also described in [**2**] and [**3**]. For the convenience of the reader we recall them in Section 1.

The notion of quadratic u-independence over an ideal I has a slight inconvenience: it works only in case I is a *prime ideal* of R (necessarily nonzero and nonmaximal in this setting). And, in general, even if I is a prime ideal, one may have large sets of elements u-independent over I, but none that are quadratically u-independent. Therefore, Lunsford's results cannot prove the existence of indecomposable finitely generated modules M with $1 < g(M) < l(M) - 1$ for large classes of valuation domains R. For example, archimedean valuation domains, where the only primes are the maximal ideal and zero.

In the present paper we show that the existence of a set of four units of S which are u-independent over any nonzero ideal I is enough to ensure the existence of an indecomposable R-module M with $l(M) = 4$ and $g(M) = 2$. The definition of M by generators and relations is the same as in [**4**] (and in [**6**]). But, since the related units of S are just u-independent and not quadratically u-independent, we need a completely different, more direct argument to prove that M is indecomposable.

1. Preliminaries. In what follows R will denote a valuation domain and P will denote its maximal ideal. Let us recall some notions and results on valuation domains and their finitely generated modules (see [**2**] and [**3**] for a general exposition).

Let M be a finitely generated R-module. There exists a finite ascending chain of submodules

$$0 = M_0 < M_1 < \cdots < M_{n-1} < M_n = M \tag{1}$$

such that

(i) every M_i is pure in M_{i+1};

(ii) M_{i+1}/M_i is cyclic.

(Here and throughout, the symbol "$<$" denotes proper inclusion.)

We may also choose (1) in such a way that, setting $A_i = \mathrm{Ann}\,(M_i/M_{i-1})$, $1 \le i \le n$, we have

$$\mathrm{Ann}\,(M) = A_1 \le A_2 \le \cdots \le A_n.$$

A *pure-composition series* for M is any chain M_i of M, as in (1), satisfying conditions (i) and (ii) as above. The sequences of ideals $A_i = \mathrm{Ann}\,(M_i/M_{i-1})$ is called the *annihilator sequence* of M. Any two pure-composition series of M are isomorphic. Therefore, M determines its annihilator sequence (up to the order).

We denote by $l(M)$ the minimal number of generators of M; $l(M)$ is the common length of the pure-composition series of M. When we say that M is n-generated, we mean that $l(M) = n$. The *Goldie dimension* of M is the minimal number of generators of any direct sum of cyclics which is essential in M; it is denoted by $g(M)$.

For an assigned valuation domain R, we denote by S a fixed maximal immediate extension of R. Recall that S is not in general determined as a ring, but it is determined as an R-module: S is the pure-injective envelope of the R-module R.

Let u be a unit of S not in R. The *breadth ideal* $B(u)$ of u is defined by

$$B(u) = \{a \in R : u \notin R + aS\}.$$

Since R is a valuation domain, it is readily checked that the above set is in fact an ideal. Recall that $I < R$ is a breadth ideal exactly if R/I is not complete the topology of its ideals.

The units $u_1, \ldots, u_n \in S \setminus R$ are said to be *u-independent* over the ideal I of R if $I = B(u_i)$ for all $i \le n$ and, for any given congruence of the form $a_0 + \sum_{i=1}^n a_i u_i \equiv 0 \bmod IS$ with $a_0, \ldots, a_n \in R$, we must have $a_0, \ldots, a_n \in P$ (see [**8**] or [**2**], [**3**]).

The notion of u-independence was used in [**8**] to construct indecomposable finitely generated R-modules as follows.

Assume that $u_1, \ldots, u_n \in S \setminus R$ are u-independent over a suitable nonzero ideal I of R. Let us choose $0 \ne a \in I$, and let $A = aR$. Let us consider the ideal $J = \{r \in R : rI < A\}$. We set $J^* = J \setminus A$; by definition $r^{-1}A > I$ for all $r \in J^*$, and one can check that $\bigcap_{r \in J^*} r^{-1}A = I$.

