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SZEGÖ POLYNOMIALS: QUADRATURE RULES ON THE UNIT CIRCLE AND ON $[-1,1]$

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Dedicated to Professor William B. Jones on the occasion of his 70th birthday

ABSTRACT. We consider some of the relations that exist between real Szegö polynomials and certain para-orthogonal polynomials defined on the unit circle, which are again related to certain orthogonal polynomials on $[-1,1]$ through the transformation $x=(z^{1/2}+z^{-1/2})/2$. Using these relations we study the interpolatory quadrature rule based on the zeros of polynomials which are linear combinations of the orthogonal polynomials on $[-1,1]$. In the case of any symmetric quadrature rule on $[-1,1]$, its associated quadrature rule on the unit circle is also given.

1. Introduction. Let $d\nu(z)$ be a Borel measure on the unit circle, i.e., $\nu(e^{i\theta})$ is real, bounded, non-decreasing with infinitely many points of increase in $0\le\theta\le 2\pi$, and let $\mu_m=\int_C z^m d\nu(z)$ be the associated moments. Then $\mu_{-m}=\overline{\mu}_m$ and

$$\begin{aligned}\mathbf{T}_n &= \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_0 \end{vmatrix} > 0 \\ &= (-1)^{\lfloor n/2\rfloor} \begin{vmatrix} \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_0 \\ \mu_{-n+2} & \mu_{-n+3} & \cdots & \mu_1 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_0 & \mu_1 & \cdots & \mu_{n-1} \end{vmatrix} \\ &= (-1)^{\lfloor n/2\rfloor}\mathbf{H}_n^{(-n+1)},\end{aligned}$$

for $n\ge 1$. Here $\lfloor n/2\rfloor$ represents the integer part of $n/2$. In relation to the above moments $\mathbf{T}_n$ are known as the Toeplitz determinants and $\mathbf{H}_n^{(-n+1)}$ are known as the Hankel determinants.

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We consider the Szegö polynomials $\{S_n\}$ associated with the measure $d\nu(z)$ defined by

$$\int_C S_n(z)\overline{S_m(z)}\, d\nu(z) = 0, \quad n \neq m.$$

These polynomials were introduced by Szegö (see, for example, [**11**]). For a good reference for basic information on these polynomials we refer to [**12**].

Since $\overline{z} = 1/z$ on the unit circle, the polynomials S_n can also be defined by

$$\int_C z^{-n+s} S_n(z) z\, d\nu(z) = 0, \quad 0 \leq s \leq n-1. \tag{1.1}$$

Hence the Szegö polynomials also satisfy the L-orthogonality property on the unit circle in relation to $z\, d\nu(z)$. Polynomials satisfying the L-orthogonality property on the positive real axis were introduced by Jones, Thron and Waadeland [**8**]. For more information on such polynomials on the unit circle see for example [**1, 3, 7**].

It is well known that the Szegö polynomials (from here on assumed to be in monic form) satisfy

$$S_n(z) = \frac{1}{\mathbf{H}_n^{(-n+1)}} \begin{vmatrix} \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_1 \\ \mu_{-n+2} & \mu_{-n+3} & \cdots & \mu_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_0 & \mu_1 & \cdots & \mu_n \\ 1 & z & \cdots & z^n \end{vmatrix}, \quad n \geq 1,$$

with $S_0 = 1$. Furthermore,

$$S_n(0) = (-1)^n \frac{\mathbf{H}_n^{(-n+2)}}{\mathbf{H}_n^{(-n+1)}}, \qquad \int_C S_n(z) z\, d\nu(z) = \frac{\mathbf{H}_{n+1}^{(-n+1)}}{\mathbf{H}_n^{(-n+1)}},$$

for $n \geq 1$. The Szegö polynomials also satisfy the system of recurrence relations

$$\begin{aligned} S_{n+1}(z) &= zS_n(z) + a_{n+1} S_n^*(z), \\ \left(1 - |a_{n+1}|^2\right) zS_n(z) &= S_{n+1}(z) - a_{n+1} S_{n+1}^*(z), \end{aligned} \tag{1.2}$$

for $n \geq 0$ (see, for example, [**5**]). Here $S_n^*(z) = z^n \overline{S}_n(1/z)$, where only the coefficients are conjugated, are the reciprocal polynomials. The numbers $a_n = S_n(0)$, $n \geq 1$, which are less than one in modulus, are known as the reflection coefficients of the Szegö polynomials.

The zeros of the Szegö polynomials are known to lie inside the open unit disk, that is they have modulus less than one.

If $S_n(0) \neq 0$, $n \geq 1$, then it follows from (1.2) that these polynomials also satisfy the three term recurrence relation

$$S_{n+1}(z) = (z + \frac{a_{n+1}}{a_n})\, S_n(z) - \frac{a_{n+1}}{a_n}\left(1 - |a_n|^2\right) z S_{n-1}(z), \quad n \geq 1,$$

where $S_0(z) = 1$ and $S_1(z) = z + a_1$.

In this manuscript we consider the Szegö polynomials with real reflection coefficients and consider the polynomials $S_n(z) \pm S_n^*(z)$ and their relations to symmetric orthogonal polynomials within the interval $[-1, 1]$. These relations, found by Delsarte and Genin in [**4**], were very nicely explored by Zhedanov [**13**]. Zhedanov uses the information contained in the relations associated with $S_n(z)+S_n^*(z)$ (or $S_n(z)-S_n^*(z)$) to derive information about S_n from the corresponding orthogonal polynomials and vice versa.

