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# A CLASS OF ABELIAN GROUPS DEFINED BY CONTINUOUS CROSS SECTIONS IN THE BOHR TOPOLOGY

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ABSTRACT. Comfort, Hernández and Trigos-Arrieta [**2**] introduced the class **ACCS**(#) of abelian groups $H$ such that the natural map $\varphi : G \to G/H$, where $G$ is the divisible hull of $H$, has a cross section $\Gamma : G/H \to G$ that is *continuous* in the Bohr topology of $G$ and $G/H$. They showed that **ACCS**(#) is closed under finite products and contains all finitely generated groups (and, of course, all divisible groups). They also gave an example of a group that does not belong to **ACCS**(#). We give further examples of groups from **ACCS**(#) (e.g., the groups of $p$-adic integers) and we find some new restraints for the groups from **ACCS**(#). This entails that large powers of nondivisible abelian groups never belong to **ACCS**(#) and gives an upper bound for the size of the reduced groups in **ACCS**(#) (roughly speaking, most of the abelian groups do not belong to **ACCS**(#)).

**1. Introduction.** The *Bohr topology* of an abelian group $G$ is the initial topology on $G$ with respect to the family of all homomorphisms of $G$ into the circle group. Following van Douwen [**6**], we write $G^{\#}$ for an abelian group $G$ equipped with its Bohr topology.

E.K. van Douwen [**6**] (cf. [**1**, p. 515]) raised the following question: *Are $G^{\#}$ and $H^{\#}$ homeomorphic as topological spaces whenever $G$ and $H$ are abelian groups of the same size?* A negative answer to this question was given independently and around the same time in [**11**], [**5**]. On the other hand, it was proved recently by Comfort, Hernández and Trigos-Arrieta [**2**] that $\mathbf{Q}^{\#}$ and $\mathbf{Z}^{\#} \times (\mathbf{Q}/\mathbf{Z})^{\#} = (\mathbf{Z} \times (\mathbf{Q}/\mathbf{Z}))^{\#}$ are homeomorphic. The proof of this quite surprising fact is related to another question of van Douwen.

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**Question 1.1.** [**6**, Question 4.12]. Is every (countable) subgroup $H$ of a group $G^{\#}$ a retract of $G^{\#}$?

Let us recall that $H$ is a retract of $G^{\#}$ if there exists a continuous map $r : G^{\#} \to H^{\#}$ such that $r(h) = h$ for every $h \in H$. Here are two instances when this occurs: i) if $H$ has finite index in $G$, then $H^{\#}$ is clopen in $G^{\#}$, hence is a retract of $G^{\#}$; ii) in a group $G$ of prime exponent $p$ every subgroup splits off algebraically, hence it is a topological direct summand in $G^{\#}$.

He asked also:

**Question 1.2.** [**6**, Question 4.13]. Is every countable closed subset of $G^{\#}$ a retract of $G^{\#}$?

This question was answered in negative by Gladdines (see Section 6.1 for a short solution), whereas Question 1.1 still remains open. This makes it interesting to consider the following notion proposed by [**2**] that leads to a modified version of Question 1.1:

**Definition 1.3.** A subgroup $H$ of an abelian group $G$ is a ccs-subgroup of $G$ if the natural map $\varphi : G \to G/H$ has a *continuous cross section*, i.e., a continuous map $\Gamma : (G/H)^{\#} \to G^{\#}$ such that $\varphi \circ \Gamma = \mathrm{id}_{|G/H}$.

It is proved in [**2**, Theorem 8] that if $H$ is a ccs-subgroup of $G$, then $H$ is a retract of $G^{\#}$ and $G^{\#}$ is homeomorphic to $(G/H)^{\#} \times H^{\#}$. It can be easily seen that a ccs-subgroup $H$ of a group $G$ is not only a retract, actually there exists a retraction $r : G \to H$ that is "linear" on each coset of $H$ [**2**, Theorem 38] (see also Fact 2.1 below for more detail).

Following [**2**] we denote by $\mathbf{ACCS}(\#)$ the class of the groups $H$ that are ccs-subgroup of any enveloping group $G$, and we refer to such groups $H$ as *ccs-groups.* It turns out [**2**, Theorem 19] that $H \in \mathbf{ACCS}(\#)$ if and only if $H$ is a ccs-subgroup of its divisible hull (or any divisible group containing $H$, cf. Corollary 2.3 (a)). Hence the study of the class $\mathbf{ACCS}(\#)$ can be considered as a version of van Douwen's Question 1.1

modified in two points: a) the retracts are understood in a stronger, linear, sense; b) the emphasis is given to the subgroup $H$ instead of the group $G$ itself.

The question of existence of non-ccs-subgroups is one of the main topics in [**2**] (quoted as Theorem 3(c) in [**2**, Abstract]). In the sequel we discuss this matter and we give contributions in the following two (opposite) directions.

a) We give necessary conditions for ccs-groups. This provides an upper bound for the size of the reduced groups in $\mathbf{ACCS}(\#)$ (so that the reduced groups in $\mathbf{ACCS}(\#)$ form a set) and shows that large powers may belong to $\mathbf{ACCS}(\#)$ only if they are divisible (Theorem 4.12). This gives a large class of new examples of non ccs-subgroups (actually the known examples from [**2**, Remark 36] are particular cases of a single example: the subgroup $\oplus_\kappa \mathbf{Z}(p)$ of the group $\oplus_\kappa \mathbf{Z}(p^2)$ for arbitrary infinite cardinal $\kappa$).

b) We establish new properties of the class $\mathbf{ACCS}(\#)$, e.g., closure with the expectation of taking extensions and direct summands. This provides some new examples of ccs-groups. This includes i) $\mathfrak{c}$ many pairwise nonisomorphic rank-one torsion-free (reduced) groups; ii) $\mathfrak{c}$ many pairwise nonisomorphic reduced groups of size $\mathfrak{c}$ each (cf. Example 3.9). The known examples of reduced ccs groups from [**2**] are only the finitely generated abelian groups and rank-one torsion-free groups $G$ such that for some infinite cyclic subgroup $C$ of $G$ the quotient $G/C$ is quasi-cyclic (i.e., only *countably many* pairwise nonisomorphic reduced ccs-groups).

In order to make the paper sufficiently self-contained and accessible both to algebra-minded and to topology-minded readers, we give detailed proofs of all our results as well as some of the results from [**2**].

**Notation and terminology.** The symbols, $\mathbf{P}, \mathbf{N}, \mathbf{Z}$ and $\mathbf{Q}$ are used for the set of primes, the set of positive integers, the group of integers and the group of rationals, respectively. The circle group $\mathbf{T}$ is identified with the quotient group $\mathbf{R}/\mathbf{Z}$ of the reals $\mathbf{R}$ and carries its usual compact topology. The cyclic group of order $n$ is denoted by $\mathbf{Z}(n)$. The $p$-adic integers are denoted by $\mathbf{Z}_p$.

We consider only abelian groups, we write $H \leq G$ if $H$ is a subgroup of $G$. Let $G$ be a group. The torsion subgroup of $G$ is denoted by $t(G)$.

The cyclic subgroup of $G$ generated by $b$ is denoted by $\langle b\rangle$. For $n \in \mathbf{N}$ and $p \in \mathbf{P}$ we put $G[n] = \{g \in G : ng = 0\}$, we denote by $t_p(G)$ the maximal $p$-torsion subgroup of $G$, and we denote by $r_p(G)$ the $p$-rank of $G$.

The symbol $\mathfrak{c}$ stands for the cardinality of the continuum, so $\mathfrak{c} = 2^{\aleph_0}$.

**1.1. Background on Bohr topologies.** It follows directly from the definition of the Bohr topology that a net $x_d \to 0$ in $G^\#$ if and only if the net $\chi(x_d) \to 0$ in $\mathbf{T}$ for every character $\chi : G \to \mathbf{T}$. Moreover, a map $f : G^\# \to H^\#$ is continuous if and only if the composition $\chi \circ f : G^\# \to \mathbf{T}$ is continuous for every character $\chi : G \to \mathbf{T}$. Let $G$ be a group of exponent $m$. Since the image of every homomorphism $G \to \mathbf{T}$ is contained in the subgroup $\mathbf{Z}(m)$ of $\mathbf{T}$, a typical subbasic open set $U_\zeta$ around 0 in $G^\#$ is given by $\ker f$ where $f : G \to \mathbf{Z}(m)$ is an arbitrary homomorphism. In other words, in this case the Bohr topology coincides with the profinite topology of $G$.