Since $B(u_i) = I$, by the definition of breadth ideal for every $i \leq n$, there exists a family $\{u_i^r : r \in J^*\}$ of units of R satisfying the condition

$$u_i^r - u_i \in r^{-1}AS, \quad \forall r \in J^*.$$

As in [**8**] (see also [**4**]), we define by generators and relations a finitely generated R-module $M = \langle x_0, x_1, \dots, x_n \rangle$, where the generators x_i are subject to the conditions:

$$\mathrm{Ann}\,(x_i) = A; \quad rx_0 = r\sum_{i=1}^{n} u_i^r x_i, \quad \forall r \in J^*.$$

Note that the above relations are consistent, since $u_i^r - u_i^s \in r^{-1}A$ whenever $r, s \in J^*$ and s divides r.

Since $u_1, \dots, u_n$ are u-independent over I, the module M turns out to be indecomposable and $(n+1)$-generated, in view of Theorem 6 of [**8**]. The annihilator sequence of M is given by $A = \cdots = A < J$. The submodule $B = \langle x_1, \dots, x_n \rangle = \langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is pure and essential in M, whence $g(M) = n$. Note that B is a direct sum of cyclic submodules since its annihilator sequence is of the form $A = \cdots = A$ (see [**2**]). Here we recall that a direct sum of cyclics which is pure and essential in M is said to be a *basic* submodule of M. All basic submodules are isomorphic.

A set $\{u_1, \dots, u_n\}$ is *quadratically u-independent* over an ideal I of R if $B(u_i) = I$ for all $i \leq n$ and, for every polynomial $f \in R[X_1, \dots, X_n]$ of degree ≤ 2, $f(u_1, \dots, u_n) \equiv 0 \bmod IS$ implies $f \in P[X_1, \dots, X_n]$ (see [**4**]).

Let us remark that, if the units $u_1, \dots, u_n$ are quadratically u-independent over I, then I is necessarily a prime ideal ([**4**, Lemma 1]).

Suppose now that we have a set of units $\{u_{ij} : 1 \leq i \leq h; 1 \leq j \leq k\}$ quadratically u-independent over a nonzero ideal I. One defines a finitely generated module related to $\{u_{ij}\}$ as follows (see [**4**]; see also [**7**]). Let the ideals A, J be as above. Since $B(u_{ij}) = I$, $r^{-1}A > I$ for all $r \in J^*$ and $\cap_{r \in J^*} r^{-1}A = I$ for all i, j there exist units $u_{ij}^r \in R$ such that

$$u_{ij}^r - u_{ij} \in r^{-1}AS, \quad \forall r \in J^*.$$

We set $M = M_{h,k} = \langle x_1, \dots, x_h, y_1, \dots, y_k\rangle$, where the generators are subject to the relations: $Ax_i = 0$, $1 \le i \le k$, and

$$ry_j = r\sum_{i=1}^{h} u_{ij}^r x_i, \quad r \in J^*, 1 \le j \le k.$$

Then we have the following result

Theorem 1.1 [**4**]. *In the above notation of the preceding module M, we have $l(M) = h + k$, $g(M) = h$ and, if $\{u_{ij}\}$ is quadratically u-independent over I, then $\mathrm{End}_R(M)$ is a local ring.*

It is useful to note that the increasing annihilator sequence of the preceding M is given by $A = \cdots = A < J = \cdots = J$ with h copies of A and k copies of J.

2. An indecomposable module M with $l(M) = 4$ and $g(M) = 2$. We will consider a 4-generated R-module M, with $g(M) = 2$, defined in the same way as in Lunsford's paper, but assuming that the units involved are just u-independent and not necessarily quadratically u-independent. We will prove that M turns out to be indecomposable, even with this weaker assumption. Note that, in this particular case, we will adopt a notation simpler than that in [**4**].

Theorem 2.1. *Let us suppose that four units of S not in R exist which are u-independent over a suitable nonzero ideal I of R. Then there exists an indecomposable 4-generated module M with Goldie dimension $g(M) = 2$.*

Proof. Let u_1, u_2, v_1, v_2 be units of S u-independent over a suitable nonzero ideal I of R. Let the ideals A, J be as in Section 1. Recall that $I = \bigcap_{r\in J^*} r^{-1}A$. Choose units u_i^r, v_j^r of R such that, for $1 \le i$, $j \le 2$,

$$u_i^r - u_i \in r^{-1}AS, \qquad v_j^r - v_j \in r^{-1}AS, \quad \forall r \in J^*. \tag{2}$$

Set $M = \langle x_1, \dots, x_r\rangle$, where the generators x_i are subject to the conditions: $\mathrm{Ann}\,(x_i) = A$ for $1 \le i \le 4$ and, for all $r \in J^*$,

$$rx_1 = r(u_1^r x_3 + v_1^r x_4), \qquad rx_2 = r(u_2^r x_3 + v_2^r x_4). \tag{3}$$

It is readily checked that $l(M) = 4$. Moreover, using Lemma 4 of [**4**] we see that $B = \langle x_3, x_4 \rangle = \langle x_3 \rangle \oplus \langle x_4 \rangle$ is a basic submodule of M, whence in particular $g(M) = 2$. Note that the increasing annihilator sequence of M is as follows:

$$A = A < J = J.$$

We shall prove that M is indecomposable in two steps.

