ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 1, 2005

TYPE SUBMODULES AND DIRECT SUM DECOMPOSITIONS OF MODULES

JOHN DAUNS AND YIQIANG ZHOU

ABSTRACT. A type decomposition of a module M over a ring R is a direct sum decomposition for which any two distinct summands have no nonzero isomorphic submodules. In this paper, we investigate when a module possesses certain kinds of type decompositions and when such decompositions are unique.

Introduction. It is well known that every torsion abelian group has a unique decomposition into its p-torsion subgroups. By Goodearl-Boyle [**4**], every nonsingular injective module E has a unique decomposition $E = E_1 \oplus E_2 \oplus E_3$ where E_1, E_2, E_3 are of types I, II, III respectively, see Definition 2.7. Why do such decompositions exist? Why are such decompositions unique? Are there any common things between these two results? All these questions will be answered in this paper. In fact, we can present a more general theory on existence and uniqueness of type decompositions of modules, so that the above results, as well as many other known results, are obtained as very special cases.

The common property for certain diverse kinds of direct sum decompositions of modules M including the two decompositions above is that any two distinct direct summands have no nonzero isomorphic submodules, or equivalently all direct summands are what we will call type submodules. The cause for the existence of such decompositions is that these modules M have a 'decomposability property' which will be discussed in detail in Section 1, while the uniqueness of such direct sum decompositions is ensured by a module property called UTC. A theory of such modules is developed in Section 2.

Throughout, all rings R are associative with identity and modules are unital right R-modules and M is an R-module. A class $\mathcal{K}$ of modules is a *type*, or *natural class*, if it is closed under isomorphic copies,

1991 AMS *Mathematics Subject Classification.* Primary 16D70, 16D80.
Research of the second author was supported by NSERC Grant OGP0194196.

submodules, arbitrary direct sums and injective hulls. A submodule N of M is a *type submodule* if, for some type $\mathcal{K}$, N is a submodule of M which is maximal with respect to $N \in \mathcal{K}$. In this case, we also say N is a type submodule of type $\mathcal{K}$. Two modules M_1 and M_2 are *orthogonal*, written $M_1 \perp M_2$, if they do not have nonzero isomorphic submodules. Equivalently, a submodule N of M is a type submodule if and only if, whenever $N \subset X \subseteq M$, there exists $0 \neq Y \subseteq X$ such that $N \perp Y$. An *atomic module* is any nonzero module A which has only one nonzero type submodule, namely A itself. A module direct sum, or module decomposition $M = \oplus_{i\in I} M_i$ is called a *type direct sum*, or *type decomposition*, if $M_i \perp M_j$ for all $i \neq j$ in I.

Let N be a submodule of M. By Zorn's lemma, there exists a submodule P of M which is maximal with respect to the property that $N \subseteq P$ and every nonzero submodule of P is not orthogonal to N. The module P is called a *type closure* of N in M and is denoted by $N^{tc} = P$, even though P need not be unique. Again by Zorn's lemma, there exists a submodule Q of M maximal with respect to $N \perp Q$. The module Q is called a *type complement* of N in M. Clearly, type closures and type complements of N in M all are type submodules of M.

For any module class $\mathcal{F}$, let $c(\mathcal{F}) = \{N : \forall\, 0 \neq X \leq N, X \not\hookrightarrow P$ for all $P \in \mathcal{F}\}$ and $d(\mathcal{F}) = \{N : \forall\, 0 \neq X \leq N, \exists\, 0 \neq Y \leq X$ and $P \in \mathcal{F}$ such that $Y \hookrightarrow P\}$. Note that both $c(\mathcal{F})$ and $d(\mathcal{F})$ are natural classes and they are Boolean complements of each other in the complete Boolean lattice of all natural classes. If $\mathcal{F} = \{N\}$, we write $d(N) = d(\{N\})$, [**3**, p. 514]. Two types $\mathcal{K}_1$ and $\mathcal{K}_2$ are orthogonal if $M_1 \perp M_2$ for all $M_1 \in \mathcal{K}_1$ and all $M_2 \in \mathcal{K}_2$, and this happens if and only if $\mathcal{K}_1 \wedge \mathcal{K}_2 = \mathbf{0}$. We define *a maximal set of pairwise orthogonal types* to be any family $\{\mathcal{K}_i : i \in I\}$ of types $\mathcal{K}_i$ such that $\vee_{i\in I}\mathcal{K}_i = \mathbf{1}$ and $\mathcal{K}_i \wedge \mathcal{K}_j = \mathbf{0}$ when $i \neq j$. The notation $N \leq_t M$ and $N \leq_e M$ denote type and essential submodules of M.

1. 2**-decomposable modules and existence of type decompositions.** The two decompositions mentioned in the beginning of the introduction are both type decompositions. Every type decomposition $M = \oplus_{i\in I} M_i$ gives a maximal set $\{c(M)\} \cup \{d(M_i) : i \in I\}$ of pairwise orthogonal types such that $M_i \in d(M_i)$, for $i \in I$, and $(0) \in c(M)$. Conversely, any decomposition $M = \oplus_{i\in I} M_i$ where $\{\mathcal{K}_i : i \in I\}$ is a maximal set of pairwise orthogonal types and $M_i \in \mathcal{K}_i$, $i \in I$, is a type decomposition. So, the study of type decompositions of modules could be started with the following definition.

Definition 1.1. A module M is called *n-decomposable* if, for any maximal set $\{\mathcal{K}_i : i = 1, \cdots, n\}$ of pairwise orthogonal types, M has a decomposition $M = \oplus_{i=1}^n M_i$ where $M_i \in \mathcal{K}_i$. The module M is called *finitely decomposable* if M is n-decomposable for every positive integer n. If M has a decomposition $M = \oplus_{i \in I} M_i$ with $M_i \in \mathcal{K}_i$ for every countable maximal set, respectively every maximal set, $\{\mathcal{K}_i : i \in I\}$ of pairwise orthogonal types, then we say M is *countably decomposable*, respectively *fully decomposable*.

Theorem 1.2. *The following are equivalent for a module M:*

(1) *M is 2-decomposable.*

(2) *Every submodule of M has a type complement Q in M such that Q is a direct summand of M.*

(3) *Every type submodule of M has a (type) complement Q in M such that Q is a direct summand of M.*

Proof. (1) $\Rightarrow$ (2). Let N be a submodule of M and let $\mathcal{K} = d(N)$. By (1), M has a decomposition $M = M_1 \oplus M_2$ where $M_1 \in \mathcal{K}$ and $M_2 \in c(\mathcal{K})$. It follows that M_2 is a type complement of N in M.

(2) $\Rightarrow$ (3). It is clear because the complements of a type submodule N in M are precisely the type complements of N in M.

(3) $\Rightarrow$ (1). Let $\mathcal{K}_1, \mathcal{K}_2$ be types such that $\mathcal{K}_1 \vee \mathcal{K}_2 = \mathbf{1}$ and $\mathcal{K}_1 \wedge \mathcal{K}_2 = \mathbf{0}$. Let N be a type submodule of M of type $\mathcal{K}_1$. By hypothesis $M = P \oplus Q$ where $P \leq M$ and $N \oplus Q \leq_e M$. Thus, N is essentially embeddable in $M/Q \cong P$. So, $P \in \mathcal{K}_1$. Since N is a type submodule, $N \cap Q = 0$ implies $N \perp Q$. It follows that $Q \in c(\mathcal{K}_1) = \mathcal{K}_2$ since N is a type submodule of type $\mathcal{K}_1$. □

A module is called TS if every type submodule is a direct summand [**12**]. A module M is said to satisfy (C_{11}) if every submodule of M has a complement Q in M such that Q is a direct summand of M [**10**]. Clearly, TS-modules and modules satisfying (C_{11}) all are 2-

decomposable. Later we will give examples of 2-decomposable modules which are neither TS nor (C_{11}).