If $H \leq G$, then $H^\#$ is a topological subgroup of $G^\#$ and the quotient topology of $G^\#/H$ coincides with the Bohr topology of $G/H$. In particular, the product topology of $G^\# \times G_1^\#$ coincides with the Bohr topology of the product $G \times G_1$.

For an ordinal $\alpha$, define $\beth_\alpha$, as usual, by $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ and for limit $\alpha$ let $\beth_\alpha = \sup_{\beta<\alpha} \beth_\beta$.

Let $m, k \in \mathbf{N}$ and let $\kappa$ be an infinite cardinal. Let $G = \oplus_\kappa \mathbf{Z}(m)$, and let $\{e_\lambda : \lambda < \kappa\}$ be the "canonical base" of $G$, i.e., $e_\lambda$ is the generator $1 + m\mathbf{Z} \in \mathbf{Z}(m)$ of the $\lambda$th copy of $\mathbf{Z}(m) = \mathbf{Z}/m\mathbf{Z}$ in $G$. For a subset $Z \subseteq \kappa$ we shall denote by $[Z]^k$ the subset of the elements of $G$ of the form $\sum_{i=1}^k e_\lambda$ where $\lambda_1, \dots, \lambda_k$ are distinct elements of $Z$.

The following theorem, proved in [**4** Straightening Theorem] (for the case $p = 2$, see also [**3**]) will be our main tool in providing *necessary* conditions for ccs-group.

**Theorem 1.4.** *If $\kappa > \beth_{2p-1}$, then every continuous finite-to-one map $\pi : (\oplus_k \mathbf{Z}(p))^\# \to H^\#$ with $\pi(0) = 0$, necessarily sends $[S]^p$ to $H[p]$ for some infinite $S \subseteq \kappa$.*

This theorem implies that if $p$ is a prime and $r_p(H) < \infty$ for an abelian group $H$, then there exists no continuous finite-to-one map $\pi : (\oplus_\kappa \mathbf{Z}(p))^\# \to H^\#$ for $\kappa > \beth_{2p-1}$. In particular, there exists no continuous finite-to-one map $\pi : (\oplus_\kappa \mathbf{Z}(2))^\# \to H^\#$ for $\kappa > 2^{2^c}$ if $r_2(H) < \infty$.

**Question 1.5.** Does there exist a continuous one-to-one map from $(\oplus_\omega \mathbf{Z}(2))^\#$ to any torsion-free group $H^\#$?

**2. Continuous cross sections in the Bohr topology.** Let $H$ be a subgroup of the abelian group $G$. We say that a retraction $r : G^\# \to H^\#$ is linear, if $r(x+h) = r(x) + h$ for every $x \in G$ and $h \in H$.

**Fact 2.1.** [**2**, Theorem 38]. There exists a linear retraction $r : G^\# \to H^\#$ if and only if $H$ is a ccs-subgroup.

Indeed, for a ccs-subgroup $H$ with continuous cross section $\Gamma : G/H \to G$ the map $r : G^\# \to H^\#$ defined by $r(x) = x - \Gamma(x+H)$ is a linear retraction. Vice versa, if $r : G \to H$ is a linear retraction, then $\Phi(\varphi(x)) = x - r(x)$ defines a continuous cross section $(G/H)^\# \to G^\#$ since for every character $\chi : G \to \mathbf{T}$ the composition $\chi \circ \Phi : (G/H)^\# \to \mathbf{T}$ is continuous. Indeed, as $(G/H)^\# \cong G^\#/H^\#$ carries the quotient topology, it suffices to prove that the composition $\chi \circ \Phi \circ \varphi : G^\# \to \mathbf{T}$ is continuous. Since $\Phi \circ \varphi = x - r(x)$, we have $(\chi \circ \Phi \circ \varphi) = \chi(x - r(x)) = \chi(x) - \chi(r(x))$. Hence, as a difference of two continuous functions ($\chi \circ r : G \to \mathbf{T}$ is continuous by the continuity of $r : G^\# \to H^\#$), we conclude that $\chi \circ \Phi \circ \varphi$ is continuous.

It is easy to see that if a subgroup $H$ of an abelian group $G$ is either finite or has a finite index, then $H$ is a ccs-subgroup [**2**].

Some items of (a) in the following lemma can be found in [**2**, Corollary 13].

**Lemma 2.2.** (a) *Let $H \leq K \leq G$.*

($\mathrm{a}_1$) *If $K$ is a* ccs-*subgroup of $G$, then $K/H$ is a* ccs-*subgroup of $G/H$.*

($\mathrm{a}_2$) *If $H$ is a* ccs-*subgroup of $G$, then $H$ is a* ccs-*subgroup of $K$ too.*

($a_3$) (*the relations "*ccs-*subgroup" is transitive*) *If $H$ is a* ccs-*subgroup of $K$ and $K$ is a* ccs-*subgroup of $G$, then $H$ is a* ccs-*subgroup of $G$.*

($a_4$) *If $H$ is a* ccs-*subgroup of $G$, then $K/H$ is a* ccs-*subgroup of $G/H$ if and only if $K$ is a ccs-subgroup of $G$.*

($a_5$) *If $K$ is a* ccs-*subgroup of $G$, then $H$ is a* ccs-*subgroup of $G$ if and only if $H$ is a* ccs-*subgroup of $K$.*

($a_6$) *The following are equivalent*: (i) *$H$ is a* ccs-*subgroup of $G$ and $K/H$ is a* ccs-*subgroup of $G/H$* (ii) *$H$ is a* ccs-*subgroup of $K$ and $K$ is a* ccs-*subgroup of $G$.*

($a_7$) *If $(K : H) < \infty$, then $H$ is a* ccs-*subgroup of $G$ if and only if $K$ is a* ccs-*subgroup of $G$.*

(b) *If $H \leq G$, then $H$ is a* ccs-*subgroup of $G$ if and only if $H$ is a* ccs-*subgroup of every product $G \times G_1$.*

(c) *If $H \leq G$ and $H' \leq G'$, then the following are equivalent*

($c_1$) *$H$ is a* ccs-*subgroup of $G$ and $H'$ is a* ccs-*subgroup of $G'$.*

($c_2$) *$H \times H'$ is a* ccs-*subgroup of $G \times G'$.*

(d) *If $\alpha$ is a cardinal and $H^\alpha$ is a* ccs-*subgroup of $G^\alpha$,* (*respectively $H^{(\alpha)}$ is a* ccs-*subgroup of $G^{(\alpha)}$*)*, then $H$ is a* ccs-*subgroup of $G$.*

*Proof.* ($a_1$) is obvious.

($a_2$) If $\Phi : G/H \to G$ is a continuous cross section, then note that for $k \in K$ one has $\Phi(k+H) \in K$ since $g = \Phi(k+H)$ satisfies $\varphi(g) = k+H$, i.e., $g - k \in H$ so $g \in K$. Therefore, $\Psi = \Phi|_{K/H}$ is the desired continuous cross section $\Psi : K/H \to K$.

($a_3$)–($a_6$) are essentially contained in [**2**] and ($a_7$) follows from ($a_3$) and ($a_4$) since finite subgroups are always ccs-subgroups [**2**, Corollary 22].

(b) If $H$ is a ccs-subgroup of some product $G \times G_1$ then, by ($a_2$), $H$ is a ccs-subgroup of $G$. Now assume that the quotient map $f : G \to G/H$ has a continuous cross section $\Phi : G/H \to G$. Then $f' = f \times \mathrm{id}_{G_1} : G \times G_1 \to G/H \times G_1$ has a continuous cross section $\Phi' = \Phi \times \mathrm{id}_{G_1} : G/H \times G_1 \to G \times G_1$.

(c) Suppose $H \times H'$ is a ccs-subgroup of $G \times G'$ and let $\Phi : G/H \times G'/H' \to G \times G'$ be a cross section. Let $i : G/H \to G/H \times G'/H'$

be the natural embedding, and let $p : G \times G' \to G$ be the natural projection. Then $p \circ \Phi \circ i : (G/H)^{\#} \to G^{\#}$ is a continuous cross section. Hence $H$ is a ccs-subgroup of $G$. Analogously, we prove that $H'$ is a ccs-subgroup of $G'$.

Vice versa, suppose that $H$ is a ccs-subgroup of $G$ and $H'$ is a ccs-subgroup of $G'$, and let $\Phi : G/H \to G$ and $\Phi : G'/H' \to G'$ be continuous cross sections. Then $\Phi \times \Phi' : G/H \times G'/H' \to G \times G'$ is a continuous cross section, so $H \times H'$ is a ccs-subgroup of $G \times G'$.