The principal aim of this paper is to study certain n-point interpolatory quadrature rules based on the zeros of the polynomial $\sum_{j=0}^{r} \lambda_j P_{n-j}(x)$, where $\{P_n\}$ are the sequence of polynomials which satisfy $P_n(x) = z^{-n/2}[S_n(z) + S_n^*(z)]$ with $x = (z^{1/2} + z^{-1/2})/2$.

2. Para-orthogonal polynomials. In [**7**] Jones, Njåstad and Thron considered the polynomials $S_n(z) + \omega_n S_n^*(z)$, where $|\omega_n| = 1$. They called these para-orthogonal polynomials and showed that their zeros are all distinct and lie on the unit circle. The proof is based on the self inversive properties of these polynomials and the conditions

$$\int_C z^{-n+s}[S_n(z) + \omega_n S_n^*(z)]\, d\nu(z) = 0, \quad 1 \leq s \leq n-1. \tag{2.1}$$

Here, restricting ourselves to only real Szegö polynomials, we consider the two special cases of monic para-orthogonal polynomials

$$R_n^{(1)}(z) = \frac{S_n(z)+S_n^*(z)}{1 + S_n(0)} \quad \text{and} \quad (z-1)R_n^{(2)}(z) = \frac{S_{n+1}(z)-S_{n+1}^*(z)}{1 - S_{n+1}(0)},$$

for $n \geq 1$. The denominators in the fractions are chosen in order to make the polynomials monic. Clearly,

$$2S_n(z) = (1+a_n)R_n^{(1)}(z) + (1-a_n)(z-1)R_{n-1}^{(2)}(z), \quad n \geq 1.$$

From (1.2), we can also derive

$$2zS_{n-1}(z) = R_n^{(1)}(z) + (z-1)R_{n-1}^{(2)}(z), \quad n \geq 1. \tag{2.2}$$

Furthermore,

Theorem 2.1. *The monic polynomials $R_n^{(i)}$, $i = 1, 2$, satisfy*

$$R_{n+1}^{(i)}(z) = (z+1)R_n^{(i)}(z) - 4\alpha_{n+1}^{(i)} z R_{n-1}^{(i)}(z), \quad n \geq 1,$$

with $R_0^{(i)}(z) = 1$, $R_1^{(i)}(z) = z + 1$ and

$$\alpha_{n+1}^{(1)} = \frac{1}{4}(1+a_{n-1})(1-a_n) > 0, \quad \alpha_{n+1}^{(2)} = \frac{1}{4}(1-a_n)(1+a_{n+1}) > 0,$$

for $n \geq 1$. Moreover these polynomials satisfy the L-orthogonality relations

$$\int_C z^{-n+s} R_n^{(1)}(z) \frac{z}{z-1}\, d\nu(z) = 0, \quad 0 \leq s \leq n-1 \tag{2.3}$$

and

$$\int_C z^{-n+s} R_n^{(2)}(z)(z-1)\, d\nu(z) = 0, \quad 0 \leq s \leq n-1. \tag{2.4}$$

The recurrence relations of $\{R_n^{(i)}\}$, $i = 1, 2$, were first established in Delsarte and Genin [**4**]. The proof of (2.3) and (2.4) can be found in Bracciali at al. [**2**].

The reason for choosing the multiplier 4 in the recurrence relation will become apparent after Theorem 2.2. The recurrence relations also confirm the self inversive property $z^n R_n^{(i)}(1/z) = R_n^{(i)}(z)$.

Extensive studies of the quadrature rules based on the zeros of para-orthogonal polynomials $S_n(z) + \omega_n S_n^*(z)$ were considered in [**1, 3**]. In these contributions the connection with quadrature rules on $[-1,1]$ have also been treated using the Szegö transformation $x=(z+z^{-1})/2$. In what follows, the transformation employed is the Delsarte and Genin transformation (see [**4**] and also [**13**])

$$x=(z^{1/2}+z^{-1/2})/2.$$

First we give some information about the sequence of polynomials $\{R_n\}$ satisfying the recurrence relation

$$R_{n+1}(z)=(z+1)R_n(z)-4\alpha_{n+1}zR_{n-1}(z),\quad n\geq 1, \tag{2.5}$$

with $R_0(z)=1$, $R_1(z)=z+1$ and $\alpha_{n+1}>0$. We have the following theorem given in Bracciali et al. [**2**].

Theorem 2.2. *Let C be the open unit circle $\{z : z=e^{i\theta}, 0<\theta<2\pi\}$. Let $\{R_n\}$ be the sequence of monic polynomials generated by the recurrence relation* (2.5). *Then the zeros of R_n are distinct* (*except for a possible double zero at $z=1$*) *and lie on $C\cup(0,\infty)$. In particular, if $\{\alpha_{n+1}\}$ is a chain sequence, then all the zeros are distinct and lie on C. In this case, there exists a positive measure $d\nu(z)$ on the unit circle such that*

$$\int_C z^{-n+s}R_n(z)\,\frac{z}{z-1}\,d\nu(z)=0,\quad 0\leq s\leq n-1.$$

The polynomials $P_n^{(1)}(x) = (4z)^{-n/2}R_n^{(1)}(z)$ and $P_n^{(2)}(x) = (4z)^{-n/2}R_n^{(2)}(z)$ satisfy the recurrence relations

$$P_{n+1}^{(i)}(x)=xP_n^{(i)}(x)-\alpha_{n+1}^{(i)}P_{n-1}^{(i)}(x),\quad n\geq 1, \tag{2.6}$$

and are orthogonal polynomials on $[-1,1]$ in relations to the measures $d\phi^{(1)}(x)=-d\nu(z)$ and $d\phi^{(2)}(x)=-(1-x^2)d\nu(z)$, respectively. Hence we can state the following theorem, which was also given in Bracciali et al. [**2**].