Theorem 1.3. *Any direct sum of* 2*-decomposable modules is* 2*-decomposable. In particular, any direct sum of atomic modules is* 2*-decomposable.*

Proof. Let $M = \oplus_{i\in I} M_i$ where each M_i is 2-decomposable. Let $\mathcal{K}$ be a natural class. Then, for each i, $M_i = X_i \oplus Y_i$ where $X_i \in \mathcal{K}$ and $Y_i \in c(\mathcal{K})$. Let $X = \oplus_{i\in I} X_i$ and $Y = \oplus_{i\in I} Y_i$. Then $M = X \oplus Y$ and $X \in \mathcal{K}$ and $Y \in c(\mathcal{K})$. Thus, M is 2-decomposable. For the last statement, note that every atomic module is 2-decomposable. $\square$

Let $Z(M) \subseteq Z_2(M)$ be the singular and second singular submodules of M and let $\widehat{M}$ denote the injective hull of M. A submodule N of M is *fully invariant* if $f(N) \subseteq N$ for all $f \in \mathrm{End}\,(M)$.

Theorem 1.4. *The following are equivalent for a module M:*

(1) *M is* 2*-decomposable.*

(2) *$M = Z_2(M) \oplus K$ where $Z_2(M)$ is* 2*-decomposable and K is nonsingular TS.*

(3) *For some fully invariant type submodule F of M, $M = F \oplus K$ where F and K both are* 2*-decomposable.*

(4) *For every fully invariant type submodule F of M, $M = F \oplus K$ where F and K both are* 2*-decomposable.*

Proof. (1) $\Rightarrow$ (4). Suppose M is 2-decomposable and F is a fully invariant type submodule of M. Then $M = X \oplus Y$ where $X \in d(F)$ and $Y \in c(F)$. Since F is fully invariant, $F = (F \cap X) \oplus (F \cap Y)$. Since $Y \perp F$, $F \cap Y = 0$, and so $F \subseteq X$. It follows that $F = X$ since F is a type submodule of type $d(F)$. Thus, we have $M = F \oplus K$ with $K = Y$ orthogonal to F.

To see F is 2-decomposable, let $\mathcal{F}$ be a natural class. We let A be a type submodule of F of type $\mathcal{F}$. Since M is 2-decomposable, $M = M_1 \oplus M_2$ where $M_1 \in d(A) \subseteq \mathcal{F}$ and $M_2 \in c(A)$. Since $A \perp M_2$,

$M_2 \in c(\mathcal{F})$. Since $F \leq M$ is fully invariant, $F = (F \cap M_1) \oplus (F \cap M_2)$, where $F \cap M_1 \in \mathcal{F}$ and $F \cap M_2 \in c(\mathcal{F})$. So, F is 2-decomposable.

To prove that K is 2-decomposable, let $\mathcal{F}$ be a natural class. We let B be a type submodule of K of type $\mathcal{F}$. Since M is 2-decomposable, $M = N_1 \oplus N_2$ where $N_1 \in d(B)$ and $N_2 \in c(B)$. Since $F \leq M$ is fully invariant, we have $F = (F \cap N_1) \oplus (F \cap N_2)$. We claim $F \cap N_1 = 0$. If not, there exist $0 \neq C_1 \leq F \cap N_1$ and $0 \neq C_2 \leq B \leq K$ with $C_1 \cong C_2$, contradicting that $F \perp K$. Hence $F = F \cap N_2$. Thus, $N_2 = F \oplus (N_2 \cap K)$ and $M = N_1 \oplus F \oplus (N_2 \cap K)$. Let π be the projection of M onto K along F. Then $N_1 \oplus F = \pi(N_1) \oplus F$. It follows that $M = \pi(N_1) \oplus F \oplus (N_2 \cap K)$ and so $K = \pi(N_1) \oplus (N_2 \cap K)$. Since $\pi(N_1) \cong K/(N_2 \cap K) \cong (K + N_2)/N_2 \hookrightarrow N_1$, we see that $\pi(N_1) \in \mathcal{F}$. Since $N_2 \in c(B)$, $N_2 \cap K \in c(\mathcal{F})$. Thus, K is 2-decomposable.

(4) $\Rightarrow$ (2). Since $Z_2(M)$ is a fully invariant type submodule of M, by (4), $M = Z_2(M) \oplus K$ where both $Z_2(M)$ and K are 2-decomposable. Since K is nonsingular, K is TS, by Example 2.2 (1).

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). By Theorem 1.3. $\square$

Part 1 of the next corollary follows from Theorem 1.4 (2) and [**12**, Corollaries 15.1–15.3]. Two modules M_1 and M_2 are *parallel*, written $M_1 \| M_2$, if every nonzero submodule of M_1 is not orthogonal to M_2 and every nonzero submodule of M_2 is not orthogonal to M_1. For example, as $\mathbf{Z}$-modules, $\mathbf{Z}_2 \oplus \mathbf{Z}_4$, $\mathbf{Z}_2$ and $\mathbf{Z}_4$ all are parallel. See [**6**, Definition 1.30] for the superspectivity of modules.

Corollary 1.5. (1) *A module M is 2-decomposable if and only if $M = A \oplus B \oplus C \oplus D \oplus E$ with a Goldie torsion 2-decomposable module A, a nonsingular TS-module B having essential socle, a nonsingular socle-free TS-module C having an essential submodule which is a direct sum of uniform submodules, a nonsingular TS-module D containing no uniform submodules and having an essential submodule which is a direct sum of atomic submodules, and a nonsingular TS-module E containing no atomic submodules.*

(2) *The decomposition of M above is unique up to superspectivity.*

Proof. We only need to prove (2). Suppose $M = A' \oplus B' \oplus C' \oplus D' \oplus E'$ is another decomposition as described in (1). Then $A = Z_2(M) = A'$. Let $M = B \oplus X$. Then $Z_2(M) = Z_2(X) \subseteq X$. Since $C' \oplus D' \oplus E'$ is nonsingular and orthogonal to B, it follows from [**11**, Lemma 3.1] applied to the projection $\pi : M = B \oplus X \to B$ with $\pi(C' \oplus D' \oplus E') = 0$ that $C' \oplus D' \oplus E' \subseteq X$. This gives that $M = B' + X$. Since $X \perp B$ and $B' \| B$, we have $X \perp B'$. It follows that $M = B' \oplus X$. Similarly we see that $M = B' \oplus Y$ implies $M = B \oplus Y$. So, B' is superspective to B. The same arguments show that C', D' and E' are superspective to C, D and E respectively. $\square$

A 2-decomposable module M that does not satisfy either of (C_{11}) and TS can be given as follows.