(d) follows from (c). □

**ACCS**(#) is closed under finite direct products, see [**2**, Corollary 20]. Now we prove that **ACCS**(#) is closed under extensions and direct summands.

**Corollary 2.3.** (a) *$H \in$* **ACCS**(#) *if and only if there exists a divisible abelian group $D$ containing $H$ as a ccs-subgroup.*

(b) *If $D$ is a divisible group containing a subgroup $H \in$* **ACCS**(#), *then a subgroup $K$ of $D$ containing $H$ belongs to* **ACCS**(#) *if and only if $K/H \in$* **ACCS**(#).

(c) **ACCS**(#) *is closed under extension.*

(d) *$H^k \in$* **ACCS**(#) *for a cardinal $\kappa$ and a group $H$ if and only if $H^\kappa$ is a* ccs-*subgroup of $D(H)^\kappa$.*

(e) *$H \times H' \in$* **ACCS**(#) *if and only if $H \in$* **ACCS**(#) *and $H' \in$* **ACCS**(#).

*Proof*. (a) Assume that a subgroup $H$ of an abelian group $G$ is a ccs-subgroup of its divisible hull $D$, and let us note that, according to ($a_2$) of Lemma 2.2, it suffices to prove that $H$ is a ccs-subgroup of the divisible hull $D_1$ of $G$. It is not restrictive to assume that $D \leq D_1$. Since $D$ splits, now (b) of Lemma 2.2 can be applied. Now assume that $H$ is a ccs-subgroup of some arbitrary divisible group $D$. It is not restrictive to assume that $D$ contains the divisible hull $D_1$ of $H$. Then $H$ is a ccs-subgroup of $D_1$ by ($a_2$) of Lemma 2.2. Hence, $H$ is a ccs-group by the above argument.

(b) follows from (a) and ($a_4$) of Lemma 2.2.

(c) follows from (b). □

In particular, the proof of (a) contains the following fact proved in [**2**, Theorem 19]: $H$ is a ccs-subgroup of any enveloping group $G$ if and only if $H$ is a ccs-subgroup of its divisible hull.

Corollary 2.3 reduces the study of ccs-groups to those that are reduced. Indeed, every group $G$ is a product $d(G) \times R$ where $d(G)$ is the maximal divisible subgroup of $G$ and $R \cong G/d(G)$ is reduced. By (e) of the above corollary $G$ is a ccs-group if and only if the reduced group $R$ is a ccs-group.

The $\mathbf{Z}$-topology of an abelian group $G$ has as a local base at 0 the family of subgroups $nG$, $n \in \mathbf{N}$.

**Corollary 2.4.** *Suppose $K$ has a subgroup $H \in \mathbf{ACCS}(\#)$.*

(a) *If the quotient $K/H$ is divisible, then $K \in \mathbf{ACCS}(\#)$.*

(b) *If $H$ is dense in the $\mathbf{Z}$-topology of $K$, then again $K \in \mathbf{ACCS}(\#)$.*

*Proof.* (a) follows from item (b) of Corollary 2.3, (b) follows from (a). □

The following easy result will be needed in the sequel.

**Lemma 2.5.** *Let $H, K \leq G$ be such that $H \cap K$ is a* ccs-*subgroup of $K$. Then $H$ is a* ccs-*subgroup of $H + K$.*

*Proof.* Let $\Phi : K/K \cap H \to K$ be a continuous cross section of the canonical map $g_0 : K \to K/H \cap K$. Then, with $f : (H + K)/H \to K/H \cap K$ the canonical isomorphism, let $\Psi = i \circ \Phi \circ f$, where $i : K \hookrightarrow H + K$ is the inclusion. Let us check that this is a continuous cross section of the canonical map $g : H + K \to (H + K)/H$. Indeed, if $x \in (H + K)/H$, then $g(\Psi(x)) = g(i \circ \Phi(f(x)) = f^{-1}(g_0(\Phi(f(x))) = x$. □

## 3. CCS-groups.

**3.1. Subgroups of Q.** The following result is the main source for ccs-groups:

**Theorem 3.1.** [**2**, Theorem 24, Corollary 26]. $\mathbf{Z}$ *is a* ccs-*subgroup of* $\mathbf{Q}$. *Consequently,* $\mathbf{Q}^{\#}$ *is homeomorphic to* $(\mathbf{Q}/\mathbf{Z})^{\#} \times \mathbf{Z}^{\#}$ *and* $\mathbf{Z}^{\#}$ *is a retract of* $\mathbf{Q}^{\#}$.

Since every finite abelian group belongs to $\mathbf{ACCS}(\#)$ and since $\mathbf{ACCS}(\#)$ is closed under finite direct products, this theorem implies that $\mathbf{ACCS}(\#)$ contains all finitely generated abelian groups [**2**, Corollary 27]. As the authors note in [**2**, Theorem 29], it easily follows from item ($a_4$) Lemma 2.2 and from Theorem 3.1 that, for every prime $p$, the subgroup $D_p = \{m/p^k \in \mathbf{Q} : m, k \in \mathbf{Z}\}$ of $\mathbf{Q}$ belongs to $\mathbf{ACCS}(\#)$. Clearly, also, finite products of such groups belong to $\mathbf{ACCS}(\#)$. This suggest the following interesting general problem:

**Problem 3.2.** Determine which subgroups of $\mathbf{Q}$ belong to $\mathbf{ACCS}(\#)$.

In the sequel we discuss the properties of these groups. Note that $H \leq \mathbf{Q}$ is a ccs-subgroup if and only if $H \in \mathbf{ACCS}(\#)$. It is not restrictive to assume $\mathbf{Z} \leq H$. Then by Lemma 2.2, $H \in \mathbf{ACCS}(\#)$ if and only if $H/\mathbf{Z} \in \mathbf{ACCS}(\#)$. So Problem 3.2 is equivalent to the description of the ccs-subgroups of $\mathbf{Q}/\mathbf{Z}$.

Following the current terminology [**7**] we call *type* an isomorphism class $\tau$ of subgroups of $\mathbf{Q}$. We say that $\tau$ is idempotent if it is the type of a rank 1 ring. Every type can be described by an equivalent class of infinite sequences of naturals or symbols $\infty$ where two sequences are declared to be equivalent when they coincide almost everywhere. For a subgroup $H \leq \mathbf{Q}$ containing $\mathbf{Z}$, the sequence in question is $\{h_p^H(1) : p \in \mathbf{P}\}$ where $h_p^H(1)$ denotes the $p$-height of 1 in $H$, i.e., the supremum of all $n$ such that $1 = p^n h_n$ for some $h_n \in H$.

Obviously, Problem 3.2 can be given also the following form: *determine the types of the subgroups of* $\mathbf{Q}$ *that belong to* $\mathbf{ACCS}(\#)$. By [**2**, Theorem 29], every idempotent type having only one $\infty$ (i.e., $D_p$ in the notation of [**2**]) belongs to this class. A similar argument shows

**Proposition 3.3.** *All idempotent types belong to the class* **ACCS**$(\#)$.

This gives immediately continuum many pairwise nonisomorphic reduced groups in **ACCS**$(\#)$ (all of them subgroups of $\mathbf{Q}$).

Since reduced subgroups of $\mathbf{Q}/\mathbf{Z}$ correspond to types (subgroups of $\mathbf{Q}$) without infinities, we shall consider in the sequel only types without infinities. It is not clear whether a type with infinitely many nonzero finite entries belongs to this class. In particular,

**Question 3.4.** Does the subgroup of $\mathbf{Q}$ generated by all fractions $1/p$, with $p$ prime, belong to **ACCS**$(\#)$?

A more precise form is the following. Let $\pi$ be a set of prime numbers. Set $H_\pi = \langle 1/p : p \in \pi\rangle$. Note that $H_\pi \cong H_{\pi'}$ if and only if the symmetric difference of $\pi$ and $\pi'$ is finite.

**Problem 3.5.** *Determine the family* $\mathfrak{J}$ *of all sets* $\pi$ *of prime numbers for which the subgroup* $H_\pi$ *of* $\mathbf{Q}$ *belongs to* **ACCS**$(\#)$.