STEP 1. M does not admit cyclic summands.

By contradiction, let us assume that $M = \langle y_1 \rangle \oplus \langle y_2, y_3, y_4 \rangle = \langle y_1 \rangle \oplus N$. Let us observe that, necessarily, $\mathrm{Ann}\,(y_1) = A$, since any basic submodule of M is isomorphic to $R/A \oplus R/A$. Let $T = (a_{ij})$ be the 4×4 invertible matrix with entries in R such that $T\underline{x} = \underline{y}$, where $\underline{x}$ denotes the column whose entries are $x_1, \dots , x_4$, and similarly for $\underline{y}$.

Our first aim is to manipulate the generators $y_1, \dots , y_4$ in such a way that we may assume, without loss of generality, that the entries a_{11}, a_{12} of T satisfy the condition: $a_{11} \equiv a_{12} \bmod P$.

Let us note that not all the entries a_{i1}, a_{i2}, $2 \leq i \leq 4$ lie in P; otherwise $|T| \in P$ in view of the Laplace rule. Therefore, there exist $s \in R$ and $2 \leq k \leq 4$ such that either $a_{11} - sa_{k1} = 0$ or $a_{12} - sa_{k2} = 0$. Thus, replacing y_1 with $y_1 - sy_k$ we may assume that in T either a_{11} or a_{12} is zero; indeed, note that $M = \langle y_1 - sy_k \rangle \oplus \langle y_2, y_3, y_4 \rangle$, since $\mathrm{Ann}\,(y_1) = \mathrm{Ann}\,(M)$.

If now $a_{11}, a_{12} \in P$ we are done, since then $a_{11} \equiv a_{12} \equiv 0 \bmod P$.

Otherwise, let us assume that a_{11} is a unit and $a_{12} = 0$ (similar arguments work in the symmetric case). Without loss of generality, let $a_{11} = 1$.

We now have to distinguish various cases.

A) There exists an index k, with $2 \leq k \leq 4$, such that a_{k1} is a unit and $a_{k2} \in P$.

In this case we may replace y_1 with $z = y_1 - a_{k1}^{-1} y_k$. We get $M = \langle z \rangle \oplus \langle y_2, y_3, y_4 \rangle$ and $z = \sum_{j=1}^{k} b_{1j} x_j$, where $b_{11}, b_{12} \in P$.

B) The case A) is never satisfied and there exists an index k, with $2 \leq k \leq 4$, such that a_{k2} is a unit and $a_{k1} \not\equiv a_{k2} \bmod P$.

Let us set $t = a_{k2}^{-1}$ and $s = 1 - a_{k1}t$. Let us note that $s \notin P$, since otherwise $a_{k1} \equiv a_{k2} \equiv t^{-1} \bmod P$. Replacing y_1 with $z = sy_1 + ty_k$, we have, as above, $M = \langle z \rangle \oplus \langle y_2, y_3, y_4 \rangle$. Now $z = \sum_{j=1}^{k} c_{1j}x_j$, where $c_{11} \equiv c_{12} \equiv 1 \bmod P$.

C) Neither A) nor B) is satisfied so that, for all $2 \leq i \leq 4$, the entries a_{i1}, a_{i2} are either both in P or (without loss of generality) $a_{i1} \equiv a_{i2} \equiv 1 \bmod P$.

Our purpose is to show that this last case C) cannot occur.

Recall that, as observed above, it is not possible that $a_{i1}, a_{i2} \in P$ for all $2 \leq i \leq 4$. Thus we may assume, without loss of generality, that $a_{21} \equiv 1 \equiv a_{22} \bmod P$. Now for $i = 3, 4$, let us replace y_i with $y_i - y_2$ exactly if $a_{i1} \equiv a_{i2} \equiv 1 \bmod P$. After these modifications of T, we may assume that $a_{31} \equiv a_{32} \equiv a_{41} \equiv a_{42} \equiv 0 \bmod P$.