Example 1.6. Let $R = \mathbf{Z} \propto (\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ be the trivial extension of $\mathbf{Z}$ and the $\mathbf{Z}$-module $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, i.e., $R = \left\{\begin{pmatrix} n & x \\ 0 & n \end{pmatrix} : n \in \mathbf{Z}, x \in \mathbf{Z}_2 \oplus \mathbf{Z}_2\right\}$ be the subring of the formal triangular matrix ring $\begin{pmatrix} \mathbf{Z} & \mathbf{Z}_2 \oplus \mathbf{Z}_2 \\ 0 & \mathbf{Z} \end{pmatrix}$. Let $I_0 = \left\{\begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix} : n \in \mathbf{Z}\right\}$, $I = \left\{\begin{pmatrix} 4n & 0 \\ 0 & 4n \end{pmatrix} : n \in \mathbf{Z}\right\}$ and $J = \left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \mathbf{Z}_2 \oplus \mathbf{Z}_2\right\}$. Set $M = M_1 \oplus M_2$ where $M_1 = R/I$ and $M_2 = R/J$. Note that U is an essential (right) ideal of R if and only if $U = V \oplus J$ for some $0 \neq V \subseteq I_0$. It follows that $J = Z(R) = Z_2(R)$. Thus, M_2 is nonsingular uniform. $M_1 = R/I$ contains an essential submodule $(I_0 + J)/I \cong (I_0/I) \oplus J$. Note that I_0/I is embeddable in J. This shows that $(I_0 + J)/I$ is singular and atomic. It follows that M_1 is Goldie torsion and atomic. Therefore, by Theorem 1.2, M is 2-decomposable. To prove that M is not TS, we only need to show that M_1 is not M_2-injective because of [**12**, Proposition 14]. Consider $f : (I + J)/J \to R/I$ given by $\begin{pmatrix} 4n & 0 \\ 0 & 4n \end{pmatrix} + J \mapsto \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix} + I$. Then f is a well-defined R-homomorphism. Suppose f extends to a homomorphism $g : R/J \to R/I$. Write $g(1_R + J) = \begin{pmatrix} m & x \\ 0 & m \end{pmatrix} + I$. Then $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + I = f(41_R + J) = g(41_R + J) = g(1_R + J)41_R = \begin{pmatrix} m & x \\ 0 & m \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} + I = 0 + I = \bar{0}$. This is a contradiction. So, M_1 is not M_2-injective. To see M does not satisfy (C_{11}), let $N = [(K + I)/I] \oplus (R/J)$ where $K = \left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in (0) \oplus \mathbf{Z}_2\right\}$. Suppose M satisfies (C_{11}). Then there exists a complement P of N in M such that P is a direct summand of M. Clearly $P \neq 0$. Since $P \cap M_2 = 0$, we have $P \hookrightarrow M_1$. This shows that P is Goldie torsion. So, $P \subseteq M_1$ and thus P is a direct summand of M_1. But it is easy to see that M_1 is indecomposable, so it must be $P = M_1$. It follows that $(K+I)/I \subseteq N \cap P$, a contradiction. So, M does not satisfy (C_{11}).

Before giving more decomposition results of 2-decomposable modules, we point out here that the proof of the uniqueness of [**12**, Proposition 11] has a gap and a proper statement of [**12**, Proposition 11] should be Proposition 1.7. We recall some definitions from [**6**, Definitions 1.24 and 1.32]. A module D is *directly finite* if D is not isomorphic to a proper direct summand of itself. A module P is *purely infinite* if $P \cong P \oplus P$. A module M is said to satisfy (T_3) if, whenever M_1 and M_2 are type submodules as well as direct summands of M such that $M_1 \oplus M_2$ is essential in M, then $M = M_1 \oplus M_2$ [**12**, p. 86].

Proposition 1.7. *Let M be a TS-module with (T_3). Then M has a decomposition, unique up to superspectivity, $M = D \oplus P$, where D and $\widehat{D}$ are directly finite, $\widehat{P}$ is purely infinite, and $D \perp P$.*

Proof. See the next proposition. □

The next two propositions extend the above result and [**12**, Proposition 13] from TS-modules to 2-decomposable modules.

Proposition 1.8. *Every 2-decomposable module M has a decomposition $M = D \oplus P$, where D and $\widehat{D}$ are directly finite, $\widehat{P}$ is purely infinite, and $D \perp P$. If in addition M satisfies (T_3), then the decomposition is unique up to superspectivity.*

Proof. Let $\mathcal{F} = \{X : X^{(\aleph_0)} \hookrightarrow M\}$ and $\mathcal{K} = c(\mathcal{F})$. Then $M = D \oplus P$ where $D \in \mathcal{K}$ and $P \in c(\mathcal{K}) = c(c(\mathcal{F})) = d(\mathcal{F})$. Then D and $\widehat{D}$ are directly finite by [**6**, Lemma 1.26], $D \perp P$ and $\widehat{P} \in d(\mathcal{F})$. By [**6**, Theorem 1.35], $\widehat{P} = E_1 \oplus E_2$ where E_1 is directly finite, E_2 is purely infinite and $E_1 \perp E_2$. We prove $E_1 = 0$ and hence $\widehat{P} = E_2$ is purely infinite. If $E_1 \neq 0$ then, since $E_1 \in d(\mathcal{F})$, there exists $0 \neq X \subseteq E_1$ such that $X \in \mathcal{F}$. Thus, $X^{(\aleph_0)} \cong N \leq M$ for some N. Since $N \in d(\mathcal{F})$, $D \perp N$. It follows that N is embeddable in P. Since $E_1 \perp E_2$, $N \perp E_2$. This implies that $X^{(\aleph_0)} \cong N$ is embeddable in E_1. Thus, E_1 is not directly finite by [**6**, Proposition 1.27], a contradiction. So, $E_1 = 0$.

Let $M = D_1 \oplus P_1$ where D_1 and $\widehat{D_1}$ are directly finite, $\widehat{P_1}$ is purely finite and $D_1 \perp P_1$. Then we have $\widehat{M} = \widehat{D} \oplus \widehat{P} = \widehat{D_1} \oplus \widehat{P_1}$. By the uniqueness of [**6**, Theorem 1.35], $\widehat{D} \cong \widehat{D_1}$ and $\widehat{P} \cong \widehat{P_1}$. Thus, $D_1 \in c(\mathcal{F})$ and $P_1 \in d(\mathcal{F})$. Now the uniqueness follows from [**12**, Lemma 6]. $\square$

A module M_1 is *square free* if $X \oplus X \not\hookrightarrow M_1$ for any nonzero module X, while a module M_2 is *square full* if, for any $0 \neq N \leq M_2$, we have $X \oplus X \hookrightarrow M_2$ for some $0 \neq X \leq N$, see [**6**, Definitions 2.34 and 2.35].

Proposition 1.9. *The module M is square free if and only if every complement submodule of M is a type submodule.*

Proof. Suppose that M is not square free. Then there exist submodules A and B of M such that $0 \neq A \cong B$ and $A \cap B = 0$. For any complement closure A^c of A in M, $A^c \cap B = 0$ and B embeds in A^c. So, A^c is not a type submodule of M.

Suppose that there exists a complement submodule N of M such that N is not a type submodule. Then there exists a proper extension P of N in M such that $N \| P$. Since N is a complement submodule of M, $N \cap X = 0$ for some nonzero submodule X of P. Since $N \| P$, X and N have nonzero isomorphic submodules, and thus M is not square free. $\square$

Proposition 1.10. *Every 2-decomposable module M has a decomposition $M = M_1 \oplus M_2$, where M_1 is square free, M_2 is square full and $M_1 \perp M_2$. If in addition M satisfies (T_3), the decomposition is unique up to superspectivity.*

Proof. The proof of [**12**, Proposition 13] works. $\square$

There exist 2-decomposable modules satisfying (T_3) but not TS.

Example 1.11. Let M be the $\mathbf{Z}$-module $\mathbf{Z}_p \oplus \mathbf{Q}$ where p is a prime number. Then, by [**10**, Example 4.2], any submodule isomorphic to a direct summand of M is a direct summand of M. Thus, for direct summands M_1, M_2 of M with $M_1 \cap M_2 = 0$, $M_1 \oplus M_2$ is a direct summand of M, see [**6**, Proposition 2.2]. So, M satisfies (T_3). But $Z_2(M) = \mathbf{Z}_p$ is not $\mathbf{Q}$-injective. Thus, M is not TS by [**12**, Proposition 14], but M is 2-decomposable by Theorem 1.3.

A module M has *finite type dimension* n, notation: $\mathrm{t.dim}(M) = n$, if there exists an essential type direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_n \leq_e M$ of atomic submodules $A_i \subseteq M$. In this case such an n is uniquely determined, see [**11**, Definition 1.1] and [**12**, p. 91]. From now on, for a module M, the annihilator of any $m \in M$ is denoted by $m^{\perp} = \{r \in R : mr = 0\}$.