If $H$ is a subgroup of $\mathbf{Q}$ containing $\mathbf{Z}$, let us denote by $\operatorname{supp} H$ the set of primes $p$ such that $r_p(H/Z) > 0$ (note that $\operatorname{supp} H$ is defined modulo a finite set of primes). Call $H$ *bounded* whenever all heights $h_p^H(1)$ are bounded. For $L \leq \mathbf{Q}/\mathbf{Z}$, let $\operatorname{supp} L = \operatorname{supp} H$ where $\mathbf{Z} \leq H \leq \mathbf{Q}$ with $H/\mathbf{Z} = L$.

**Lemma 3.6.** (a) *The subgroup* $H_\pi$ *of* $\mathbf{Q}$ *belongs to* **ACCS**$(\#)$ *if and only if the subgroup* $L_\pi = H_\pi/\mathbf{Z}$ *of* $\mathbf{Q}/\mathbf{Z}$ *belongs to* **ACCS**$(\#)$.

(b) *If* $H$ *and* $L$ *are* ccs*-subgroups of* $\mathbf{Q}/\mathbf{Z}$, *then also* $H + L$ *is a* ccs*-subgroup; if* $K \leq \mathbf{Q}/\mathbf{Z}$ *is reduced and* ccs, *then every subgroup of* $K$ *is* ccs *as well.*

(c) $\mathfrak{J}$ *is an ideal of the Boolean algebra* $2^{\mathbf{P}}$ *containing the ideal of all finite subsets of* $\mathbf{P}$.

(d) *For a subset* $\pi$ *of* $\mathbf{P}$ TFAE:

($d_1$) $\pi \in \mathfrak{J}$;

($d_2$) $H_\pi \in \mathbf{ACCS}(\#)$;

($d_3$) *some* ccs-*subgroup* $H$ *of* $\mathbf{Q}/\mathbf{Z}$ *has support containing* $\pi$;

($d_4$) *every bounded subgroup* $H$ *of* $\mathbf{Q}$ *with* $\operatorname{supp} H \subseteq \pi$ *is* ccs.

*Proof.* (a) was explained above.

(b) Since $\mathbf{Q}/\mathbf{Z}$ is divisible a subgroup of $\mathbf{Q}/\mathbf{Z}$ is a ccs-subgroup if and only if it belongs to the class $\mathbf{ACCS}(\#)$ (cf. Corollary 2.3 (b)). Let us first prove the assertion of (b) for groups with $H \cap L = 0$. This follows directly from Corollary 2.3(d). Otherwise, note that each of the groups $H, L$ splits in a direct sum of two groups with disjoint supports: $H = H' + H''$ and $L = L' + L''$ where $\operatorname{supp} H'' = \operatorname{supp} L''$ and $H', L'$ have disjoint supports. As direct summands of ccs-subgroup, both $H'$ and $L'$ are ccs-subgroups of $Q/Z$, hence by the first part of the argument $H' + L'$ is a ccs-subgroup of $Q/Z$. Now note that $H'' + L''$ and $H' + L'$ have disjoint supports; hence, it suffices only to prove that $H'' + L'' \in \mathbf{ACCS}(\#)$. Here again we can split each one of these two subgroups into a direct sum of two submodules: $H'' = H_1 + H_2$ and $L'' = L_1 + L_2$ with pairwise disjoint supports in each decomposition. Moreover, we shall assume that every summand in $H_1$ contains the corresponding summand in $L_1$ so that $L_1 \leq H_1$.

Analogously, choose $H_2, L_2$ such that every summand in $L_2$ contains the corresponding summand in $H_2$ so that $L_2 \geq H_2$. Consequently, $H_1 + L_1 = H_1$ and $H_2 + L_2 = L_2$. Therefore,

$$H'' + L'' = H_1 + L_2 \in \mathbf{ACCS}(\#).$$

Now assume that $K = \oplus_{p\in\pi}\mathbf{Z}(p^{n_p}) \in \mathbf{ACCS}(\#)$ and $H \leq K$. Then there exists a sequence $m_p \leq n_p$ such that $H = \oplus_{p\in\pi}\mathbf{Z}(p^{m_p})$. Now let $L = \oplus_{p\in\pi}\mathbf{Z}(p^{n_p - m_p})$ so that $K/L \cong H$. Since $(\mathbf{Q}/\mathbf{Z})/L \cong \mathbf{Q}/\mathbf{Z}$, we conclude that $H$ is isomorphic to the subgroup $K/L$ of $\mathbf{Q}/\mathbf{Z}$ that is ccs by Lemma 2.2 ($a_1$).

(c) If $\pi \in \mathfrak{J}$ and $\pi'$ is a subset of $\pi$, then $L_{\pi'}$ is a direct summand of $L_\pi$ so a ccs-subgroup of $L_\pi$. By assumption $L_\pi$ is a ccs-subgroup of $\mathbf{Q}/\mathbf{Z}$ so, by transitivity, also $L_{\pi'}$ is a ccs-subgroup of $\mathbf{Q}/\mathbf{Z}$. Now suppose that $\pi, \pi' \in \mathfrak{J}$. Then $\pi \cup \pi' \in \mathfrak{J}$ since $L_{\pi\cup\pi'} = L_\pi + L_{\pi'}$ so that (b) applies to give $L_{\pi\cup\pi'} \in \mathbf{ACCS}(\#)$.

(d) ($d_1$) and ($d_2$) are equivalent by definition. For every positive natural $m$, let $H_{\pi,m} = \langle 1, p^m : p \in \pi\rangle$. If $H_\pi \in \mathbf{ACCS}(\#)$, then also $H_{\pi,m} \in \mathbf{ACCS}(\#)$ (argue by induction and note that $H_{\pi,1} = H_\pi$ and $H_{\pi,m}/H_{\pi,m-1} \cong H_\pi/\mathbf{Z} \in \mathbf{ACCS}(\#)$). Now to prove that ($d_2$) implies ($d_4$), suppose that $H$ is a bounded subgroup of $\mathbf{Q}$ with $\operatorname{supp} H \in \mathfrak{J}$. Then all $h_p(H)$ are bounded. Let $\pi_i = \{p \in P : h_p^H(1) = i\}$. Then $H/\mathbf{Z} = \oplus_{i=1}^s H_{\pi_i,i}/\mathbf{Z} \in \mathbf{ACCS}(\#)$ by (b).

Obviously ($d_4$) implies ($d_3$). Finally ($d_3$) implies ($d_2$) by Lemma 2.2 since every subgroup $H$ with support $\pi$ contains the subgroup $H_\pi$. ◻

We conclude these subsections with the following

*Remark* 3.7. (a) If $\mathbf{Z} \le H \le \mathbf{Q}$ is a reduced subgroup, then $H/\mathbf{Z}$ is a ccs-subgroup of $\mathbf{Q}/\mathbf{Z}$ if and only if there is a continuous cross section $\Phi : \mathbf{Q}/H \to \mathbf{Q}/\mathbf{Z}$. Let $\Phi_p$ denote the restriction of $\Phi$ to $t_p(\mathbf{Q}/H) \cong \mathbf{Z}(p^\infty)$. The image of $\Phi_p$ need not be contained in $t_p(\mathbf{Q}/\mathbf{Z}) \cong \mathbf{Z}(p^\infty)$ but, if we compose with the projection $\pi_p : \mathbf{Q}/\mathbf{Z} \to \mathbf{Z}(p^\infty)$ the so-obtained composition $\Psi_p$ sends $t_p(\mathbf{Q}/H)$ to $t_p(\mathbf{Q}/\mathbf{Z})$ and $\Psi_p$ is a cross section of the canonical projection $\varphi_p : t_p(\mathbf{Q}/\mathbf{Z}) \to t_p(\mathbf{Q}/H)$. Nevertheless, we cannot claim that the complex map $\mathbf{Q}/H \to \mathbf{Q}/\mathbf{Z}$ obtained by "gluing" continuous cross sections of the $\varphi_p$s is always continuous with respect to the Bohr topology of the codomain.

(b) Since every reduced subgroup of $\mathbf{Q}/\mathbf{Z}$ has the form $H_f = \oplus_p \mathbf{Z}(p^{f(p)})$ for some function $f : \mathbf{P} \to \mathbf{N}$, one can consider also the family $\mathcal{I}$ of all functions $f : \mathbf{P} \to \mathbf{N}$ such that the corresponding reduced subgroup $H_f$ of $\mathbf{Q}/\mathbf{Z}$ is a ccs-subgroup. It is easy to see that (b) and (d) imply that $\mathcal{I}$ is an ideal of the lattice $\mathbf{N}^{\mathbf{P}}$ of all functions $f : \mathbf{P} \to \mathbf{N}$. Clearly, $\mathcal{I}$ contains the ideal of all functions $f : \mathbf{P} \to \mathbf{N}$ with finite support (i.e., vanishing almost everywhere).