Thus we get the following congruence

$$|T| \equiv \begin{vmatrix} 1 & 0 & a_{13} & a_{14} \\ 1 & 1 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{vmatrix} \bmod P.$$

Since $|T|$ is a unit, it follows that $\left| \begin{smallmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{smallmatrix} \right|$ is a unit.

We deduce that N must contain elements of the form

$$p(ax_1 + bx_2) + x_3; p(cx_1 + dx_2) + x_4,$$

for suitable $p \in P$ and $a, b, c, d \in R$. Since A is a principal ideal, we may choose $t \in p^{-1}A \setminus A$. Then we have $tp(ax_1 + bx_2) = 0 = tp(cx_1 + dx_2)$, and we get $N \supseteq \langle tx_3 \rangle \oplus \langle tx_4 \rangle = tB$. But tB is an essential submodule of M and so it cannot be contained in a proper direct summand. This is the desired contradiction which shows that case C) cannot occur.

From examination of cases A), B) and C), we conclude that it is not restrictive to assume that $a_{11} \equiv a_{12} \bmod P$.

For every $r \in J^*$, let

$$\underline{b}_{-1}^r = (b_1^r, \dots, b_4^r) = (-1, -1, u_1^r + u_2^r, v_1^r + v_2^r)T^{-1}.$$

Then we get

$$r\underline{b}^r_{-1}\underline{y} = r\underline{b}^r_{-1}T\underline{x} = -rx_1 - rx_2 + r(u_1^r + u_2^r)x_3 + (v_1^r + v_2^r)x_4 = 0.$$

From the above relation and $Ry_1 \cap N = 0$, we deduce that $rb_1^r y_1 = 0$ so that $b_1^r \in r^{-1}A$ for all $r \in J^*$. On the other hand, by the definition of b_1^r, we have

$$b_1^r|T| = -c_1 - c_2 + c_3(u_1^r + u_2^r) + c_4(v_1^r + v_2^r),$$

where the c_i are the cofactors of the first row of T. Thus the second member of the above relation lies in $r^{-1}A$. From the relations (2) and since $I = \bigcap_{r\in J^*} r^{-1}A$, we get

$$-c_1 - c_2 + c_3(u_1 + u_2) + c_4(v_1 + v_2) \equiv 0 \bmod IS.$$

Then the u-independence of u_1, u_2, v_1, v_2 gives $c_1 + c_2, c_3, c_4 \in P$. Now the assumption on a_{11}, a_{12} gives

$$|T| = \sum_{i=1}^{4} a_{1i}c_i \equiv a_{11}(c_1 + c_2) + a_{13}c_3 + a_{14}c_4 \equiv 0 \bmod P.$$

This is the desired contradiction which shows that M cannot admit cyclic direct summands.

STEP 2. M does not contain 2-generated direct summands.

We argue by contradiction, assuming that $M = N_1 \oplus N_2$ where the N_i are 2-generated. In view of Step 1, we may assume without loss of generality that the N_i are indecomposable. By the uniqueness of the annihilator sequence of M, we deduce that the annihilator sequence of each N_i is given by: $A < J$. Let $N_1 = \langle y_1, y_2\rangle$, $N_2 = \langle y_3, y_4\rangle$ and let $T = (a_{ij})$ be the invertible matrix such that $T\underline{x} = \underline{y}$ (notations as in Step 1).

Let us first examine N_1 (we refer to [**6**] for a thorough description of 2-generated indecomposable R-modules). We may assume without loss of generality that $Ry_2 \cong R/A$ is basic in N_1 and that, for all $r \in J^*$,

$$ry_1 = rw_r y_2, \tag{4}$$

for suitable units w_r in R. A unit w exists in S such that $w - w_r \equiv 0$ modulo $r^{-1}AS$ for all $r \in J^*$. Since N_1 is indecomposable, necessarily $w \notin R$. Using (3) and (4), from $y_i = \sum_{j=1}^{4} a_{ij}x_j$, we get for all $r \in J^*$,

$$r(a_{11}u_1^r + a_{12}u_2^r + a_{13} - w_r(a_{21}u_1^r + a_{22}u_2^r + a_{23}))x_3 \\ + r(a_{11}v_1^r + a_{12}v_2^r + a_{14} - w_r(a_{21}v_1^r + a_{22}v_2^r + a_{24}))x_4 = 0.$$