Theorem 1.12. *The following hold for a module M.*

(1) *If every type direct summand of M is 2-decomposable, then M is finitely decomposable. In particular, every TS-module is finitely decomposable.*

(2) *Suppose that for any chain $m_1^{\perp} \subseteq m_2^{\perp} \subseteq \cdots$ of right ideals, where $m_i \in M$, $t.\mathrm{dim}\,[\oplus_{i=1}^{\infty} R/m_i^{\perp}] < \infty$.*

(a) *If every type direct summand of M is 2-decomposable, then M is countably decomposable.*

(b) *M is fully decomposable if and only if M is a direct sum of atomic modules.*

(c) *If M is TS, then M is fully decomposable.*

Proof. Let $I = \{1 < 2 < \ldots < n < \ldots\} \subseteq \{i : 1 \leq i < \omega\}$. By induction assume that for some $0 \leq j < \omega$ we have that $M = (\oplus_{i=1}^{j} M_i) \oplus N_j$, where M_i is a type submodule of M of type $\mathcal{K}_i$ and $(\oplus_{i=1}^{j} M_i) \perp N_j$. Then N_j is a type submodule of M. By hypothesis, N_j is 2-decomposable. Thus, $N_j = M_{j+1} \oplus N_{j+1}$, where $M_{j+1} \in \mathcal{K}_{j+1}$, $N_{j+1} \in c(\mathcal{K}_{j+1})$. Then, by [**12**, Lemma 1], M_{j+1} is a type submodule of M of type $\mathcal{K}_{j+1}$. Thus again $M = (\oplus_{i=1}^{j+1} M_i) \oplus N_{j+1}$ with $(\oplus_{i=1}^{j+1} M_i) \perp N_{j+1}$.

(1) If the index set is the finite set $\{1, \ldots, n\}$, take $j = n$. Since $\vee_{i \leq n} \mathcal{K}_i = \mathbf{1}$, $N_{n+1} = 0$, and so $M = \oplus_{i=1}^{n} M_i$. The last statement follows because every type direct summand of a TS-module is TS [**12**, Lemma 4].

(2) We first note that, by the proof of [**12**, Proposition 18], it follows from our assumption that every local type summand of M, i.e., a direct sum $\oplus_{i \in I} X_i$ in M with all X_i type submodules such that $\oplus_{i \in F} X_i$ is a summand of M for any finite subset F of I, is a type submodule.

(a) Let $I = \{i : 1 \leq i < \omega\}$. Then by induction, $\oplus_{i<\omega} M_i \oplus C \leq_e M$ for some $C \leq M$, where M_i is a type submodule of M of type $\mathcal{K}_i$, and $(\oplus_{i<\omega} M_i) \perp C$. This shows that $\mathcal{K}_i \wedge d(C) = \mathbf{0}$ for all $i < \omega$. Since any complete Boolean lattice satisfies a limited infinite distributive law, $d(C) = d(C) \wedge (\vee_{i<\omega} \mathcal{K}_i) = \vee_{i<\omega} (d(C) \wedge \mathcal{K}_i) = \mathbf{0}$, from which we conclude that also $C = 0$. Next, by the note above, the local type summand $\oplus_{i<\omega} M_i \leq_e M$ is a type submodule, and hence in particular, a complement submodule. Hence $M = \oplus_{i<\omega} M_i$.

(b) Suppose that $M = \oplus_{t \in T} X_t$ where all X_t are atomic modules. Let $\{\mathcal{K}_i : i \in I\}$ be a maximal set of pairwise orthogonal types. Note that each X_t is in some unique $\mathcal{K}_i$. For each $i \in I$, let $N_i = \oplus\{X_t : t \in T$ and $X_t \in \mathcal{K}_i\}$ or $N_i = 0$ if no X_t is in $\mathcal{K}_i$. Then $N_i \in \mathcal{K}_i$ and $M = \oplus_{i \in I} N_i$. So, M is fully decomposable.

Suppose that M is fully decomposable. The hypothesis implies that every nonzero submodule of M contains an atomic submodule. So, there exists a family $\{X_i : i \in I\}$ of atomic submodules of M such that $X_i \perp X_j$ if $i \neq j \in I$ and $X = \oplus_{i \in I} X_i \leq_e M$. Then $\{c(X)\} \cup \{d(X_i) : i \in I\}$ is a maximal set of pairwise orthogonal types, and thus $M = P \oplus (\oplus_{i \in I} M_i)$ where $P \in c(X)$ and $M_i \in d(X_i)$. It must be that $P = 0$ and all M_i are atomic.

(c) Note that every local type summand of M is a direct summand. If M is TS, then, by [**12**, Proposition 16], $M = \oplus_{i \in I} M_i$ where each M_i is not a type direct sum of two nonzero submodules. Each M_i is still TS, so it must be atomic. $\square$

A ring R is said to satisfy (right) t-acc if, for any ascending chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ of right ideals, $\mathrm{t.dim}(\oplus_i R/I_i) < \infty$. The rings satisfying t-acc were characterized in [**12**]. The next theorem gives some new characterizations of these rings.

Theorem 1.13. *The following are equivalent for a ring R*:

(1) *R satisfies t-*acc.

(2) *Every module whose type direct summands are* 2*-decomposable is countably decomposable.*

(3) *Every TS-module is fully* (*or countably*) *decomposable.*

(4) *Every injective module is fully* (*or countably*) *decomposable.*

Proof. (1) $\Rightarrow$ (2) + (3) + (4). By Theorem 1.12.

(3) $\Rightarrow$ (4). Obvious.

(2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1). Suppose that every injective module is countably decomposable. By [**12**, Theorem 22], it suffices to show that, for any set $\{E_i : i \in I\}$ of pairwise orthogonal injective modules, $\oplus_{i\in I} E_i$ is injective. By [**6**, Theorem 1.7], we can assume that I is a countable set. Let $E = E(\oplus_{i\in I} E_i)$. Then $\{c(E)\} \cup \{d(E_i) : i \in I\}$ is a countable maximal set of pairwise orthogonal types. Since E is countably decomposable, $E = A \oplus (\oplus_{i\in I} A_i)$ where $A \in c(E)$ and $A_i \in d(E_i)$. It must be that $A = 0$. So $E = A_i \oplus (\oplus_{j\neq i} A_j)$ and $\oplus_{j\neq i} A_j \in c(E_i) = c(d(E_i))$. On the other hand, $E = E_i \oplus B_i$ for some B_i. Because $E_i \perp E_j$ whenever $i \neq j$ in I, $E_i \perp B_i$ and so $B_i \in c(E_i)$. Thus, we have $E = E_i \oplus B_i = A_i \oplus (\oplus_{j\neq i} A_j)$ where $E_i, A_i \in d(E_i)$ and $B_i, \oplus_{j\neq i} A_j \in c(E_i)$. By [**12**, Lemma 6], E_i is perspective to A_i. Thus, $E_i \cong A_i$ for all $i \in I$. It follows that $\oplus_{i\in I} E_i \cong \oplus_{i\in I} A_i = E$ is injective. □

By Theorem 1.13, for any ring R without t-acc, there exists a finitely decomposable R-module which is not countably decomposable. We do not know if every 2-decomposable module is always finitely decomposable and if every countably decomposable module is always fully decomposable. But if $R = \mathbf{Z}$ or a commutative Dedekind domain, every 2-decomposable module is always fully decomposable as the next theorem shows.

Theorem 1.14. *For $R = \mathbf{Z}$, the following are equivalent for an abelian group M:*

(1) *M is 2-decomposable.*

(2) *M is fully decomposable.*

(3) *M is a direct sum of a torsion abelian group and a torsion free abelian group.*

(4) *Every direct summand of M is 2-decomposable.*

Proof. (1) $\Rightarrow$ (3). By Theorem 1.4.

(3) $\Rightarrow$ (2). Note that every torsion free abelian group is atomic and every torsion abelian group is a direct sum of atomic modules. Then, if (3) holds, M is a direct sum of atomic modules. By 2(b) of Theorem 1.12, M is fully decomposable.

(2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1). Obvious.