**3.2. Nonrational groups.** A solution to Problem 3.2 will lead to the solution of the problem of determining all completely decomposable torsion-free abelian ccs-groups of finite rank as such groups are isomorphic to finite products of subgroups of $\mathbf{Q}$. More precisely, if $G = H_1 \oplus \cdots \oplus H_n$ where $H_i \le \mathbf{Q}$, then $G \in \mathbf{ACCS}(\#)$ if and only if all $H_i \in \mathbf{ACCS}(\#)$. By Lemma 2.2 this leads to a solution also in the case of almost completely decomposable torsion-free abelian groups. Indeed, such a group $G$ has a finite index subgroup $H$ that is completely decomposable of finite rank. By Lemma 2.2 ($a_7$) $G$ is a ccs-subgroup of its divisible hull $Q$ if and only if $H$ is a ccs-subgroup of $Q$.

**Example 3.8.** There are torsion-free groups in $\mathbf{ACCS}(\#)$ that are not finitely generated and have no nontrivial $p$-divisible subgroups for any prime $p$. For example, take a subgroup $G$ of $\mathbf{Q}^2$ containing $\mathbf{Z}^2$ without nontrivial $p$-divisible subgroups and such that $G/\mathbf{Z}^2 \cong \mathbf{Z}(p^\infty)$ (follow the construction of primitive torsion-free finite rank groups in Kurosch [**12**]). Note that these groups are indecomposable. The groups of $p$-adic integers present an example of an indecomposable reduced ccs-group of size $\mathfrak{c}$ (see below for the proof of the fact that they are ccs-groups).

It is possible to prove a counterpart of Lemma 3.6 for finite rank torsion-free groups, i.e., subgroups of $\mathbf{Q}^n$. Nevertheless, we prefer to omit it given the fact that very few are known in the basic case of rank one groups.

Now we give a family of $\mathfrak{c}$ many pairwise nonisomorphic reduced torsion-free ccs-groups of size $\mathfrak{c}$. We shall see later that maybe this is the largest possible size of reduced ccs-groups (cf. Theorem 5.3).

**Example 3.9.** Let $H = \prod_{p\in\mathbf{P}} \mathbf{Z}_p$. Then $H$ is a reduced torsion-free ccs-group. Indeed the divisible hull $D$ of $H$ has a subgroup $C \cong \mathbf{Q}$ (the divisible hull of the cyclic subgroup generated by $(1_p) \in H$) such that $H + C = D$ and $C \cap H \cong \mathbf{Z}$ is a ccs-subgroup of $C$ (by Theorem 3.1). Therefore, by Lemma 2.5, also $H$ is a ccs-subgroup of $S$. This proves that $H$ is a ccs-group. In this argument $H$ can be replaced by a subproduct $N_\pi = \prod_{p\in\pi} \mathbf{Z}_p$ where $\pi$ is a set of prime numbers. Now the divisible hull $D$ of $H_\pi$ again has a subgroup $C \cong \mathbf{Q}$ (the divisible hull of the cyclic subgroup generated by $(1_p)_{p\in\pi} \in H$) such that $H + C = D$. Now $C \cap N_\pi = \mathbf{Q}_\pi$, the subring of $\mathbf{Q}$ generated by all $1/p$ for $p \notin \pi$. By Proposition 3.3 $\mathbf{Q}_\pi$ is a ccs-subgroup of $\mathbf{Q}$ so we are through again with Lemma 2.5. Another proof will be given below.

It follows from this example that finite products $N_{\pi_1} \times \cdots \times N_{\pi_n}$ are ccs, hence all products $\prod_p \mathbf{Z}_p^{n_p}$ with bounded $n_p$ are ccs, but we do not know if this remains true for *unbounded* sequences $n_p$ (cf. Question 6.8).

We can collect these observations in the following more general result.

**Proposition 3.10.** *Let $n \in \mathbf{N}$. For every prime $p$, let $M_p$ be an $n$-generated $\mathbf{Z}_p$-module. Then $G = \prod_p M_p \in \mathbf{ACCS}(\#)$.*

*Proof.* In case $n = 1$ the group $G$ has the form $\prod_p \mathbf{Z}_p/I_p$, where each $I_p$ is either 0 or $I_p = p^n\mathbf{Z}_p$ for some $n$. Since there is a copy of $\mathbf{Z} \in \mathbf{ACCS}(\#)$ in $G$ that is dense in the $\mathbf{Z}$-topology of the product $G$, Corollary 2.3 applies to give $G \in \mathbf{ACCS}(\#)$. For $n > 1$, write $G$ as a product of $\leq n$ groups for which the previous argument applies. A direct proof is also possible by noting that in the general case $G$ (being isomorphic to a quotient of $(\prod_p \mathbf{Z}_p)^n$) contains a finitely generated subgroup $F$ that is dense in the $\mathbf{Z}$-topology of $G$. Since $F \in \mathbf{ACCS}(\#)$, Corollary 2.3 applies again. □

If $G$ is a torsion abelian group and $H$ is a ccs-subgroup of $G$, then $t_p(H)$ is a ccs-subgroup of $t_p(G)$ for every prime $p$ by Lemma 2.2(c). We do not know whether the converse is also true. If such a criterion holds true, then all subgroups of $\mathbf{Q}/\mathbf{Z}$ are ccs, and consequently all subgroups of $\mathbf{Q}$ are ccs (cf. Question 6.6).

**4. Restraints for ccs-subgroups.** It is proved in [**2**, Theorem 35] that $G_p = \oplus_\omega \mathbf{Z}(p)$ is not a ccs-subgroup of $\oplus_\omega \mathbf{Z}(p^2)$ by proving that whenever $k$ is a multiple of $p$ but not of $p^2$ and $\pi : \{0\} \cup [\omega]^k \to (\oplus_\omega \mathbf{Z}(p^2))^\#$ is a continuous map with $\pi(0) = 0$ and $\pi(s_1) - \pi(s_2) \notin G_p \leq \oplus_\omega \mathbf{Z}(p^2)$ for $s_1 \neq \sigma_2$ in $[\omega]^k$, then $\pi(s)$ has not order $p^2$ for some $s \in [\omega]^k$ ([**2**, Theorem 34]). In connection with this last fact we mention that a direct application of Theorem 1.4 gives a similar result to [**2**, Theorem 34] providing new examples of non-ccs subgroups:

**Theorem 4.1.** [**3**]. *If $G$ is an abelian group such that $|G[p^2]| > \beth_{2p-1}$ for some prime $p$, then $H = G[p]$ is not a* ccs-*subgroup of $G$.*

*Proof.* Assume that the canonical map $f : G \to G/H$ admits a continuous cross section $\Phi : G/H \to G$. Since $G/H$ contains the subgroup $G[p^2]/H \cong \mathbf{Z}(p)^{(\kappa)}$ with $\kappa > \beth_{2p-1}$, by the straightening Theorem 1.4 there exists an infinite set $Z$ of $\kappa$ such that $\Phi$ restricted to $[Z]^2$ is injective with image contained in $H$. So $\Phi$ sends an infinite set to $H$, a contradiction since $f$ vanishes on $H$. $\square$

In particular, for $\kappa > \beth_{2p-1}$ the group $\oplus_\kappa \mathbf{Z}(p)$ is not a ccs-subgroup of $(\oplus_k \mathbf{Z}(p^2))^\#$. (Note that $\Phi(s) \in G[p]$ for every cross section $\Phi$ and every $s \in [Z]^2$, so $\Phi(s)$ cannot have order $p^2$.) We prove a much more general result below (cf. Lemma 4.3 and Corollary 4.4).

**Corollary 4.2.** *Let $\kappa > \beth_{2p-1}$ and let $p$ be a prime number. Then the subgroup $G[p]$ of the group $G = \oplus_\kappa \mathbf{Z}(p^\infty)$ is not a* ccs-*subgroup.*

The next lemma will be needed in Sections 4.1–4.2.