Theorem 2.3. *Let $d\phi^{(1)}(x)$ and $d\phi^{(2)}(x)$ be two positive measures on $[-1,1]$ such that*

$$d\phi^{(2)}(x)=(1-x^2)d\phi^{(1)}(x).$$

Let the respective monic orthogonal polynomials $P_n^{(1)}$ and $P_n^{(2)}$ associated with these measures satisfy

$$P_{n+1}^{(i)}(x) = xP_n^{(i)}(x) - \alpha_{n+1}^{(i)} P_{n-1}^{(i)}(x), \quad n \geq 1.$$

Let

$$2zS_{n-1}(z) = R_n^{(1)}(z) + (z-1)R_{n-1}^{(2)}(z), \quad n \geq 1,$$

where $R_n^{(1)}(z) = (4z)^{n/2}P_n^{(1)}(x(z))$ and $R_n^{(2)}(z) = (4z)^{n/2}P_n^{(2)}(x(z))$. Then S_n are the monic Szegö polynomials associated with the measure $d\nu(z) = -d\phi^{(1)}(x(z))$. Furthermore, the reflection coefficients $a_n = S_n(0)$ can be generated by

$$a_n = 1 - \frac{4\alpha_{n+1}^{(1)}}{1+a_{n-1}} \quad \text{or} \quad a_{n+1} = -1 + \frac{4\alpha_{n+1}^{(2)}}{1-a_n}, \quad n \geq 1.$$

Given explicitly, for all $n \geq 1$,

$$a_{2n-1} = 2\frac{\alpha_{2n-1}^{(2)}\alpha_{2n-3}^{(2)}\cdots\alpha_3^{(2)}\mu_0^{(2)}}{\alpha_{2n-1}^{(1)}\alpha_{2n-3}^{(1)}\cdots\alpha_3^{(1)}\mu_0^{(1)}} - 1, \quad a_{2n} = 2\frac{\alpha_{2n}^{(2)}\alpha_{2n-2}^{(2)}\cdots\alpha_2^{(2)}}{\alpha_{2n}^{(1)}\alpha_{2n-2}^{(1)}\cdots\alpha_2^{(1)}} - 1.$$

Here $\mu_0^{(i)}$, $i = 1, 2$, are the respective moments of order zero.

3. Quadrature rules. Let us consider the interpolatory quadrature rule based on the zeros of the polynomial

$$P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x] = \sum_{j=0}^{r} \lambda_j \, P_{n-j}^{(1)}(x),$$

where $\lambda_0 = 1$, $\lambda_j \in \mathbf{R}$ for $j = 1, 2, \cdots, r$ and $r \leq n$. We assume that the parameters λ_j are chosen such that the zeros of $P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x]$ are all real, distinct and lie within $(-1, 1)$.

For example, if $r = 1$ then all the zeros of $P_n[\lambda_0, \lambda_1; x]$ are real, distinct and at least $n-1$ of them lie within $(-1, 1)$. Since $\lambda_0 = 1$, the necessary and sufficient condition for all the zeros to be inside $(-1, 1)$ is $-P_n^{(1)}(1)/P_{n-1}^{(1)}(1) < \lambda_1 < -P_n^{(1)}(-1)/P_{n-1}^{(1)}(-1)$.

If $r = 2$, then from $P_n[\lambda_0, \lambda_1, \lambda_2; x] = (x + \lambda_1)P_{n-1}^{(1)}(x) - (\alpha_n^{(1)} - \lambda_2)P_{n-2}^{(1)}(x)$, all the zeros are real, distinct and at least $n-2$ of them lie within $(-1,1)$, provided that $\alpha_n^{(1)} > \lambda_2$. In addition if $\lambda_1 = 0$, then the condition $\alpha_n^{(1)} > \lambda_2 \geq 0$ is sufficient for all the zeros to be within $(-1,1)$.

For larger values of r, in general, it is difficult to study the polynomial $P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x]$ as shown above and obtain a condition which guarantees its zeros to be real, distinct and within $(-1,1)$. In section 5 we obtain conditions, that not only give the requirement on the zeros, but also guarantee that the weights of the interpolatory quadrature rule based on these zeros are all positive.

Let $x_{n,k}$, $k = 1, 2, \cdots, n$ be the zeros of $P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x]$, assumed to be real, distinct and in decreasing order. Then the interpolatory quadrature rule on these zeros,

$$\int_{-1}^{1} f(x)\, d\phi^{(1)}(x) = \sum_{k=1}^{n} \omega_{n,k} f(x_{n,k}), \tag{3.1}$$

holds for $f \in \mathbf{P}_{2n-r-1}$. The weights $\omega_{n,k}$ of this quadrature rule can be given by

$$\omega_{n,k} = \frac{O_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x_{n,k}]}{P_n'[\lambda_0, \lambda_1, \cdots, \lambda_r; x_{n,k}]}, \tag{3.2}$$

where $O_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x] = \sum_{j=0}^{r} \lambda_j O_{n-j}(x)$, with $O_n(x) = \int_{-1}^{1} (x-t)^{-1} [P_n^{(1)}(x) - P_n^{(1)}(t)]\, d\phi^{(1)}(t)$.