(3) $\Rightarrow$ (4). Write $M = A \oplus B$ where A is torsion and B is torsion free. Let N be any direct summand of M. Write $M = N \oplus N'$. Since $Z_2(M) \leq M$ is fully invariant, $Z_2(M) = [Z_2(M) \cap N] \oplus [Z_2(M) \cap N'] = Z_2(N) \oplus Z_2(N')$. Thus $N = N \cap [Z_2(N) \oplus Z_2(N') \oplus K] = Z_2(N) \oplus \{N \cap [Z_2(N') \oplus K]\}$, where the last summand is torsion free. Thus, N satisfies (3). By the equivalence (1) $\Leftrightarrow$ (3), B is 2-decomposable. $\square$

The next proposition gives a partial answer to the question of when a direct sum of atomic modules is TS. For any $M = \oplus_{i \in I} M_i$ and any $j \in I$, set $M(I - j) = \oplus\{M_i : i \in I, i \neq j\}$.

Proposition 1.5. *Let $0 \neq N \leq_t M = \oplus_{i \in I} M_i$ be any type submodule where all M_i, $i \in I$ are atomic and, for all $j \in I$, $\mathrm{Hom}_R(\widehat{M(I-j)}, \widehat{M_j}) = 0$. Then $N = \oplus\{M_i : i \in I, M_i \subseteq N\}$.*

Proof. Define $I(0) = \{i \in I : M_i \subseteq N\} \subseteq I(1) = \{i \in I \mid \pi_i N \neq 0\}$. Note that $\oplus_{i \in I(0)} M_i \subseteq N \subseteq \oplus_{i \in I(1)} M_i$. It suffices to prove that for any $j \in I(1)$, $j \in I(0)$ so that $I(0) = I(1)$. Choose the notation so that $j = 1$ and let $K = M(I - 1)$. Then $M = M_1 \oplus K$ and $\widehat{M} = \widehat{M_1} \oplus \widehat{K}$. Let $\pi : \widehat{M} \longrightarrow \widehat{M_1}$ and $\rho : \widehat{M} \longrightarrow \widehat{K}$ be the corresponding projections.

Take $C \leq N$ with $(N \cap M_1) \oplus (N \cap K) \oplus C \leq_e N$. Note that $K \cap C = K \cap N \cap C = 0$ and $M_1 \cap C = M_1 \cap N \cap C = 0$, so $\ker(\pi) \cap C = 0$ and $\ker(p) \cap C = 0$. Thus $C \cong \pi C \cong pC$. It follows that $C \in d(M_1)$ and $C \in d(K)$. But since $M_1 \perp K$, and $C \in d(M_1) \cap d(K) = \{(0)\}$, $C = 0$. Thus $(N \cap M_1) \oplus (N \cap K) \leq_e N$.

Case 1. $N \cap M_1 = 0$. Thus also $N \cap \widehat{M_1} = 0$. From $(N \cap M_1) \oplus (N \cap K) \leq_e N$ we get that $N \cap K \leq_e N$. For $n, n' \in N$, if $pn = pn'$, then $p(n - n') = 0$, $n - n' \in N \cap \widehat{M_1} = 0$, or $n = n'$. Consequently the restriction $p|_N : N \to K$, $n \mapsto pn$ is monic. Let g be the monic inverse map $g : p(N) \to N$, where $g(pn) = n$. Define $\varphi = \pi g$. Thus there exists a map $\widehat{\varphi} : \widehat{K} \to \widehat{M_1}$ that extends φ. If $p(N) = 0$, then $N \subseteq \widehat{M_1}$, and hence $N \subseteq N \cap \widehat{M_1} = 0$. So let $p(N) \neq 0$, in which case $gp(N) = N$. If $\pi|_N : N \to \widehat{M_1}$ is not zero, then $\varphi(pN) = \pi(gp(N)) = \pi N \neq 0$. Hence $0 \neq \widehat{\varphi} \in \mathrm{Hom}_R(\widehat{K}, \widehat{M_1}) = 0$ contradicts the hypothesis. Therefore $\pi(N) = 0$, which contradicts that $1 \in I(1)$.

Case 2. $N \cap M_1 \neq 0$. If $(N \cap M_1) \oplus D \leq M_1$ with D nonzero, then by the atomicity of M_1, $N \cap M_1$ and D have a common nonzero isomorphic submodule. But since $N + D = N \oplus D$, it follows that $N \perp D$, and this is a contradiction. So, $N \cap M_1 \leq_e M_1$. Form $N \subseteq \widehat{N} \subseteq \widehat{M}$, and since $N \cap M_1 \subset N$, choose some injective hull of $N \cap M_1$ inside $\widehat{N}$, i.e., $\widehat{M_1} \cong \widehat{N \cap M_1} \subseteq \widehat{N}$. Since $\widehat{M_1} < \widehat{M}$ is fully invariant by the hypothesis, $\widehat{M}$ contains a unique injective hull of M_1, i.e., $\widehat{M_1} = \widehat{N \cap M_1} \subseteq \widehat{N}$. Since $N < M$ is a complement, and since $N \leq_e \widehat{N} \cap M$, necessarily $N = \widehat{N} \cap M$. Since also $\widehat{M_1} \cap M = M_1$, we get that

$$\widehat{M_1} \subseteq \widehat{N} \Longrightarrow M \cap \widehat{M_1} \subseteq M \cap \widehat{N} \Longleftrightarrow M_1 \subseteq N.$$

Thus $1 \in I(0)$. Consequently $I(0) = I(1)$. □

We conclude this section by proving a type analogue of a result of Müller and Rizvi in [**7**].

Theorem 1.16. *If every type direct summand of M is 2-decomposable, then M has a decomposition $M = M_1 \oplus M_2$, where M_1 is essential over a type direct sum $\oplus_{i \in I} N_i$ of atomic type summands of M and*

M_2 contains no atomic submodules. Moreover, if in addition M satisfies the condition that, for any two type direct summands A and B with $A \cap B = 0$, $A \oplus B$ is a direct summand of M, then the decomposition is unique in the sense that if M has another decomposition $M = M_1' \oplus M_2'$, where M_1' is essential over a type direct sum $\oplus_{j\in J} N_j'$ of atomic type summands of M and M_2' contains no atomic submodules, then $M_1 \cong M_1'$, $M_2 \cong M_2'$ and there is a bijection $\theta : I \to J$ such that $N_i \cong N'_{\theta(i)}$ for all $i \in I$.

Proof. The existence: Let $\mathcal{K}$ be the class of the modules containing no atomic submodules. Then $\mathcal{K}$ is a natural class and $c(\mathcal{K})$ is the class of the modules N such that every nonzero submodule of N contains an atomic submodule. Thus, there exists $M_1 \in c(\mathcal{K})$ and $M_2 \in \mathcal{K}$ such that $M = M_1 \oplus M_2$. It follows that M_1 contains an essential submodule $X = \oplus_{i\in I} X_i$ where each X_i is atomic. Without loss of generality, we may assume that $X_i \perp X_j$ for all $i \neq j$ in I. By the hypothesis, M_1 is 2-decomposable. For each $k \in I$, let $Z_k = \oplus\{X_i : i \in I, i \neq k\}$. Then, by Theorem 1.2, there exists a type complement N_k of Z_k in M_1 such that N_k is a summand of M_1 (and hence of M). Since $X_k \oplus Z_k \leq_e M_1$, it can easily be proved that $N_k \| X_k$. So, N_k is an atomic type summand of M and $\oplus_{i\in I} N_i$ is a type direct sum. To see that $\oplus_{i\in I} N_i$ is essential in M_1, let $(\oplus_{i\in I} N_i) \cap Y = 0$ where Y is a submodule of M_1. Since each N_i is a type submodule, $Y \perp N_i$ for all $i \in I$. Thus, $Y \perp X_i$ for all $i \in I$. It follows that $Y \perp (\oplus_{i\in I} X_i)$ and so $Y \cap (\oplus_{i\in I} X_i) = 0$. Since $\oplus_{i\in I} X_i$ is essential in M_1, we have $Y = 0$.