**Lemma 4.3.** *Let $p$ be a prime, and let $H$ be a subgroup of an abelian group $G$ such that $G[p] \cap H$ has finite index in $G[p]$ while $|(G/H)[p]| > \beth_{2p-1}$. Then $H$ is not a* ccs-*subgroup of $G$.*

*Proof.* Assume that $H$ is a ccs-subgroup of $G$, and let $\Phi : (G/H)^\# \to G^\#$ be a continuous cross section such that $\Phi(0) = 0$. By our assumption there exists a subgroup $L \leq G/H$ such that $L \cong \oplus_\kappa \mathbf{Z}(p)$ with $\kappa > \beth_{2p-1}$. Now let $\pi = \Phi|_L$. Then to the continuous injective map $\pi : L^\# \to G^\#$ we can apply Theorem 1.4 to claim that there exists an infinite $Z \subseteq \kappa$ such that $\pi$ sends $[Z]$ injectively into $G[p]$. Since the subgroup $G[p] \cap H$ of $G[p]$ has finite index, there exists an infinite subset $Z'$ of $Z$ such that $\pi$ sends $Z'$ into a coset of $G[p] \cap H$. On the other hand, being a cross section of the canonical map $G \to G/H$, the image of $\Phi$ meets every coset of $H$ into a single element, a contradiction. Therefore, $H$ is not a ccs-subgroup of $F$. $\square$

**Corollary 4.4.** *If $H$ is a ccs-group with divisible hull $D$ then, for every $p$, one has $r_p(D/H) \leq \beth_{2p-1}$.*

**4.1. An upper bound for the size of reduced ccs-groups.** Now we show that the reduced ccs-groups are relatively small.

**Lemma 4.5.** *Let $p$ be a prime number, let $H$ be a reduced subgroup of a $p$-torsion divisible group $D$ and let $\alpha = r_p(D/H)$. Then $H$ is finite with $r_p(H) \leq \alpha$ when $\alpha$ is finite, otherwise $|H| \leq \alpha^\omega$.*

*Proof.* Let $F'$ be an essential subgroup of $D/H$ of exponent $p$ so that $r_p(F') = \alpha$. Then there exists a subgroup $F \leq D$ such that $(F + H)/H = F'$ and $r(F) = a$. There exists a divisible subgroup $D_1$ of $D$ such that $F \leq D_1$ and $r_p(D_1) = \alpha$. Therefore, $(D_1 + H)/H$ is essential in $D/H$ and divisible. Hence it coincides with $D/H$ so that $D_1 + H = D$. Now $F'' = D_1 \cap H$ is reduced of $p$-rank $\leq \alpha$ and $(D_1 + H)/D_1 \cong H/F''$ is divisible.

Assume $\alpha$ is infinite. By a theorem of Szele [**7**, Proposition 26.2] there exists a pure subgroup $L$ of $H$ containing $F''$ of size $\alpha$. Let $B$ be a basic subgroup of $L$. Then $B$ is also a basic subgroup of $H$ since $L$ is pure in $H$ (so $B$ is pure in $H$) and $H/B$ is divisible (note that $H/B$ has a divisible subgroup $L/B$ such that $(H/B)/(L/B) \cong H/L$ is divisible). By a theorem of Kulikov [**7**, Corollary 34.4] $|H| \leq |B|^\omega \leq \alpha^\omega$.

Now assume that $\alpha$ is finite. Then $F''$ is finite as a reduced group of finite $p$-rank. From the divisibility of the quotient $H/F''$ we deduce $pH + F'' = H$ so $p^{m+1}H + p^m F'' = p^m H$ for every $m \in \mathbf{N}$. Choose $m$ with $p^m F'' = 0$. Then $p^m H$ is divisible, but as a subgroup of $H$ it is also reduced. Hence, $p^m H = 0$. Thus $p^m(H/F'') = 0$ and, consequently, $H/F'' = 0$ by divisibility of $H/F''$. This proves that $H = F''$ is finite and $r_p(H) \leq \alpha$. □

One cannot hope to prove $|H| \leq r_p(D/H)$ or $r_p(H) \leq r_p(D/H)$ in the above lemma. Indeed, let $H$ be the torsion subgroup of the product $P = \prod_n \mathbf{Z}(p^n)$ considered as a subgroup of the power $P' = \mathbf{Z}(p^\infty)^\omega$. Let $D$ be the torsion subgroup of $P'$ so that $D$ is the divisible hull of $H$ in $P'$ and $D = \oplus_\omega \mathbf{Z}(p^\infty) + H$. Then $H$ is reduced and $r_p(D/H) = \omega$ but $r_p(H) = |H| = 2^\omega$.

**Theorem 4.6.** *Let $R$ be a reduced* ccs*-group. Then $|R| \leq \beth_{\omega+1}$ and $|t_p(R)| \leq \beth_{2p-1}$ for every prime $p$ (so $|t(R)| \leq \beth_\omega$).*

*Proof.* Let $D$ be the divisible hull of $R$. Then Corollary 4.4 gives

$$r_p(D/R) \leq \beth_{2p-1} \quad \text{for every prime } p. \tag{1}$$

If $R$ is torsion-free, then $D$ is torsion-free too and (1) gives $r_p(R/pR) \leq \beth_{2p-1}$ for every prime $p$, since $r_p(D/R) = r_p(R/pR)$ for every prime $p$. Further, the inequality $r_p(R/p^nR) \leq \beth_{2p-1}$ for every $n \in \mathbf{N}$ can be proved by induction. Since $(p_1 \dots p_k)^n R = \cap_{i=1}^{k} p_i^n R$ for distinct primes $p_1, \dots, p_k$, we conclude also that $|R/(p_1 \dots p_k)^n R| < \beth_\omega$. Consequently, $|R/mR| < \beth_\omega$ for every $m \in \mathbf{N}$. Now $\cap_{m=1}^{\infty} mR = 0$ as $R$ is reduced, therefore $R$ embeds in the product of the groups $R/mR$, thus $R$ has size $|R| \leq \beth_\omega^\omega \leq \beth_{\omega+1}$.

In the general case Lemma 4.5 and (1) yield $|t_p(R)| \leq \beth_{2p-1}^\omega = \beth_{2p-1}$. Let $D = t(D) \oplus D_1$ be the splitting of $D$ with a torsion-free divisible group $D_1$. Then $R_1 = R \cap D_1$ is torsion-free and essential in $D_1$, hence $R' = t(R) \oplus R_1$ is essential in $R$. Therefore, $|R| = |t(R)||R_1|$. By the above argument $|R_1| \leq \beth_{\omega+1}$ and $|t(R)| \leq \beth_\omega$, hence $|R| \leq \beth_{\omega+1}$. $\square$

**4.2. ACCS(#) does not contain nondivisible large powers.** Here we prove a theorem about the relation between ccs-groups and divisible groups. The following lemmas will be used in the characterization, given below, of the divisible groups as those groups $G$ such that all powers of $G$ belong to **ACCS**(#).

**Lemma 4.7.** *Let $p$ be a prime, and let $H$ be a subgroup of an abelian group $G$ such that $G[p] \subseteq H$. If $H^\kappa$ is a ccs-subgroup of $G^\kappa$ for $\kappa \geq \beth_{2p-1}$, then $H$ contains the subgroup $\{x \in G : p^n x \in H \text{ for some } n \in \mathbf{N}\}$.*

*Proof.* Assume that there exists $x \in G$ such that $px \in H$ but $x \notin H$. Then $(G/H)[p] \neq 0$ so that $(G^\kappa/H^\kappa)[p] = (G/H)[p]^\kappa$ has size $\leq 2^\kappa > \beth_{2p-1}$. By Lemma 4.3 applied to $G^\kappa$ and its subgroup $H^\kappa$ we conclude that $H^\kappa$ is not a ccs-subgroup of $G$, a contradiction. $\square$

**Corollary 4.8.** $\mathbf{Z}^k \notin$ **ACCS**(#) *for* $\kappa \geq \beth_3$.

It follows from Corollary 4.8 that, for $\kappa \geq \beth_3$, the subgroup $\mathbf{Z}^\kappa$ of $\mathbf{Q}^\kappa$

is not a ccs-subgroup. In the sequel we give a large class of examples of such subgroups.

Recall that $H \leq G$ is *saturated* if $nx \in H$ with $x \in G$ and $n \neq 0$ implies $x \in H$.