The polynomials O_n, which are known as the associated polynomials of the orthogonal polynomials $P_n^{(1)}$, satisfy the same recurrence relation

$$O_{n+1}(x) = xO_n(x) - \alpha_{n+1}^{(1)} O_{n-1}(x), \quad n \geq 1;$$

however, with the initial conditions $O_0(z) = 0$ and $O_1(z) = \mu_0^{(1)} = \int_{-1}^{1} d\phi^{(1)}(x)$.

Now we consider the symmetric quadrature rule

$$\int_{-1}^{1} f(x)\, d\phi^{(1)}(x) = \sum_{k=1}^{n} \hat{\omega}_{n,k} f(\hat{x}_{n,k}), \tag{3.3}$$

obtained when $\hat{x}_{n,k}$ are distinct zeros of $P_n[\lambda_0, 0, \lambda_2, \cdots, 0, \lambda_{2l}; x]$. Here $2l \leq n$ and $\lambda_{2j-1} = 0$ for $j = 1, 2, \cdots, l$. Clearly, $\hat{x}_{n,k} = -\hat{x}_{n,n+1-k}$

and $\hat{\omega}_{n,k} = \hat{\omega}_{n,n+1-k}$ and that the quadrature rule holds when $f \in \mathbf{P}_{2n-2l-1}$.

Theorem 3.1. *Let $d\nu(z) = -d\phi^{(1)}(x(z))$, and let $z_{n,k} = \left\{\hat{x}_{n,k} + i\sqrt{1-\hat{x}_{n,k}^2}\,\right\}^2$. Then the quadrature rule* (3.3) *holds for $f \in \mathbf{P}_{2n-2l-1}$ if and only if the quadrature rule*

$$\int_C F(z)\,d\nu(z) = \sum_{k=1}^{n} \hat{\omega}_{n,k} F(z_{n,k}),$$

holds for $F(z) \in \mathrm{Span}\,\{z^{-n+l+1}, z^{-n+l+2}, \cdots, z^{n-l-2}, z^{n-l-1}\}$.

Proof. The nodes $z_{n,k}$ are the zeros of the polynomial $(4z)^{n/2}P_n[\lambda_0, 0, \lambda_2, \cdots, 0, \lambda_{2l}; x(z)]$. We first show that (3.3) holds for $f \in \mathbf{P}_{2n-2l-1}$ if and only if

$$\int_{-1}^{1} \left\{x + i\sqrt{1-x^2}\,\right\}^{2s+1} \frac{d\phi^{(1)}(x)}{2i\sqrt{1-x^2}} = \sum_{k=1}^{n} \left\{\hat{x}_{n,k} + i\sqrt{1-\hat{x}_{n,k}^2}\,\right\}^{2s+1} \frac{\hat{\omega}_{n,k}}{2i\sqrt{1-\hat{x}_{n,k}^2}},$$

for $-n+l \le s \le n-l-1$. The quadrature rule (3.3), which is symmetric, is satisfied for any function $f = p + g$, where p is an even polynomial of degree less than or equal to $2n-2l-2$ and g is any odd function.

We consider the function $f_s(x) = \frac{1}{i\sqrt{1-x^2}}\left\{x + i\sqrt{1-x^2}\right\}^{2s+1}$. Using the binomial expansion we can write

$$f_s(x) = g_s(x) + p_{2s}(x),$$

for $s \ge 0$ and since $\{x + i\sqrt{1-x^2}\}\{x - i\sqrt{1-x^2}\} = 1$,

$$f_{-s-1}(x) = g_s(x) - p_{2s}(x),$$

for $s \ge 0$. Here,

$$g_s(x) = \frac{x}{\sqrt{x^2-1}} \sum_{j=0}^{s} \binom{2s+1}{2j} (x^2-1)^j x^{2s-2j},$$

which is an odd function and

$$p_{2s}(x) = \sum_{j=0}^{s} \binom{2s+1}{2j+1} (x^2-1)^j x^{2s-2j},$$

which is an even polynomial of degree 2s with leading coefficient equal to 2^{2s}. Thus (3.3) implies (3.4).

On the other hand, since $p_{2s}(x) = [f_s(x) - f_{-s-1}(x)]/2$, $s = 0, 1, \cdots, n-l-1$, form a basis for all even polynomials of degree less than or equal to $2n-2l-2$, (3.4) also implies (3.3).

Now the substitution $2x = z^{1/2} + z^{-1/2}$ in (3.4) leads to

$$\int_C z^s \frac{z}{z-1} [-d\phi^{(1)}(x(z))] = \sum_{k=1}^{n} \hat{\omega}_{n,k} \frac{z_{n,k}}{z_{n,k}-1} z_{n,k}^s,$$

for $s = -n+l, -n+l+1, \cdots, n-l-2, n-l-1$. Hence the observation that if $F(z) \in \mathrm{Span}\{z^{-n+l+1}, z^{-n+l+2}, \cdots, z^{n-l-2}, z^{n-l-1}\}$ then $(1-z^{-1})F(z) \in Span\{z^{-n+l}, z^{-n+l+2}, \cdots, z^{n-l-2}, z^{n-l-1}\}$ concludes the proof of the theorem. □

The case $l = 0$ in this theorem shows the connection between the n-point Gaussian quadrature rule defined on $(-1, 1)$ and the n-point Szegö quadrature rule.