The uniqueness: Suppose M_1', M_2' and $\oplus_{j\in J} N_j'$ are as assumed above. Since M is a 2-decomposable module satisfying (T_3), by [**12**, Lemma 6], $M_1 \cong M_1'$ and $M_2 \cong M_2'$. Note that both type direct sums $\oplus_{i\in I} N_i$ and $\oplus_{j\in J} N_j'$ are essential in M_1 and all N_i, N_j' are atomic. It follows that, for each $i \in I$, there exists a unique $j_i \in J$ such that $N_i \| N'_{j_i}$, and, for each $j \in J$, there exists a unique $i_j \in I$ such that $N_j' \| N_{i_j}$. Then $\theta : I \longrightarrow J$ defined by $\theta(i) = j_i$ is a bijection. Since N_i and $N'_{\theta(i)}$ both are type summands of M_1, write $M_1 = N_i \oplus A = N'_{\theta(i)} \oplus B$ with $N_i \perp A$ and $N'_{\theta(i)} \perp B$. Then N_i and $N'_{\theta(i)}$ are in $d(N_i)$ and A, B are in $c(N_i)$. By [**12**, Lemma 6], $N_i \cong N'_{\theta(i)}$. $\square$

2. UTC-modules and uniqueness of type decompositions. If $M = \oplus_{i \in I} M_i$ where each $M_i \in \mathcal{K}_i$ and $\{\mathcal{K}_i : i \in I\}$ is a maximal set of pairwise orthogonal types, then each M_i is a type submodule of M of type $\mathcal{K}_i$. Thus, an obvious sufficient condition for this decomposition to be unique is that M has a unique type submodule of type $\mathcal{K}$ for every natural class $\mathcal{K}$. This observation leads us to introduce and study UTC-modules. A partial homomorphism from M to N is a homomorphism from a submodule of M to N.

Theorem 2.1. *The following are equivalent for a module M:*

(1) *M has a unique type submodule of type $\mathcal{K}$ for every natural class $\mathcal{K}$.*

(2) *For every natural class $\mathcal{K}$, $\Sigma\{X \subseteq M : X \in \mathcal{K}\} \in \mathcal{K}$, i.e., M has a largest submodule in $\mathcal{K}$.*

(3) *Every submodule has a unique type closure in M.*

(4) *For any nonzero submodule N of M, if $C_1 \neq C_2$ are two closures of N in M then there exists $0 \neq X \subseteq C_1 + C_2$ such that $C_1 \cap X = 0$ and $X \hookrightarrow N$.*

(5) *There does not exist an R-module X and a proper essential submodule Y of X such that $X \perp (X/Y)$ and $X \oplus (X/Y)$ embeds in M.*

(6) *Every partial endomorphism $f : A \longrightarrow M$ with $f(A) \perp A$, $ker(f)$ is a complement submodule of A.*

Proof. (1) $\Leftrightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Let N be a submodule of M and $\mathcal{K} = d(N)$. Then $\mathcal{K}$ is a natural class. For any type closure N^{tc} of N in M, we have $N^{tc} \in \mathcal{K}$. Thus $N^{tc} \subseteq P$ where $P = \Sigma\{X \subseteq M : X \in \mathcal{K}\} \in \mathcal{K}$. By the definition of N^{tc}, $P = N^{tc}$. Therefore, P is the only type closure of N in M.

(3) $\Rightarrow$ (4). Suppose that a nonzero submodule N of M has closures $C_1 \neq C_2$ in M. Let $\mathcal{K} = d(N)$ and $P = C_1 + C_2$. Then $N \leq_e C_1$ and $N \leq_e C_2$. It is easy to see that C_1^{tc} and C_2^{tc} are type closures of N in M. By (3), $C_1^{tc} = C_2^{tc}$. So, $P \subseteq C_1^{tc} \in \mathcal{K}$. Since $C_1 \neq C_2$, $C_1 \cap A = 0$ for some $0 \neq A \subseteq P$. Then $A \in \mathcal{K}$. It follows that $X \hookrightarrow N$ for some $0 \neq X \subseteq A$. Thus, (4) is proved.

(4) $\Rightarrow$ (2). Let $\mathcal{K}$ be a natural class. To show (2), it suffices to show that for any submodules X and Y of M, if X and Y are in $\mathcal{K}$ then so is $X+Y$. By Zorn's lemma, there exists a submodule P maximal with respect to $X \subseteq P \in \mathcal{K}$ and a submodule Q maximal with respect to $Y \subseteq Q \in \mathcal{K}$. Then P and Q are complement submodules of M, $P \cap Q \leq_e P$ and $P \cap Q \leq_e Q$. So, P and Q both are closures of $P \cap Q$ in M. If $P \neq Q$, by (4), there exists $0 \neq X \subseteq P+Q$ such that $P \cap X = 0$ and $X \hookrightarrow P \cap Q$. Then $X \in \mathcal{K}$ and $P \subset P \oplus X \in \mathcal{K}$, a contradiction. So $P = Q$ and thus $X + Y \subseteq P \in \mathcal{K}$.

(5) $\Rightarrow$ (1). Suppose (1) does not hold. Then there exist type submodules $T_1 \neq T_2$ of M of type $\mathcal{K}$ for a natural class $\mathcal{K}$. It follows that $T_1 \cap T_2 \neq 0$, $T_1 \cap T_2 \leq_e T_i$ for $i = 1, 2$, and $T_1 \cap T_2$ is not essential in $T_1 + T_2$. Thus, there exists $0 \neq A \subseteq T_1 + T_2$ such that $T_1 \cap T_2 \cap A = 0$. It follows that $T_i \cap A = 0$ for $i = 1, 2$. Since each T_i is a type submodule of M, we have $T_i \perp A$. We see that $A = A/(T_1 \cap A) \cong (A + T_1)/T_1 \subseteq (T_2 + T_1)/T_1 \cong T_2/(T_1 \cap T_2)$. Then $A \cong B/(T_1 \cap T_2)$ for some B with $T_1 \cap T_2 \leq_e B \subseteq T_2$. Note that $B \perp A$, and so $B \cap A = 0$ and $B \oplus A \subseteq M$.

(3) $\Rightarrow$ (5). Suppose there exists an embedding $X \oplus (X/Y) \xrightarrow{\alpha} M$ where Y is a proper essential submodule of X and $X \perp (X/Y)$. Take $x \in X$ but $x \notin Y$ and let $m_1 = \alpha(x)$ and $m_2 = \alpha(x+Y)$. Then $m_1R \perp m_2R$. To see this, let $m_1aR \cong m_2bR$ for some $a, b \in R$. It follows that $\alpha(xaR) \cong \alpha((x+Y)bR)$. This gives that $xaR \cong (x+Y)bR$. It must be $xaR = 0$ since $X \perp (X/Y)$. So, $m_1aR = 0$. Thus, $m_1R \perp m_2R$. Moreover, $m_1^{\perp} \subseteq m_2^{\perp}$ and $m_2^{\perp}/m_1^{\perp} \leq_e R/m_1^{\perp}$. We next prove $m_2 = 0$, which gives a contradiction. Define $\beta : m_1R \to m_2R$ by $\beta(m_1r) = m_2r$, $r \in R$. Then β is a homomorphism and $\ker(\beta) = m_1m_2^{\perp}$. Let L be a type closure of $\ker(\beta)$ in m_1R. Define $f : m_1R \to m_1R \oplus m_2R$ $(\subseteq M)$ by $f(x) = x + \beta(x)$, $x \in m_1R$. Then f is a monomorphism. Since L is a type closure of $\ker(\beta)$ in m_1R, $f(\ker(\beta))$ is parallel to $f(L)$. This gives that $\ker(\beta)$ is parallel to $f(L)$. Let L^{tc} and $f(L)^{tc}$ be the type closures of L and $f(L)$ in M respectively. Then both L^{tc} and $f(L)^{tc}$ are type closures of $\ker(\beta)$ in M. By (3), $L^{tc} = f(L)^{tc}$. It follows that $L + f(L)$ is a parallel extension of L. Note L is a type submodule of m_1R. Since $m_1R \perp m_2R$, L is a type submodule of $m_1R \oplus m_2R$. This implies that $L = L + f(L)$, i.e., $f(L) \subseteq L$. It follows that $\beta(L) \subseteq L$.

