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JERZY ŁOŚ AND A HISTORY OF ABELIAN GROUPS IN POLAND

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1. A birthday. The Polish mathematical school before 1939, that is, before the Second World War, concentrated most of its research potential and mathematical activity on a few branches including set theory, number theory, topology, functional analysis, complex analysis, foundation of mathematics and logic. Algebra was not represented as an independent field. In a strict sense, algebra did not exist at that time in Poland. However, many of the leading Polish mathematicians, in particular K. Borsuk (1905–1982), K. Kuratowski (1896–1980) and W. Sierpiński (1882–1969), realized perfectly well this abnormal situation because they realized at least the importance of algebraic methods in topology. Unfortunately, some of the distinguished Polish analysts did not understand the role of abstract algebra in modern mathematics. One of them was Władysław Orlicz (1903–1990), who in the late seventies said to me several times: "Algebra does not simplify things, it just makes them much more complicated than they are".

Fortunately, after the Second World War in 1945, Kuratowski (who was the head of the Polish Academy of Sciences) and Borsuk initiated a discussion on a restoration program for Polish mathematics, including the existence of algebra. It seems to me that Kuratowski's view of the role algebra plays in modern mathematics was influenced a lot by his former student Samuel Eilenberg (1913–1998), who was one of the leading persons at that time in developing homological algebra in cooperation with S. MacLane and E. Cartan (see the papers [**11**], [**12**], [**13**] and the book [**7**]).

We recall that Eilenberg was born 30 September 1913 in Warsaw. He received the Ph.D. degree from Warsaw University shortly before the Second World War under the supervision of Kuratowski. On 27 April 1939 he emigrated from Poland to the United States (see [**34**] for details).

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The program of restoring the Polish school of mathematics after the Second World War was discussed and created by the First Polish Congress of Science organized in Warsaw between 29 June and 2 July 1951. As a consequence, the Polish Academy of Science was obligated by the Congress to create a modern algebra research program in Poland. At the end of 1951 the Mathematical Institute of the Academy decided to create an algebra research group at the six-year-old Nicholas Copernicus University in Toruń. The nomination for the leader of the Toruń group was given to Dr. Jerzy Maria Łoś (1920–1998), 31 years old, then an almost unknown logician of Wrocław University. This was the birthday of algebra in Poland.

J. Łoś was born 22 March 1920 in Lwów. He received a Master's degree in Philosophy in 1947 from the Marie Curie-Skłodowska University in Lublin and a doctorate in 1949 from the University of Wrocław under the supervision of Jerzy Słupecki, a logician, on the basis of a thesis entitled, *Matrix Theory of Multivalued Logic*. Below please find a caricature of Łoś made in 1952 by Leon Jeśmanowicz (1914–1989), a mathematician at Nicholas Copernicus University and a former Ph.D. student of Anthony Zygmund. The caricature is taken from the book [**22**]. Let us add that Jeśmanowicz was a student of Anthony Zygmund in the Stefan Batory University in Vilnius until 1942, when Zygmund emigrated from Poland to the United States.

Jerzy Łoś

2. Model theory and abelian groups in Poland. Łoś moved from Wrocław to Toruń at the beginning of 1952. He was appointed an assistant professor in mathematics at Nicholas Copernicus University. In Spring 1952 he formed an algebra research group in Toruń consisting of seven young mathematicians and promising undergraduate students of mathematics. In particular, the following three persons were appointed to the group, two students Stanisław Balcerzyk (born 29 June 1932) and Edward Sąsiada (1924–1999), and Józef Słomiński (born in 1929), a teacher of mathematics in a secondary school of Toruń. Research seminars and summer workshops were the main activity of the group. In 1953, one of the first workshops was devoted to representations of topological groups and harmonic analysis. Among many others, the following people attended the workshop: Aleksiewicz, Borsuk, Hartman, Białynicki-Birula, Hulanicki, Mostowski (junior), Mycielski and Suliński.