As a first example of an application of the above theorem, we immediately obtain from the Gauss-Chebyshev rule, the following quadrature rule

$$\int_0^{2\pi} F(e^{i\theta})\, d\theta = \frac{2\pi}{n} \sum_{k=1}^{n} F(e^{i\frac{2k-1}{n}\pi}),$$

which holds for

$$F(z) \in \mathrm{Span}\{z^{-n+1}, z^{-n+2}, \cdots, z^{n-2}, z^{n-1}\}.$$

This quadrature rule has also been given in [**6**] with some numerical results.

As another example we consider the Chebyshev-Fejér rule, which is exact for $f \in \mathbf{P}_{2n-2l-1}$, where $l = \lfloor n/2 \rfloor$, the integer part of $n/2$. From this we obtain the quadrature rule

$$\int_0^{2\pi} F(e^{i\theta}) \sin(\theta/2)\, d\theta = \sum_{k=1}^{n} \hat{\omega}_{n,k} F(e^{i(2k-1/n)\pi}),$$

where

$$\hat{\omega}_{n,k} = \frac{4}{n} \left\{ 1 - 2 \sum_{m=1}^{\lfloor n/2 \rfloor} \frac{\cos[(m(2k-1)/n)\pi]}{4m^2-1} \right\},$$

which holds for $F(z) \in \text{Span}\,\{z^{-n+\lfloor n/2\rfloor+1}, z^{-n+\lfloor n/2\rfloor+2}, \cdots, z^{n-\lfloor n/2\rfloor-2}, z^{n-\lfloor n/2\rfloor-1}\}$.

4. Further considerations on quadrature rules. Recall that $P_n^{(1)}(x) = (4z)^{-n/2} R_n^{(1)}(z)$, where $x = (z^{1/2} + z^{-1/2})/2 = \cos(\theta/2)$ and $R_n^{(1)}(z) = [S_n(z) + S_n^*(z)]/[1 + S_n(0)]$. Since S_n are polynomials with real coefficients, we can also write

$$P_n^{(1)}(x) = \frac{2^{-(n-1)}}{1 + S_n(0)} \,\mathbf{Re}\,[z^{-n/2} S_n(z)], \quad n \geq 1,$$

where $z = e^{i\theta}$ and $x = \cos(\theta/2)$.

We now consider the polynomials $\tilde{S}_n$ that satisfy the recurrence relation

$$\tilde{S}_{n+1}(z) = z\tilde{S}_n(z) - a_{n+1}\tilde{S}_n^*(z), \quad n \geq 0.$$

with $\tilde{S}_0(z) = 1$. Clearly, $\tilde{S}_n(0) = -a_n$, $n \geq 1$. $\tilde{S}_n$ are known as the associated polynomials of the Szegö polynomials S_n, where $S_n(0) = -\tilde{S}_n(0) = a_n$ for $n \geq 1$. Like the polynomials S_n, the zeros of $\tilde{S}_n$ also lie inside the open unit disk.

Now if we consider the polynomials, $\tilde{P}_n^{(2)}(x) = (4z)^{-n/2}\tilde{R}_n^{(2)}(z) = (4z)^{-n/2}(z-1)^{-1}[\tilde{S}_{n+1}(z) - \tilde{S}_{n+1}^*(z)]/[1 - \tilde{S}_{n+1}(0)]$, then a comparison of the corresponding recurrence relations confirms that $O_n(x) = \mu_0^{(1)}\tilde{P}_{n-1}^{(2)}(x)$ for $n \geq 1$. Consequently, we can also write

$$O_n(x) = \frac{2^{-(n-1)}\mu_0^{(1)}}{1 + S_n(0)} \frac{\text{Im}\,[z^{-n/2}\tilde{S}_n(z)]}{\sin(\theta/2)}, \quad n \geq 1,$$

where $z = e^{i\theta}$ and $x = \cos(\theta/2)$.

We now define the sequence of polynomials $\{Q_{k,n}\}_{k=0}^{n}$ by the recurrence relation

$$Q_{k,n}(z) = zQ_{k-1,n}(z) + za_{n+1-k}Q^*_{k-1,n}(z), \quad k = 1, 2, \cdots, n,$$

with $Q_{0,n}(z) = Q^*_{0,n}(z) = 1$. Thus $Q_{k,n}$, which is monic and of degree k, satisfy

$$Q^*_{k,n}(z) = Q^*_{k-1,n}(z) + a_{n+1-k}Q_{k-1,n}(z), \quad k = 1, 2, \cdots, n \tag{4.1}$$

and we obtain the following lemma.

Lemma 1. *For $|z| \le 1$ we have*

$$0 < 1-|a_{n+1-k}| \le \left|\frac{Q^*_{k,n}(z)}{Q^*_{k-1,n}(z)}\right| \le 1+|a_{n+1-k}|, \quad k = 1, 2, \cdots, n \tag{4.2}$$

and that the zeros of $Q_{k,n}$, $k = 1, 2, \cdots, n$, lie inside the open unit disk. Moreover, for any k such that $1 \le k \le n$,

$$Q_{k,n}(z)S_{n-k}(z) + Q^*_{k,n}(z)S^*_{n-k}(z) = S_n(z) + S^*_n(z)$$

and

$$Q_{k,n}(z)\tilde{S}_{n-k}(z) - Q^*_{k,n}(z)\tilde{S}^*_{n-k}(z) = \tilde{S}_n(z) - \tilde{S}^*_n(z).$$

Proof. From (4.1), clearly $0 < 1 - |a_n| \le \left|Q^*_{1,n}(z)\right| \le 1 + |a_n|$ and hence the zero of $Q_{1,n}$ is inside the unit disk. Now suppose that for some k, $1 < k \le n$, that the zeros of $Q_{k-1,n}$ lie within the open unit disk. Hence from (4.1),

$$\frac{Q^*_{k,n}(z)}{Q^*_{k-1,n}(z)} = 1 + a_{n+1-k}\,\frac{Q_{k-1,n}(z)}{Q^*_{k-1,n}(z)},$$

where $\left|Q_{k-1,n}(z)/Q^*_{k-1,n}(z)\right| \le 1$ for $|z| \le 1$. Thus $Q^*_{k,n}(z)/Q^*_{k-1,n}(z)$ satisfies (4.2) and consequently the zeros of $Q_{k,n}$ are inside the open unit disk.