Now $Rx_3 \cap Rx_4 = 0$ and $I = \bigcap_{r\in J^*} r^{-1}A$ whence, using (2) and $w - w_r \in r^{-1}AS$, we get the following congruences modIS:

$$a_{11}u_1 + a_{12}u_2 + a_{13} \equiv w(a_{21}u_1 + a_{22}u_2 + a_{23})$$
$$a_{11}v_1 + a_{12}v_2 + a_{14} \equiv w(a_{21}v_1 + a_{22}v_2 + a_{24}).$$

From the above congruences we finally reach the following relation which does not depend upon w:

$$(5) \quad (a_{11}u_1 + a_{12}u_2 + a_{13})(a_{21}v_1 + a_{22}v_2 + a_{24}) \\ \equiv (a_{21}u_1 + a_{22}u_2 + a_{23})(a_{11}v_1 + a_{12}v_2 + a_{14}) \bmod IS.$$

Let us now consider the minors of the first two rows of T, $\lambda_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}$, $1 \le i < j \le 4$. Developing (5) we obtain

$$(6) \quad \lambda_{34} + u_1\lambda_{14} + u_2\lambda_{24} - v_1\lambda_{13} - v_2\lambda_{23} \\ + (u_1v_2 - u_2v_1)\lambda_{12} \equiv 0 \bmod IS.$$

Let us now repeat our argument replacing N_1 with N_2 and, considering the minors of the third and fourth rows of T, $\mu_{ij} = \begin{vmatrix} a_{3i} & a_{3j} \\ a_{4i} & a_{4j} \end{vmatrix}$, $1 \le i < j \le 4$. We obtain a relation corresponding to (6), namely,

$$(7) \quad \mu_{34} + u_1\mu_{14} + u_2\mu_{24} - v_1\mu_{13} - v_2\mu_{23} + (u_1v_2 - u_2v_1)\mu_{12} \equiv 0 \bmod IS.$$

Let us now examine the various possible cases, comparing λ_{12} with μ_{12}.

A) λ_{12} and μ_{12} are associated elements of R. In this case we may assume without loss of generality that $\lambda_{12} = \mu_{12}$.

Subtracting (7) from (6) and, using the u-independence of u_1, u_2, v_1, v_2, we get

$$\lambda_{ij} \equiv \mu_{ij} \bmod P,$$

for all i, j with $(i, j) \neq (1, 2)$. Moreover, $\lambda_{12} \equiv \mu_{12} \bmod P$, by hypothesis. But then, applying the generalized Laplace rule, we get

$$\begin{aligned} |T| &= \sum (\lambda_{ij}\mu_{hk} - \lambda_{hk}\mu_j)(-1)^{i+j} \\ &\equiv \sum (\lambda_{ij}\lambda_{hk} - \lambda_{hk}\lambda_{ij})(-1)^{i+j} \equiv 0 \bmod P, \end{aligned}$$

which is impossible.

B) $\lambda_{12} = p\mu_{12}$, for some $p \in P$.

Let us multiply relation (7) by p and subtract the result from (6). Making use of the u-independence, we get

$$\lambda_{ij} - p\mu_{ij} \equiv 0 \bmod P,$$

for all i, j with $(i, j) \neq (1, 2)$. Moreover, obviously $\lambda_{12} \in P$. But then $|T| \in P$, since all the minors of the first two rows of T are in P, which is impossible.

C) $q\lambda_{12} = \mu_{12}$ for some $q \in P$.

In this case we get a contradiction with an argument similar to that in B).

We have thus seen that from $M = N_1 \oplus N_2$ it always follows that $|T| \in P$. This is the desired contradiction which implies that M cannot contain two-generated direct summands. $\square$

We have already observed in the Introduction that there are no quadratically u-independent sets over a nonzero ideal I when the valuation domain R is archimedean. Results in a paper by Facchini and Zanardo [**1**] show the existence of other interesting valuation domains with no quadratically u-independent sets. Namely, for all $n > 0$, there is a valuation domain R having a nonzero prime ideal I such that S contains n units u-independent over I and satisfying the following property: for all $\eta \in S$ we have $\eta^2 \in R$. Therefore, any quadratic u-independence is out of the question.

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