Since $m_1R \perp m_2R$, we have $\beta(L) = 0$. Thus, $Ker(\beta) = L$ is a type submodule of m_1R. Since $m_1R/\ker(\beta) \cong m_2R$, the fact that $m_1R \perp m_2R$ implies that $\ker(\beta) = m_1R$. Hence $m_2 = \beta(m_1) = 0$.

(6) $\Rightarrow$ (5). Suppose (5) does not hold. Then there exists $0 \neq X$ and a proper essential submodule Y of X such that $X \perp (X/Y)$ and $X \oplus (X/Y) \overset{h}{\hookrightarrow} M$ and $\pi : X \to X/Y$ the quotient map. Let $A = h(X)$ and $f = h \circ \pi \circ h^{-1}$. Then $f : A \longrightarrow M$ is well defined, $f(A) = h(X/Y)$ and $\ker(f) = h(Y)$. So, $\ker(f)$ is not a complement submodule of A, but $f(A) \perp A$.

(5) $\Rightarrow$ (6). Suppose there exists $f : A \to M$ such that $f(A) \perp A$, but $\ker(f)$ is not a complement submodule of A. Replacing A by a complement closure of $\ker(f)$ in A, we can assume without loss of generality that $\ker(f)$ is properly essential in A. Note that $f(A) \cap A = 0$, and thus $A \oplus [A/\ker(f)] \hookrightarrow M$. So, (5) fails to hold. $\square$

A module M is called a UTC-module (UTC for unique type closure) if M satisfies any of the equivalent conditions in Theorem 2.1.

Example 2.2. (1) All nonsingular modules are UTC.

(2) A module is a UC-*module* if every submodule has a unique complement closure [**9**]. All UC-modules are UTC.

(3) All atomic modules are UTC.

(4) For $R = \mathbf{Z}$, an abelian group M is UTC if and only if either M is torsion, or M is torsion free. This can easily be verified using Theorem 2.1 (5).

Theorem 2.1 (6) shows that submodules of a UTC-module are UTC. Next, using ideas of Camillo and Zelmanowitz in [**1**], we determine when an essential extension of a UTC-module is UTC, and when a type direct sum of UTC-modules is UTC. For a submodule X of M, if X is itself a UTC-module then X is called a UTC-submodule.

Proposition 2.3. *Let M_i be an ascending chain of* UTC-*submodules of M. Then $\cup M_i$ is a* UTC-*submodule. In particular, every module contains maximal* UTC-*submodules.*

Proof. Suppose that $\cup M_i$ is not UTC. Then there exists a partial homomorphism $f : A \to \cup M_i$ such that $f(A) \perp A$ but $\ker(f)$ is not a complement submodule of A. We can assume that $\ker(f)$ is properly essential in A. Take $a \in A$ but $a \notin \ker(f)$. Then $f(a) \in M_i$ for some i. Let $A' = \ker(f) + aR$. Thus $f : A' \to M_i$ is such that $f(A') \perp A'$ and $\ker(f)$ is not a complement submodule of A'. So, M_i is not UTC. $\square$

The next example shows that an essential extension of a UTC-module may not be UTC.

Example 2.4. Let $M = \oplus_{i=1}^{\infty} \mathbf{Z}/p_i\mathbf{Z}$ where p_i is the ith prime number. Let $R = \{\left(\begin{smallmatrix} n & x \\ 0 & n \end{smallmatrix}\right) : n \in \mathbf{Z}, x \in M\}$. R is a ring under the usual addition and multiplication of matrices, and $\mathrm{Soc}\,(R) = \{\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) : x \in M\}$ is essential in R_R. Since $\mathrm{Soc}\,(R_R)$ is semi-simple, it is clearly UTC. To see R_R is not UTC, let $N = \oplus_{i \geq 2} \mathbf{Z}/p_i\mathbf{Z}$, $A = \{\left(\begin{smallmatrix} n & x \\ 0 & n \end{smallmatrix}\right) : n \in 2\mathbf{Z}, x \in \mathbf{N}\}$ and $B = \{\left(\begin{smallmatrix} n & x \\ 0 & n \end{smallmatrix}\right) : n \in 4\mathbf{Z}, x \in \mathbf{N}\}$. Then A and B are R-modules, $A \perp (A/B)$ and $A \oplus (A/B)$ embeds in R_R.

For a module M, define $\varphi_t(M) = \{X \leq \widehat{M} :$ for $Y \leq X$ and $f \in \mathrm{End}\,(\widehat{M})$, $f(Y) \perp Y$ and $f(Y \cap M) = 0$ implies $f(Y) = 0\}$. As in [**1**, p. 253], define $\varphi(M) = \{X \leq \widehat{M}$: for $Y \leq X$ and $f \in \mathrm{End}\,(\widehat{M})$, $f(Y) \cap Y = 0$ and $f(Y \cap M) = 0$ implies $f(Y) = 0\}$.

Theorem 2.5. *For a module M, the following hold.*

(1) *$M \in \varphi(M) \subseteq \varphi_t(M)$.*

(2) *$\varphi_t(M)$ has maximal elements.*

(3) *If M is UTC, then every $X \in \varphi_t(M)$ is* UTC.

(4) *If $X \leq_e \widehat{M}$ and $X \notin \varphi_t(M)$ then X is not* UTC.

Proof. The proof of [**1**, Theorem 8] works. $\square$

The next result is the type analogue of [**1**, Theorem 13] which gives a sufficient and necessary condition for a direct sum of modules to be UC.

Theorem 2.6. *Let $M = \oplus_{i \in I} M_i$ where $M_i \perp M_j$ whenever $i \neq j$. Then M is* UTC *if and only if each M_i is* UTC *and every partial homomorphism between two distinct M_i is zero.*

Proof. "$\Rightarrow$." For any $h : A_i \longrightarrow M_j$ where $A_i \leq M_i$ and $i \neq j$, we have $A_i \perp h(A_i)$. By Theorem 2.1, $\ker(h)$ is a complement submodule of A_i. Let B_i be a complement of $\ker(h)$ in A_i. Thus, B_i is isomorphic to an essential submodule of $A_i/\ker(h)$ which embeds in M_j. Since $M_i \perp M_j$, it must be that $A_i/\ker(h) = \bar{0}$, i.e., $h = 0$.

"$\Leftarrow$." By Proposition 2.3, it suffices to show that M is UTC whenever $|I| < \infty$. We proceed by induction on $|I|$.

Case 1. $|I| = 2$, i.e., $M = M_1 \oplus M_2$. Let $A \subseteq M$ and $f : A \to M$ be a homomorphism with $f(A) \perp A$. We need to show that $\ker(f)$ is a complement submodule of A by Theorem 2.1. Replacing A by a complement closure of $\ker(f)$ in A, we may assume that $\ker(f) \leq_e A$. We want to show that $f = 0$. Let π_i be the projection of M onto M_i, $i = 1, 2$.

Subcase 1. $\pi_1 f(A) = 0$, i.e., $f(A) \subseteq M_2$. The map $f : A \cap M_2 \longrightarrow M_2$ has an essential kernel $\ker(f) \cap M_2$. Since M_2 is UTC, it must be $f(A \cap M_2) = 0$ by Theorem 2.1. So, there is a natural epimorphism $A/A \cap M_2 \longrightarrow A/\ker(f) \to 0$ with $A/\ker(f) \hookrightarrow M_2$. But, there is a monomorphism $A/A \cap M_2 \overset{\bar{\pi}}{\hookrightarrow} M_1$ where $\bar{\pi}(\bar{a}) = \pi_1(a), a \in A$. Since every partial homomorphism from M_1 to M_2 is zero, $A/\ker(f) = \bar{0}$, and so $f = 0$. Similarly, $f = 0$ if $\pi_2(A) = 0$.