The remaining results of the above lemma follow from the applications of the recurrence relations of S_n, $\tilde{S}_n$ and $Q_{k,n}$. $\square$

The results given by the above lemma are similar to that of Peherstorfer [**9**, Lemma 3] and some subsequent results in that paper. The main difference is in the way the recurrence relation for $Q_{k,n}$ is chosen. In our case, different to [**9**], multiplication by z occurs in both terms in the righthand side.

As a consequence of Lemma 1, since $Q_{k,n}S_{n-k}$ and $Q_{k,n}\tilde{S}_{n-k}$ are polynomials of degree n with real coefficients, we can also write

$$P_n^{(1)}(x) = \frac{2^{-(n-1)}}{1+S_n(0)} \operatorname{Re}[z^{-n/2}Q_{k,n}(z)S_{n-k}(z)], \quad n \geq 1,$$

and

$$O_n(x) = \frac{2^{-(n-1)}\mu_0^{(1)}}{1+S_n(0)} \frac{\operatorname{Im}[z^{-n/2}Q_{k,n}(z)\tilde{S}_{n-k}(z)]}{\sin(\theta/2)}, \quad n \geq 1,$$

for $0 \leq k \leq n$. Furthermore,

$$P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x] = \sum_{j=0}^{r} \lambda_j \frac{2^{-(n-j-1)}}{1+S_{n-j}(0)} \operatorname{Re}[z^{-\frac{(n-j)}{2}} Q_{k_j,n-j}(z)S_{n-j-k_j}(z)]$$

and

$$O_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x] = \sum_{j=0}^{r} \lambda_j \frac{2^{-(n-j-1)}\mu_0^{(1)}}{1+S_{n-j}(0)} \frac{\operatorname{Im}[z^{-\frac{(n-j)}{2}} Q_{k_j,n-j}(z)\tilde{S}_{n-j-k_j}(z)]}{\sin(\frac{\theta}{2})},$$

where $0 \leq k_j \leq n-j$. In particular, letting $k_j = r-j$, we obtain

$$P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x] = 2^{-(n-1)} \operatorname{Re}[z^{-n/2}q_r(z)S_{n-r}(z)]$$

and

$$O_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x] = \frac{2^{-(n-1)}\mu_0^{(1)}}{\sin(\theta/2)} \operatorname{Im}[z^{-n/2}q_r(z)\tilde{S}_{n-r}(z)],$$

where

$$q_r(z) = \sum_{j=0}^{r} \frac{\lambda_j\, 2^j z^{j/2}}{1+a_{n-j}} Q_{r-j,n-j}(z).$$

5. Positive quadrature rules. We now give some conditions that ensure for the quadrature rule (3.1) the following well desired property.

$$1 > x_{n,1} > x_{n,2} > \cdots > x_{n,n} > -1 \\ \text{and} \quad \omega_{n,k} > 0, \quad 1 \le k \le n. \tag{5.1}$$

The nodes $x_{n,k}$ belonging to the interval of integration $(-1, 1)$ is desirable because it is not practical to assume that the integrand f be defined outside this interval. The weights $\omega_{n,k}$ being positive has the following nice implication due to Pólya [**10**]. Since (3.1) is an interpolating quadrature rule for any n, the positiveness of the weights guarantees that

$$\sum_{k=1}^{n} \omega_{n,k} f(x_{n,k}) \longrightarrow \int_{-1}^{1} f(x)\, d\phi^{(1)}(x),$$

for all continuous functions.

Now it is easily verified (see for example Lemma 1 of [**9**]) that the weights $\omega_{n,k}$ of the quadrature rule (3.1) are positive if the n zeros of $P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x]$ and the $n-1$ zeros of $O_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x]$ interlace.

We show that the zeros of $P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x]$ are inside $(-1, 1)$ and that the above interlacing property holds if the zeros of the $2r$-th degree polynomial $\tilde{q}_{2r}(w) = q_r(w^2)$ are inside the open unit disk. In order to show this, we use the following lemma.

Lemma 2. *Let $m \in \mathbf{N}$ and $l \in \mathbf{Z}$ with $2|l| \le m$. Assume that the real polynomial $s_m(z)$ of degree m has all its zeros in the open unit disk $|z| < 1$. Then for $x = \cos(\theta)$ and $z = e^{i\theta}$,*

- *the polynomial $p_1(x) = \mathrm{Re}\,[z^{-l} s_m(z)]$ has $m-l$ simple zeros $x_j^{(1)} = \cos(\theta_j^{(1)})$ in $(-1, 1)$,*
- *the polynomial $p_2(x) = \mathrm{Im}\,[z^{-l} s_m(z)]/\sin(\theta)$ has $m-l-1$ zeros $x_j^{(2)} = \cos(\theta_j^{(2)})$ in $(-1, 1)$;*
- *the zeros of $p_1(x)$ and $p_2(x)$ interlace, i.e., $0 < \theta_1^{(1)} < \theta_1^{(2)} < \theta_2^{(1)} < \cdots < \theta_{m-l-1}^{(2)} < \theta_{m-l}^{(1)} < \pi$.*

This lemma was stated and proved in Peherstorfer [**9**]. We state a modification of this lemma which leads to symmetric polynomials in the real line.