Łoś and his group of algebraists concentrated their research and education activity on the following areas:

(a) Foundations of mathematics, model theory and universal algebra.

(b) Abelian group theory.

(c) Associative ring theory and module theory.

In 1953–54, Łoś concentrated his research activity mainly on abelian groups and model theory. As a consequence he discovered his fundamental results in model theory: the ultraproduct theorem and a compactness theorem. His results on this subject are published in [**25**–**29**]. Let us present in a simple form his main ideas and results (see [**8**], [**21**] and [**29**]). For this we recall that a family $\mathcal{F} \subseteq 2^I$ in the Boolean algebra $(2^I, \cup, \cap)$ of all subsets of a fixed set I is called a filter, if the following three conditions are satisfied:

(a) $I \in \mathcal{F}$;

(b) if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$; and

(c) if $X \in \mathcal{F}$ and $A \in 2^I$, then $X \cup A \in \mathcal{F}$.

If, in addition, $\mathcal{F}$ does not contain the empty set and is a maximal filter, then $\mathcal{F} \subseteq 2^I$ is called an ultrafilter.

Let $\{R_j\}_{j \in I}$ be a family of groups, respectively rings, semigroups, etc., with index set I, and let $\mathcal{F} \subseteq 2^I$ be an ultrafilter. The ultraproduct of the family $\{R_j\}_{j\in I}$ is defined to be the set

$$\Big(\prod_{j\in I} R_j\Big)\Big/\mathcal{F} = \Big(\prod_{j\in I} R_j\Big)\Big/\sim$$

of all equivalence classes of sequences $\{r_j\}_{j\in I} \in \prod_{j\in I} R_j$ with respect to the relation

$$\{r_j\}_{j\in I} \sim \{r'_j\}_{j\in I} \iff \text{the set } \{j \in I; r_j = r'_j\} \text{ belongs to } \mathcal{F}.$$

The concept of ultraproduct appears in [**29**] under the name "operation (**P**)." The importance of this construction is established by the following famous Łoś Ultraproduct theorem (see [**8**], [**21**] and [**29**]). We present it here in a simplified form.

Theorem U. *Let $\{R_j\}_{j\in I}$ and $\mathcal{F}$ be as above, and let σ be a first order sentence in group theory, (respectively ring theory, semigroup theory, ...). The sentence σ is satisfied in $\prod_{j\in I} R_j/\mathcal{F}$ if and only if σ is satisfied in R_i for almost all $i \in I$, that is, the set $I_\sigma = \{i \in I; \sigma$ is satisfied in $R_i\}$ belongs to $\mathcal{F}$.*

An important consequence of the Łoś Ultraproduct theorem is the following version of the Łoś Compactness theorem.

Theorem C. *Let Σ be a set of first order sentences in a theory* (*of groups, rings, semigroups, etc.*) *such that, for any finite subset Σ' of Σ, a model exists, i.e., a group, a ring, a semigroup, etc., in which all sentences from Σ' are satisfied. Then a model, i.e., a group, a ring, a semigroup, etc., exists in which all sentences from Σ are satisfied.*

The fundamental ideas of Łoś in the new fields of model theory and universal algebra published in [**25–29**] were influenced by earlier results of Mostowski, Marczewski, Ryll-Nardzewski, Rasiowa and Sikorski. Several ideas of Łoś in universal algebra were later developed by his student Słomiński in his Ph.D. thesis [**51**] and habilitation thesis [**52**].

In the middle of the fifties, Łoś and his Toruń group had close contact with abelian group theory people in Budapest, and in particular with

László Fuchs and Tibor Szele. Very important steps in this activity were the Prague Abelian Group Theory Conference in 1955 and a one month visit of Fuchs to Poland in June 1958. He gave a series of lectures on abelian groups at Nicholas Copernicus University in Toruń and then visited the universities in Warsaw and Wrocław, and later the Jagiellonian University in Kraków. An influence of that visit is seen in the results of Łoś on slender groups and ω_σ-limits (see [**15**, Vol. I, Chap. VII.42]). The paper [**16**] is also a trace of it.