**Corollary 4.9.** *Let $H$ be a subgroup of an abelian group $G$ containing the socle of $G$. If $H^\kappa$ is a* ccs-*subgroup of $G^\kappa$ for some cardinal $\kappa \geq \beth_\omega$, then $H$ is saturated, hence it contains the torsion subgroup of $G$.*

**Corollary 4.10.** *Let $\kappa \geq \beth_\omega$ be a cardinal, and let $H$ be a subgroup of a torsion-free abelian group $G$ such that $H^\kappa$ is a* ccs-*subgroup of $G^\kappa$. Then $H$ is a pure subgroup of $G$.*

**Corollary 4.11.** *If $H$ is an essential subgroup of an abelian group $G$ such that $H^\kappa$ is a* ccs-*subgroup of $G^\kappa$ for $\kappa > \beth_\omega$, then $H = G$.*

**Theorem 4.12.** *Let $G$ be an abelian group, and let $D$ be its divisible hull. Then the following are equivalent for $G$*:

(a) *There exists a cardinal $\kappa \geq \beth_\omega$ such that $G^\kappa$ is a* ccs-*subgroup of $D^\kappa$.*

(b) *$G$ is divisible*;

(c) *$G^\kappa$ is a* ccs-*subgroup of $D^\kappa$ for every cardinal $\kappa$.*

(d) *$G^\kappa \in$* **ACCS**(#) *for every cardinal $\kappa$.*

(e) *$G^{(\kappa)} \in$* **ACCS**(#) *for every cardinal $\kappa$.*

*Proof.* Applying Corollary 4.11 to $G$ and its divisible hull $D$, we conclude that a) implies b). Clearly, b) implies d) and d) implies c) which in turn trivially implies a). This proves the theorem. $\square$

This theorem provides a wealth of non-ccs-groups. Indeed, for every nondivisible abelian group $G$, the power $G^{\beth_\omega}$ of $G$ and all its powers cannot be ccs-groups. In particular, this shows that the class of ccs-groups is not closed under taking infinite powers (see [**2**, Theorem 45] for an example of group $G \in \mathbf{ACCS}(\#)$ such that the *countable* power of $G$ fails to belong to $\mathbf{ACCS}(\#)$ ).

Actually one can prove under a stronger hypothesis (cf. Theorem 5.5) that every abelian group $H$ such that $H^{(\omega)} \in \mathbf{ACCS}(\#)$ is divisible (in fact, then $r_p(G) < \infty$ for every torsion-free abelian group $G \in \mathbf{ACCS}(\#)$ and for every prime $p$, cf. Corollary 5.2).

In order to measure the failure of $\mathbf{ACCS}(\#)$ to be closed under products one can define also $H \le G$ to be *$\rho$-ccs-subgroup* for a cardinal $\rho$ if $H^\rho$ is a ccs-subgroup of $G^\rho$. For $\rho < \omega$, clearly every ccs-subgroup of $G$ is also a $\rho$-ccs-subgroup of $G$. More generally, by Lemma 2.2, every $\rho'$-ccs-subgroup of $G$ is also $\rho$-ccs-subgroup of $G$ when $\rho' \ge \rho$. In analogy, call $G$ *$\rho$-ccs-group* when $G$ is a $\rho$-ccs-subgroup of its divisible hull $D(G)$. Clearly this property is equivalent to $G^\rho \in \mathbf{ACCS}(\#)$, so it can be considered as a weak version of "divisible" (equivalent to $\beth_\omega$-ccs-group according to Theorem 4.12). Note also that the weakest version, namely, ccs-, or equivalently $n$-ccs-groups) is satisfied by all finitely generated abelian groups. Put $\lambda(G)$, respectively $\lambda_\omega(G)$, to be the least cardinal $\lambda$ such that $G^\lambda \notin \mathbf{ACCS}(\#)$, respectively $G^{(\lambda)} \notin \mathbf{ACCS}(\#)$. For divisible groups $D$ one has to put $\lambda(D) = \infty$. Hence a nondivisible group $G$ has always $\lambda(G) \le \lambda_w(G) \le \beth_\omega$. Under GCH one has also $\lambda_\omega(G) \le 2^{\lambda(G)}$. As a corollary of Theorem 4.12 one can prove that if $H$ is a subgroup of a divisible group $G$, then $H$ is a $\beth_\omega$-ccs-subgroup of $G$ if and only if $H$ is a direct summand of $G$ (if and only if $H$ is divisible).

**5. ACCS(#) under the strong straightening theorem.** Let us consider now the following conjecture stronger than Theorem 1.4:

**Conjecture SST** (Strong straightening theorem). *For every prime number $p$ and for every continuous map $\pi : (\oplus_\omega \mathbf{Z}(p))^\# \to H^\#$ with $\pi(0) = 0$, there exists an infinite set $S \subseteq \omega$ such that $\pi([S]^p) \subseteq H[p]$.*

This conjecture implies, in particular, that there is no 1–1 map $\pi : (\mathbf{Q}/\mathbf{Z})^{(\omega)} \to H$ continuous in the Bohr topology, if $H$ is an abelian group with $r_p(H) < \infty$ for all $p \in \mathbf{P}$.

Now we show that if Conjecture SST holds true, then a similar proof can prove the following stronger version of Lemma 4.3 (roughly speaking, $r_p(G/H) < \infty$ for every essential ccs-subgroup $H$ of some group $G$).

**Lemma 5.1.** *Let $p$ be a prime, and let $H$ be a subgroup of an abelian group $G$ such that $G[p] \cap H$ has finite index in $G[p]$, while $r_p(G/H)$ is infinite. Then $H$ is not a ccs-subgroup of $G$.*

*Proof.* Arguing for a contradiction, assume that $H$ is a ccs-subgroup of $G$, i.e., there exists a continuous cross section $\Phi : G/H \to G$ to the canonical map $G \to G/H$. By the hypothesis the quotient $G/H$ contains a subgroup $L \cong \oplus_\omega \mathbf{Z}(p)$. To the restriction $\pi$ of $\Phi$ to the subgroup $L$ apply the strong straightening theorem to find an infinite $Z \subseteq \omega$ such that $\pi$ sends $[Z]^p$ into $G[p]$. This is impossible since $\Phi$ is a cross section of $G \to G/H$ and $G[p] \leq H$ (as $H$ is essential in $G$). □

**Corollary 5.2.** *Let $H$ be a reduced* ccs*-group with divisible hull $D$. Then, under* SST*-conjecture all $r_p(D/H)$ are finite, so that $r_p(H) < \infty$ for every $p$.*

*Proof.* $H$ contains $D[p]$ for every prime $p$. Hence the hypothesis $H \in \mathbf{ACCS}(\#)$ implies that $r_p(D/H) < \infty$ in view of Lemma 5.1. Now Lemma 4.5 applies to give $r_p(H) < \infty$ for every $p$. □

Now we see that, under the assumption of SST the reduced ccs-groups are necessarily small, i.e., of size $\leq \mathfrak{c}$.

**Theorem 5.3.** *Let $H$ be a reduced* ccs*-group. Under the assumption of* SST *$r_p(H)$ is finite for every $p$,* (*so that $t(H)$ is countable*) *and $|H| \leq \mathfrak{c}$.*

*Proof.* Let $D$ be the divisible hull of $H$. Then, under the assumption of SST, one has $r_p(D/H) < \infty$ for every prime $p$ by Corollary 5.2. Let us consider first the case of a torsion-free group $H$. Since $r_p(D/H) = r_p(H/pH)$, we conclude $r_p(H/pH) < \infty$. Further, $r_p(H/p^nH) < \infty$ for every $n \in \mathbf{N}$. Since $(p_1 \dots p_k)^n H = \cap_{i=1}^k p_i^n H$ for distinct prime numbers $p_1, \dots, p_k$ and $n \in \mathbf{N}$, we conclude also that $H/(p_1 \dots p_k)^n H$

is finite. Consequently, $H/mH$ is finite for every $m \in \mathbf{N}$. Now $\cap_{n=1}^{\infty} mH = 0$ as $H$ is reduced, therefore $H$ embeds in the product of the finite groups $H/mH$, thus $H$ has size $|H| \leq \mathfrak{c}$.

In the general case, $r_p(H) < \infty$ for every prime $p$ according to the above corollary. Here we split $D = t(D) \times D_1$, where $D_1$ is torsion-free. Then $t(D)$ is countable and $H_1 = H \cap D_1$ is a reduced subgroup of $D_1$ with $r_p(D_1/H_1) \leq r_p(D/H) < \infty$. Thus $|H_1| \leq \mathfrak{c}$ by the above argument. Since $H_1$ is essential in $D_1$ we conclude that also $|D_1| \leq \mathfrak{c}$, so that $|D| \leq \mathfrak{c}$. Thus $|H| \leq \mathfrak{c}$ too. □

Example 3.9 shows that $|H| \leq \mathfrak{c}$ cannot be improved.

**Corollary 5.4.** *If Conjecture* SST *holds true and* $H \in \mathbf{ACCS}(\#)$ *is a bounded torsion abelian group, then* $H$ *is finite.*

Now we see the impact of the Strong Straightening Theorem on Theorem 4.12.