Lemma 3. *Let $s_n(z)$ be a real polynomial of degree n with all its zeros in the open unit disk $|z| < 1$. Then for $x = \cos(\theta/2)$ and $z = e^{i\theta}$,*

- $p_1(x) = \mathrm{Re}\,[z^{-n/2}s_n(z)]$ *is a real and symmetric polynomial of degree n that has all its zeros simple and inside $(-1, 1)$;*
- $p_2(x) = \mathrm{Im}\,[z^{-n/2}s_n(z)]/\sin(\theta/2)$ *is a real and symmetric polynomial of degree $n-1$ that has all its zeros simple and inside $(-1, 1)$;*
- *the zeros of $p_1(x)$ and $p_2(x)$ interlace.*

Proof. It is easy to verify that both p_1 and p_2 are real and symmetric polynomials. Let $2l \in \mathbf{Z}$ such that $2|l| \le n$. Then $\mathrm{Re}\,[z^{-l}s_n(z)]$ (respectively $\mathrm{Im}\,[z^{-l}s_n(z)]$) has a zero at $z = e^{i\alpha}$, $\alpha \in [0, 2\pi)$, if and only if

$$z^{n-2l}\,\frac{s_n(z)}{s_n^*(z)} = -1 \quad (\text{respectively } +1).$$

Note that, different to Lemma 2, we also allow l to be a half integer. Now the above results is equivalent to

$$\arg z^{n-2l} + \arg \frac{s_n(z)}{s_n^*(z)} = (2k-1)\pi \quad (\text{respectively } 2k\pi),$$

for $k = 0, 1, \dots$. Since $[\arg s_n(e^{i\alpha})/s_n^*(e^{i\alpha})]$ increases from 0 to $2n\pi$ as α varies from 0 to 2π, both $\mathrm{Re}\,[z^{-l}s_n(z)]$ and $\mathrm{Im}\,[z^{-l}s_n(z)]$ have $2n-2l$ zeros in $[0, 2\pi)$ and their zeros interlace. Hence if we take $l = n/2$, we obtain precisely the results of the lemma. □

We now consider the function $s_n(z) = q_r(z)S_{n-r}(z)$. This can be considered as a polynomial of degree $m = 2n$ in $w = z^{1/2} = e^{i\theta/2}$. Then

$$s_n(w^2) = q_r(w^2)S_{n-r}(w^2) = \tilde{q}_{2r}(w)S_{n-r}(w^2).$$

Suppose that the zeros of $\tilde{q}_{2r}(w)$, which is a polynomial of degree $2r$ in w, are inside the open unit disk. Which means, all the zeros of $s_n(w^2)$, which is a polynomial of degree $m = 2n$ in w, are inside the open unit disk. Consequently, from Lemma 2, with $l = m/2$ and $x = \cos(\theta/2)$, the polynomial $P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x] = 2^{-(n-1)}\mathrm{Re}\,[z^{-n/2}q_r(z)S_{n-r}(z)]$ has $m - l = n$ zeros $x_{n,j} = \cos(\theta_j^{(1)}/2)$ in $(-1, 1)$ and the polynomial $p_2(x) = \mathrm{Im}\,[z^{-n/2}q_r(z)S_{n-r}(z)]/\sin(\theta/2)$ has $m - l - 1 = n - 1$ zeros $\cos(\theta_j^{(2)}/2)$ in $(-1, 1)$. Moreover, these zeros interlace, i.e., $0 < (1/2)\theta_1^{(1)} < (1/2)\theta_1^{(2)} < (1/2)\theta_2^{(1)} < \cdots < (1/2)\theta_{n-1}^{(2)} < (1/2)\theta_n^{(1)} < \pi$.

When $\lambda_{2j-1} = 0$, $j = 1, 2, \cdots$, i.e., when we consider the symmetric quadrature rule (3.3), then the above results can be obtained from Lemma 3.

Now from $\mathrm{Re}\,\{a\}\mathrm{Re}\,\{b\} + \mathrm{Im}\,\{a\}\mathrm{Im}\,\{b\} = \mathrm{Re}\,\{a\,\overline{b}\}$, when $|z| = 1$,

$$\begin{aligned}
&\mathrm{Re}\,[z^{-n/2}q_r(z)S_{n-r}(z)]\,\mathrm{Re}\,[z^{-n/2}q_r(z)\tilde{S}_{n-r}(z)] \\
&\qquad + \mathrm{Im}\,[z^{-n/2}q_r(z)S_{n-r}(z)]\,\mathrm{Im}\,[z^{-n/2}q_r(z)\tilde{S}_{n-r}(z)] \\
&\quad = |q_r(z)|^2\mathrm{Re}\,[S_{n-r}(z)\overline{\tilde{S}_{n-r}(z)}] \\
&\quad = 2|q_r(z)|^2\prod_{j=1}^{n-r}(1 - a_j^2) > 0,
\end{aligned}$$

since $q_r(z) \neq 0$ for $|z| = 1$. Considering the above relation at the zeros $x_{n,k}$ of $P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x]$, i.e., at the points $z_{n,k} = e^{i\theta_k^{(1)}}$, where $x_{n,k} = \cos(\theta_k^{(1)}/2)$, we obtain that the zeros of $P_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x]$ and the zeros of $O_n[\lambda_0, \lambda_1, \cdots, \lambda_r; x]$ interlace. Thus we can state the following theorem.