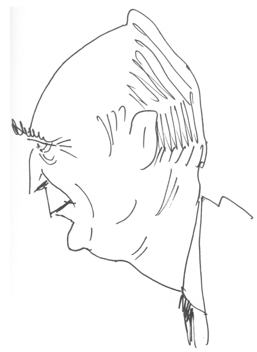
Activity in abelian group theory led to several nice results on purity in groups [**30**], [**31**], on algebraically compactness [**1**], [**32**], on pathological decompositions [**40**–**43**] and in particular to a negative solution of the Kaplansky First Test Problem [**45**]. In 1959, Sąsiada received his doctorate on the basis of an extended version of [**43**] and Balcerzyk on the basis of [**2**]. The paper [**3**] is the habilitation thesis of Balcerzyk. The ideas on purity were later developed by students of Balcerzyk and Sąsiada: in the paper [**23**] for modules over a ring, and in [**50**] for Grothendieck categories.

Let us also mention nice results obtained in Wrocław by Hulanicki [**17**–**19**] on the structure of compact groups and related problems (see [**15**, Vol. I, Corollary 42.2] and the Notes at the end of Chapter VIII of the book [**15**, Vol. 1]).

3. A future step. In 1959–61, the main interest of the Toruń algebra group was switching to ring theory, module theory, algebraic topology, algebraic number theory and homological algebra. Łoś was not very active in the new areas, the leading role in Toruń group was played then by Balcerzyk and Sąsiada. On the next page are two caricatures of Balcerzyk (left) and Sąsiada (right).

In the early sixties and the seventies, Balcerzyk concentrated his research and seminar activity mainly on algebraic topology and geometry, on commutative algebra (see the book [**6**]) and on homological algebra (see the book [**5**]). One of his main results on this subject, influenced by the famous results of Cohen [**9**] and [**10**], is contained in the paper [**4**]. Following Pierce [**37**] and Osofsky [**36**], Balcerzyk has shown in [**4**] that the global homological dimension of group algebras of abelian groups depends on the validity of the continuum hypothesis. As a consequence one gets the following algebraic version of the continuum hypothesis.

Balcerzyk



Sąsiada

Theorem. *Let* $\mathbf{R}$ *be the field of real numbers,* $m \geq 1$*, an integer, and let* G *be an abelian group such that its maximal torsion subgroup* $T(G)$ *is of cardinality* $\mathfrak{c} = 2^{\aleph_0}$*, the continuum. The global homological dimension* $\mathrm{r.gl.\,dim}\,\mathbf{R}[G]$ *of the group* $\mathbf{R}$*-algebra* $\mathbf{R}[G]$ *equals* $\mathrm{rank}\,(G) + m + 1$ *if and only if* $\mathfrak{c} = \aleph_m$*, that is, a version of the continuum hypothesis holds.*

An analogue of this result for the pure global homological dimension of group rings $\mathbf{R}[G]$ can be deduced from [**23**, Theorem 2.6].

A cooperation agreement between the Academies of Sciences of the Soviet Union and Poland allowed E. Sąsiada to spend the academic year 1960/1961 in the Moscow Łomonosov University where he was mainly influenced by Kurosh and his ring theory problems. As a result, Sąsiada constructed a nice example of a simple radical ring, that is, an example of an associative ring R with nonzero multiplication and without an identity element such that R coincides with its Jacobson radical $J(R)$.