**Theorem 5.5.** *If Conjecture* SST *holds true then, for every abelian group* $H$*, the following are equivalent*:

(a) $H^{\omega} \in \mathbf{ACCS}(\#)$

(b) $H^{(\omega)} \in \mathbf{ACCS}(\#)$

(c) $H$ *is divisible.*

*Proof.* Obviously (c) $\rightarrow$ (a) and (c) $\rightarrow$ (b). To prove (b) $\rightarrow$ (c) assume that $H^{(\omega)} \in \mathbf{ACCS}(\#)$ and $H$ is not divisible. Let $D$ be the divisible hull of $H$. Then $D/H$ is torsion, hence our assumption $D \neq H$ yields that $r_p(D/H) > 0$ for some prime $p$. Then $D^{(\omega)}$ is the divisible hull of $H^{(\omega)}$ and the quotient $D^{(\omega)}/H^{(\omega)}$ has infinite $r_p(D^{(\omega)}/H^{(\omega)})$, a contradiction (cf. Corollary 5.2). A slight modification of this argument proves also the implication (a) $\rightarrow$ (c). □

## 6. Concluding remarks.

**6.1. A new proof of Gladdines' theorem.** Since every subgroup of $(\oplus_{\kappa}\mathbf{Z}(p))^{\#}$ splits topologically (being a subspace), every subgroup of these groups is a retract even in a stronger sense.

Let us recall that $\mathcal{D}_\omega$ is the (closed) subset of $(\oplus_\omega \mathbf{Z}(2))^\#$ consisting of 0 and all elements of $\oplus_\omega \mathbf{Z}(2)$ whose support is a doubleton in $\omega$, i.e., $\mathcal{D}_\omega = \{0\} \cup [\omega]^2$.

**Theorem 6.1.** [**8**]. *$\mathcal{D}_\omega$ is not a retract of $(\oplus_\omega \mathbf{Z}(2))^\#$.*

*Proof.* Assume that $r : (\oplus_\omega \mathbf{Z}(2))^\# \to \mathcal{D}_\omega$ is a retract and identify the nonzero elements of $\mathcal{D}_\omega$ with the respective pair $(m, n)$. Define $\mu : \mathcal{D}_\omega \to (\oplus_\omega \mathbf{Z}(3))^\#$ by $\mu(0) = 0$ and $\mu(m, n) = e_m - e_n$, where $\{e_n : n = 1, 2, \dots\}$ is the canonical base of $\oplus_\omega \mathbf{Z}(3)$. Then $\mu$ is continuous ([**3**]), so that taking the composition $\mu \circ r$ we obtain a continuous map $\pi : G_2^\# \to G_3^\#$ that sends 0 to 0 and nonzero elements of $G_2^\#$ to elements of $G_3^\#$ whose support is a doubleton. By [**5**, Main lemma], there exists an infinite subset $Z \subseteq \omega$ such that $\pi$ vanishes on $[Z]^2$, i.e., $\pi(m, n) = 0$ on $Z$. On the other hand, $r$ restricted to $[Z]^2$ is the identity of $[Z]^2$, hence $\pi$ restricted to $[Z]^2$ coincides with $\mu$ restricted to $[Z]^2$, a contraction (since $\mu$ is injective). $\square$

The proof of Gladdines [**8**] goes in a different way. It was published in 1995, when the nonhomeomorphisms problem of van Douwen was still open.

**6.2. Some open questions.** We believe that for some groups $G$ one can lower the test powers in Theorem 4.12 down to $\kappa = \omega$ or at least $\kappa = \mathfrak{c}$ without any recourse to Conjecture SST (e.g., when $G$ is a torsion-free group with a 2-pure cyclic subgroup, then $\lambda(G) \leq 2^{2^{\mathfrak{c}}}$):

**Question 6.2.** Let $G$ be an abelian group. Does $G^{\mathfrak{c}} \in \mathbf{ACCS}(\#)$ imply that $G$ is divisible? What about $G^\omega \in \mathbf{ACCS}(\#)$?

In particular, we conjecture a negative answer to the first of the following questions:

**Question 6.3.** Does $\mathbf{Z}^{\mathfrak{c}} \in \mathbf{ACCS}(\#)$? What about $\mathbf{Z}^\omega \in \mathbf{ACCS}(\#)$?

The question $\mathbf{Z}^{(\omega)} \in \mathbf{ACCS}(\#)$ about the "least" torsion-free group of infinite rank is also open. A negative answer to Question 1.5 for $\kappa = \omega$ will imply $\mathbf{Z}^{(\omega)} \notin \mathbf{ACCS}(\#)$ (since otherwise $\mathbf{Q}^{(\omega)}$ is Bohr homeomorphic to the product $(\mathbf{Z} \times \mathbf{Q}/\mathbf{Z})^{(\omega)}$).

**Question 6.4.** Let $G$ be an abelian group. When is the torsion subgroup $t(G)$ of $G$ a ccs-subgroup of $G$? Does there exist a (necessarily nonsplitting) abelian ccs-group $G$ such that $t(G)$ is not a ccs-subgroup of $G$?

If $H_\pi \notin \mathbf{ACCS}(\#)$ for some $\pi \subseteq \mathbf{P}$, then $G = \prod_{p\in\pi} \mathbf{Z}(p) \in \mathbf{ACCS}(\#)$ (by Proposition 3.10) can be a counter-example.

**Question 6.5.** Does there exist a reduced ccs-group of size $> \mathfrak{c}$?

Note that the answer to this question is negative if Conjecture SST holds true (Theorem 5.3).

Roughly speaking, all known examples of non-ccs-groups are either too large (of size $> \mathfrak{c}$) or contain infinite direct sums (as $\oplus_\omega \mathbf{Z}(p)$, cf. [**2**]). We do not know whether $\mathbf{Q}$ contains a non-ccs-subgroup:

**Question 6.6.** Are all subgroups of $\mathbf{Q}$ ccs-subgroups?

If this is the case, then $\mathbf{ACCS}(\#)$ contains all almost completely decomposable torsion-free abelian groups. At the opposite end, we have

**Question 6.7.** Does $\mathbf{Q}/\mathbf{Z}$ contain any infinite reduced ccs-subgroups (i.e., does $\mathfrak{J}$ contain any infinite set $\pi$?)

**Question 6.8.** Is the product $\prod_p \mathbf{Z}_p^{n_p}$ a ccs-group for every sequence $n_p \in \mathbf{N}$?

We believe that Corollary 5.4 holds true independently on Conjecture SST (an appropriate modification of the proof of [**2**, Theorem 35] should work).

It seems that the following problem is the "true" algebraic counterpart of van Douwen's question 1.1.

**Problem 6.9.** Describe the abelian groups $G$ such that every subgroup of $G$ is a ccs-subgroup.

Let us conclude with the following question that still remains open.

**Question 6.10.** [**2**, Question 37]. Is $(\oplus_\omega \mathbf{Z}(p))^\#$ a retract of $(\oplus_\omega \mathbf{Z}(p^2))^\#$?

We do not know even if $\oplus_\omega \mathbf{Z}(2)$ is a retract of $(\oplus_\omega \mathbf{Z}(4))^\#$. Of course, this question has two versions: one considers $\oplus_\omega \mathbf{Z}(2)$ as a subgroup, so that the question is whether the subgroup $(\oplus_\omega \mathbf{Z}(4))[2]$ of $\oplus_\omega \mathbf{Z}(4)$ is a retract of $(\oplus_\omega \mathbf{Z}(4))^\#$. The weaker version is intended as: is $(\oplus_\omega \mathbf{Z}(2))^\#$ homeomorphic to a retract of $(\oplus_\omega \mathbf{Z}(4))^\#$? We do not know the answer to this question. Finally, we do not know whether $(\oplus_\omega \mathbf{Z}(2))^\#$ is homeomorphic to $(\oplus_\omega \mathbf{Z}(4))^\#$.

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*Note added in proof.* Recently Givens and Kunen (*Chromatic Numbers and Bohr Topologies,* Topology Appl., to appear) proved that if $K$ is an infinite abelian group of a given prime exponent, then $G^\#$ and $K^\#$ are homeomorphic if and only if $G$ is the product of $K$ and some finite group. In particular $(\bigoplus_\omega \mathbf{Z}(2))^\#$ is not homeomorphic to $(\bigoplus_\omega \mathbf{Z}(2))^\#$.

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