Theorem 5.1. *If* $\tilde{q}_{2r}(w) = \sum_{j=0}^{r}(1 + a_{n-j})^{-1}\lambda_j 2^j w^j Q_{r-j,n-j}(w^2)$, *which is a polynomial of degree* $2r$ *in* $w = z^{1/2}$ *with real coefficients, has all its zeros inside the open unit disk then* (5.1) *holds for the nodes and weights of the quadrature rule* (3.1).

Pursuing even further the ideas presented in [**9**], we obtain the following results which provide some sufficient conditions for (5.1) to hold.

Theorem 5.2. *Let* $\lambda_j \in \mathbf{R}$, $j = 0, 1, \cdots, r$, *and let* $j_0 = 0 < j_1 < \cdots < j_{\hat{r}}$ *be those indices for which* $\lambda_{j_v} \neq 0$ *for* $v = 0, 1, \cdots \hat{r}$. *Define*

$\Lambda_0 = 1$ *and*

$$\Lambda_j = 2^j|\lambda_j|\frac{|1+a_n|}{|1+a_{n-j}|}\,\frac{\prod_{k=0}^{r-j-1}(1+|a_{n-j-k}|)}{\prod_{k=0}^{r-1}(1-|a_{n-k}|)},$$

for $j = 1, 2, \cdots, r$. *Then* (5.1) *holds if* $\sum_{j=1}^{r}\Lambda_j < 1$.

In particular, (5.1) *holds if* $\Lambda_{j_v} \geq 2\Lambda_{j_{v+1}}$ *for* $v = 0, 1, \cdots, \hat{r}-2$ *and* $\Lambda_{j_{\hat{r}-1}} > \Lambda_{j_{\hat{r}}}$.

Proof. Consider the polynomial $\tilde{q}_{2r}(w) = \sum_{j=0}^{r}\frac{\lambda_j 2^j w^j}{1+a_{n-j}}Q_{r-j,n-j}(w^2)$, which is real and of degree exactly $2r$ in w. Hence,

$$\begin{aligned}\tilde{q}^*_{2r}(w) &= \sum_{j=0}^{r}\frac{\lambda_j 2^j w^j}{1+a_{n-j}}\,w^{2r-2j}Q_{r-j,n-j}(1/w^2)\\ &= \sum_{j=0}^{r}\frac{\lambda_j 2^j w^j}{1+a_{n-j}}\,Q^*_{r-j,n-j}(w^2).\end{aligned}$$

Clearly the zeros of $\tilde{q}_{2r}(w)$ are in $|w| < 1$ is equivalent to saying that the zeros of $\tilde{q}^*_{2r}(w)$ are in $|w| > 1$. Suppose that there is a zero ζ of $\tilde{q}^*_{2r}(w)$ inside $|w| < 1$. We show that this contradicts $\sum_{j=1}^{r}\Lambda_j < 1$. We have

$$\tilde{q}^*_{2r}(\zeta) = \sum_{j=0}^{r}\frac{\lambda_j 2^j \zeta^j}{1+a_{n-j}}\,Q^*_{r-j,n-j}(\zeta^2) = 0.$$

Since $Q^*_{r,n}(z)$ has no zeros inside $|z| < 1$, we can write

$$1 = |\lambda_0| = \left|\sum_{j=1}^{r}\lambda_j 2^j\zeta^j\,\frac{1+a_n}{1+a_{n-j}}\,\frac{Q^*_{r-j,n-j}(\zeta^2)}{Q^*_{r,n}(\zeta^2)}\right|.$$

Hence the application of the first part of Lemma 1 gives $1 \leq \sum_{j=1}^{r}\Lambda_j$, which is a contradiction to the criteria given by the theorem. Thus if $\Lambda_0 > \sum_{j=1}^{r}\Lambda_j$ holds then all the zeros of $\tilde{q}_{2r}(w)$ are inside $|w| < 1$ and consequently the nodes and weights of the quadrature rule (3.1) satisfy (5.1).

Now the proof of the remaining results of the theorem follows since $\Lambda_{j_v} \geq \Lambda_{j_{v+1}}$ for $v = 0, 1, \cdots, \hat{r}-2$ and $\Lambda_{j_{\hat{r}-1}} > \Lambda_{j_{\hat{r}}}$, is a sufficient condition for $\Lambda_0 > \sum_{j=1}^{r}\Lambda_j$ to hold. $\square$

Since the symmetric quadrature rule (3.3) is a special case of the quadrature rule (3.1), application of the above theorem to (3.3) gives

Corollary 5.2.1. *Let*

$$\hat{\Lambda}_j = 2^{2j}|\lambda_{2j}|\frac{|1+a_n|}{|1+a_{n-2j}|}\,\frac{\prod_{k=0}^{2l-2j-1}(1+|a_{n-2j-k}|)}{\prod_{k=0}^{2l-1}(1-|a_{n-k}|)}, \quad j=1,2,\cdots,l.$$

Then the nodes and weights of the symmetric quadrature rule (3.3) *satisfy*

$$-1<\hat{x}_{n,k}<1 \quad \textit{and} \quad \hat{\omega}_{n,k}>0, \tag{5.2}$$

for $1\le k\le n$, *if* $\sum_{j=1}^{l}\hat{\Lambda}_j<1$.

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