We recall from [**20**] and [**14**, Preface] that $J(R)$ is the two-sided ideal of R consisting of all elements $r \in R$ with the property that, for every $x \in R$, there exists $y \in R$ such that the following equality holds

$$xr + y + xry = rx + y + yrx = 0.$$

If, in addition, the ring R has an identity element $1 \in R$, then the above equality gets the following well-known form

$$(1 + xr)(1 + y) = 1 = (1 + y)(1 + rx).$$

It follows that, in this case, $J(R)$ is the intersection of all maximal ideals of R and, in particular, we get $J(R) \neq R$.

A construction of a simple radical ring without an identity element was presented by Sąsiada at the meeting of the Moscow Mathematical Society on 20 December 1960. Then the result was announced in his one-page long note [**46**]. It became the main part of Sąsiada's habilitation thesis written in 1961.

In 1962 a manuscript containing a construction of a simple radical ring and detailed proofs was submitted by Sąsiada for publication in Fundamenta Mathematicae. Unfortunately, the letter including the manuscript was lost and never delivered to the editor, but Sąsiada did not notice this fact before the middle of 1963. Since, by that time, he was already interested and deeply involved in other kinds of problems on stochastics, dynamical systems and ergodic theory, he did not resubmit the paper to Fundamenta Mathematicae. Fortunately, two years later P.M. Cohn was interested in the solution of the simple radical ring problem. After a letter correspondence with Sąsiada, Cohn completed some of Sąsiada's notes and wrote an article published in 1967 as a work by Sąsiada and Cohn [**48**]. This was in fact the last algebraic paper published by Sąsiada, because the article [**47**] contains his earlier results on abelian groups. Later he published only a few papers on ergodic theory problems and led a seminar on this subject in Toruń. A nice trace of his activity in this area is the Toruń research group in dynamical systems, now represented by Kwiatkowski and Lemańczyk.

E. Sąsiada retired in 1994. He passed away in Toruń 23 February 1999 at the age of 74.

In the early sixties, Jerzy Łoś moved from Toruń to Warsaw. At the same time he finished his research on algebra and model theory. In 1961 he was appointed as a professor in the Institute of Theoretical Informatics of the Polish Academy of Sciences in Warsaw, where he retired in 1990.

In 1961, Łoś decided to change over to applications of mathematical methods in economics. The origin of his interest in this new area and the inspiration for this activity in application of mathematics came from his mathematical discussions with Hugon Steinhaus during his study at Wrocław University in the forties. For detailed information on the activity of Łoś in the area of applications of mathematical methods in economics, the reader is referred to [**53**].

Łoś was a heavy smoker. In his private life as well as in his mathematical activity, he was an imperial-type character with very strong and individualistic opinions. He was a vigorous critic of anything he did not approve of. One of his hobbies was collecting china. He was a high-class expert in this field and a co-author of a book on Polish china.

J. Łoś was elected in 1964 as a corresponding member to the Polish Academy of Sciences, and in 1983 as an associate member.

Jerzy Maria Łoś died in Warsaw 1 June 1998 at the age of 78.

Note added in proof. After the paper was submitted to the Editor, Professor Paul M. Cohn sent me the following interesting information related to his joint paper with E. Sąsiada on a simple radical ring construction mentioned in Section 3:

> I hope you will allow me to point out that I had actually met E. Sąsiada. In 1964 Yitz Herstein held an algebra study session at the University of Chicago which I attended for about six months (until December). Towards the end Edward Sąsiada also came to Chicago and in conversation mentioned his work on simple radical rings. We discussed it quite a bit and after he had left I worked out a fairly simple example which I sent to him in Poland; I knew he was going to write a paper on it and told him that he was welcome to include my example. His reply was fairly brief, saying in effect that as I now had a simple example, he would not publish his work. I then wrote back saying that he ought to be at least a joint author and if he raised no objection, I would write the whole up as a joint paper. Of course I knew he was a man of few words, and when after a longish interval I had had nothing from him, I wrote up the joint paper, which of course you know. I hope that clarifies the whole episode; if you ever have the chance to write about it again, you are welcome to include these details. With best regards, Paul Cohn.

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