Subcase 2. $\pi_1 f(A) \neq 0$ and $\pi_2 f(A) \neq 0$. Thus, either $\pi_1 f(A) \cap A \not\leq_e \pi_1 f(A)$ or $\pi_2 f(A) \cap A \not\leq_e \pi_2 f(A)$, for otherwise, $A \cap [(\pi_1 f(A) \oplus \pi_2 f(A)] \leq_e (\pi_1 f(A) \oplus \pi_2 f(A)$ which leads $A \cap f(A) \neq 0$, contradicting $A \perp f(A)$. So, we may assume $\pi_1 f(A) \cap A \not\leq_e \pi_1 f(A)$. Thus, $[\pi_1 f(A) \cap A] \cap Y_0 = 0$ for some $0 \neq Y_0 \subseteq \pi_1 f(A)$. Let $A_0 = (\pi_1 f)^{-1}(Y_0)$ and then $\pi_1 f|_{A_0} : A_0 \longrightarrow Y_0$ gives a partial homomorphism from M to M_2. Clearly the kernel of $\pi_1 f|_{A_0}$ is $\ker(f)$ which is essential in A_0. By Subcase 1, $\pi_1 f|_{A_0} = 0$. Thus, $Y_0 = 0$, a contradiction.

Case 2. $|I| = n > 2$. Then $M = \oplus_{i=1}^{n} M_i$. By the induction hypothesis, $Z = \oplus_{i=2}^{n} M_i$ is UTC. Then $M = M_1 \oplus Z$ with $M_1 \perp Z$, and every partial homomorphism from M_1 to Z is zero. Next we prove every partial homomorphism from Z to M_1 is zero, and thus the claim follows from Case 1.

Let $B \subseteq Z$ and $g : B \to M_1$ be a homomorphism. We prove $g = 0$ by induction on n. Suppose $g \neq 0$. Since $Z \perp M_1$, $\ker(g)$ is not a complement submodule of B. Replacing B by a complement closure of $\ker(g)$ in B, we may assume $\ker(g) \leq_e B$. Let $W = \oplus_{i=3}^{n} M_i$ and π be the projection of Z onto M_2. By induction hypothesis, the restriction of g on $A \cap W$ is zero. Thus, $g(B \cap W) = 0$ and so there is an epimorphism $B/B \cap W \to B/\ker(g) \longrightarrow 0$ with $B/\ker(g) \hookrightarrow M_1$. But there is a monomorphism $B/B \cap W \overset{\bar{\pi}}{\hookrightarrow} M_2$ where $\bar{\pi}(\bar{b}) = \pi(b), b \in B$. Since every partial homomorphism from M_2 to M_1 is zero, $B/\ker(g) = \bar{0}$, and so $g = 0$. The proof is complete. □

Note that $\mathbf{Z} \oplus \mathbf{Z}_2$ is not UTC though $\mathbf{Z}, \mathbf{Z}_2$ are UTC and $\mathbf{Z} \perp \mathbf{Z}_2$. For a UTC-module M, if M is 2-decomposable, then M is TS because any type submodule of M is the unique type closure of its unique type complement in M and hence is a direct summand of M. If N is a submodule of a UTC-module M such that N is fully invariant in $\widehat{M}$ (equivalently, N is quasi-injective), then N has a unique complement closure, in M, which is a type submodule of M, by Theorem 2.1. Next, we are back to type decompositions of modules.

Definition 2.7 [**3**]. We use $X \subseteq^{\oplus} Y$ to mean that X is a direct summand of module Y. Let E be an injective module. Then E is said to be *abelian* if $E = P_1 \oplus P_2 \oplus V$ with $P_1 \cong P_2$ implies $P_1 = P_2 = 0$. The module E is of *type I* if for all $0 \neq N \subseteq^{\oplus} E$, there exists $0 \neq X \subseteq^{\oplus} N$ such that X is abelian. Next the module E is of *type* III if for all $0 \neq N \subseteq^{\oplus} E, P \cong P \oplus P$ for some $0 \neq P \subseteq^{\oplus} N$. Lastly, E is of *type* II provided that for all $0 \neq N \subseteq^{\oplus} E$, N is not abelian, and there exists $0 \neq X \subseteq^{\oplus} N$ such that $P \ncong P \oplus P$ for all $0 \neq P \subseteq^{\oplus} X$.

A module M is said to be of type I, respectively type II or type III, if and only if $\widehat{M}$ is of type I, respectively type II or type III.

Let $\mathcal{I}_1$, respectively $\mathcal{I}_2$ or $\mathcal{I}_3$, be the class of all R-modules of type I, respectively type II or type III. Then, by [**3**], $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ form a maximal set of pairwise orthogonal types.

Corollary 2.8. *Every* 2*-decomposable module* M *has a decomposition* $M = M_1 \oplus M_2$ *where* M_1 *contains an essential direct sum of uniform submodules and* M_2 *contains no uniform submodules. The decomposition is unique if* M *is in addition* UTC.

Proof. Let $\mathcal{K}$ be the class of all modules containing an essential direct sum of uniform submodules. Then $\mathcal{K}$ is a natural class and $c(\mathcal{K})$ is the class of all modules containing no uniform submodules. Since M is 2-decomposable and UTC, the existence and uniqueness of the decomposition follow. $\square$

Corollary 2.9. *Every finitely decomposable module* M *has a decomposition* $M = M_1 \oplus M_2 \oplus M_3$ *with* $M_1 \in \mathcal{I}_1$, $M_2 \in \mathcal{I}_2$ *and* $M_3 \in \mathcal{I}_3$. *The decomposition is unique if* M *is in addition* UTC.

Acknowledgments. The first author thanks the hospitality of Memorial University of Newfoundland during a research visit paid by the second author's NSERC grant. During this visit part of this work was done.

REFERENCES

1. V.P. Camillo and J.M. Zelmanowitz, *Dimension modules*, Pacific J. Math. **91** (1980), 249–261.

2. J. Dauns, *Unsaturated classes of modules*, in *Abelian groups and modules*, (Colorado Springs, CO, 1995), 211-225, Lecture Notes in Pure and Appl. Math., **182**, Dekker, New York, 1996.

3. ———, *Module types*, Rocky Mountain J. Math. **27** (1997), 503–557.

4. K.R. Goodearl and A.K. Boyle, *Dimension theory for nonsingular injective modules*, Mem. Amer. Math. Soc., vol. 7, Amer. Math. Soc., Providence, 1976.

5. M.A. Kamal and M.J. Müller, *Extending modules over commutative domains*, Osaka J. Math. **25** (1988), 531–538.

6. S.H. Mohamed and B.J. Müller, *Continuous and discrete modules*, London Math. Soc. Lectures Note Ser., vol. 147, Cambridge Univ. Press, 1990.

7. B.J. Müller and S.T. Rizvi, *On the decomposition of continuous modules*, Canad. Math. Bull. **25** (1982), 296–301.

8. ———, *On injective and quasi-continuous modules*, J. Pure Appl. Algebra **28** (1983), 197–210.

9. P.F. Smith, *Modules for which every submodule has a unique closure*, in *Ring theory*, World Sci. Publishing, River Edge, NJ, 1993.

10. P.F. Smith and A. Tercan, *Generalizations of* CS-*modules*, Comm. Algebra **21** (1993), 1809–1847.

11. Y.Zhou, *Nonsingular rings with finite type dimension*, in *Advances in ring theory*, Birkhäuser, Boston, 1997.

12. ———, *Decomposing modules into direct sums of submodules with types*, J. Pure Appl. Algebra **138** (1999), 83–97.

Tulane University, New Orleans, Louisiana 70118-5698, USA
E-mail address: dauns@tulane.edu

Memorial University of Newfoundland, St. John's, NF A1C 5S7, Canada