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ANDERSON'S CONJECTURE FOR DOMAINS WITH FRACTAL BOUNDARY

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ABSTRACT. The inequality

$$\liminf_{r\to 1} \frac{\operatorname{Re} b(r\zeta)}{\int_0^r |b'(p\zeta)|\,d\rho} > 0$$

is shown to hold for all ζ in a set $E \subset \mathbf{T}$ with Hausdorff dimension 1, when b lies in a special class of Bloch functions first considered by Jones.

1. Introduction and background. A function f, defined and analytic in the unit disk, is called a Bloch function if

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbf{D}} (1-|z|^2)|f'(z)| < \infty.$$

We write $f \in \mathcal{B}$. The following proposition, which establishes a close connection between Bloch functions and conformal mappings, is well known, see [**2, 3**].

Proposition 1.1. *If g is a univalent function in $\mathbf{D}$ and $f = \log g'$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \le 6$. Conversely, if $\|f\|_{\mathcal{B}} \le 1$, then there exists a univalent function g such that $f = \log g'$.*

Functions in the Bloch space are Lipschitz mappings from the disk with the hyperbolic metric to the complex plane with the Euclidean metric

$$|b(z_1) - b(z_2)| \le C\|b\|_{\mathcal{B}} d(z_1, z_2).$$

This is easily seen by integration because the hyperbolic distance between two points z_1 and z_2 in the unit disk is defined as

$$d(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1-|z|^2}$$

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where the infimum is over all rectifiable arcs joining z_a and z_2 in $\mathbf{D}$. See [**5**], for example, for the basic facts on the hyperbolic metric. In this paper the hyperbolic metric will always be denoted by d and the Lipschitz property of Bloch functions from the hyperbolic to the Euclidean metric will be used several times.

Let $D_R(z) \subset \mathbf{D}$ denote a disk with hyperbolic center z and hyperbolic radius R. In Section 2 we will consider Bloch functions b with the property

$$\mathcal{M}(\varepsilon, R) : \inf_{z \in \mathbf{D}} \Big(\sup_{w \in D_R(z)} (1 - |w|^2)|b'(w)| \Big) > \varepsilon > 0,$$

where R and ε are positive. We will show that, if $b \in \mathcal{B}$ has $M(\varepsilon, R)$ for some $\varepsilon > 0$ and $R > 0$ and if $b = \log f'$ for some univalent f, then

$$\int_0^1 |f''(r\zeta)|\, dr < \infty \quad \forall \zeta \in E,$$

where $E \subset \mathbf{T}$ has Hausdorff dimension one. This will follow from the inequality

$$\liminf_{r \to 1} \frac{\operatorname{Re} b(r\zeta)}{\int_0^r |b'(p\zeta)|\, d\rho} > 0 \quad \forall \zeta \in E.$$

That the above inequality holds on a dense set of points for any Bloch function follows from the recent result of Jones and Mueller, [**9**]. Here we are interested in the question of the metric size of the set E.

2. The Anderson conjecture for a class of domains considered by Jones. In [**1**] Anderson conjectured that a univalent function f has

$$\int_0^1 |f''(r\zeta)|\, dr < \infty$$

for some $\zeta \in \partial\mathbf{D}$.

The conjecture was recently verified by Jones and Mueller [**9**], but the problem of the size of the set on which the function f' has finite radical variation remains open. It is expected that the conjecture should hold for a set with Hausdorff dimension one. In this section we show that, in case the mapping is onto a domain with fractal boundary, the set has the expected size.

At the end of the note we will also point out how the result of Bourgain [**4**] implies the dimension one property when the mapping is onto a type of domain which is in a certain sense of the opposite extreme behavior.

Let $b = -\log f'$. We claim that

$$\liminf_{r\to 1} \frac{\operatorname{Re} b(r\zeta)}{\int_0^r |b'(\rho\zeta)|\,d\rho} > 0 \Longrightarrow \int_0^1 |f''(r\zeta)|\,dr < \infty.$$

This was remarked in [**10**].

Proof of claim. We have

$$\int_0^1 |f''(r\zeta)|\,dr = \int_0^1 |b'(r\zeta)| \exp(-\operatorname{Re} b(r\zeta))\,dr,$$

and we may assume

$$\int_0^1 |b'(r\zeta)|\,dr = +\infty.$$

Choose $r_n \to 1$ such that

$$\int_{r_{n-1}}^{r_n} |b'(r\zeta)|\,dr = 1, \quad \forall\, n,$$

and such that

$$\liminf_{r\to 1} \frac{\operatorname{Re} b(r\zeta)}{\int_0^r |b'(\rho\zeta)|\,d\rho} > c' > 0, \quad \forall\, r \geq r_0.$$

We have

$$\begin{aligned}\int_0^1 |f''(r\zeta)|\,dr \leq & \int_0^{r_0} |f''(r\zeta)|\,dr \\ & + \sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} |b'(r\zeta)| \exp\left(-c' \int_0^{r_{n-1}} |b'(t\zeta)|\,dt\right) dr \\ \leq & \int_0^{r_0} |f''(r\zeta)|\,dr + \sum_{n=1}^{\infty} \exp(-cn)\end{aligned}$$

for some $c > 0$. We will prove the following

Theorem 2.1. *Let $b \in \mathcal{B}$ have the property $\mathcal{M}(\varepsilon, R)$ for some $\varepsilon > 0$ and some $R > 0$ as explained in Section 1. Assume that $b(0) = 0$. There is a set $E \subset \mathbf{T}$ with Hausdorff dimension one such that*

$$\liminf_{r \to 1} \frac{\operatorname{Re} b(r\zeta)}{\int_0^r |b'(\rho\zeta)|\, d\rho} > 0$$

for all $\zeta \in E$.

We remark, following [**8**], that for a domain whose boundary is everywhere wrinkled on all scales, any Riemann mapping corresponds to a Bloch function with $\mathcal{M}(\varepsilon, R)$. To be precise, let $b = \log f'$ for some univalent f mapping $\mathbf{D}$ onto a domain Ω, and define the Koebe transform of f as

$$F_{z_0}(z) = \frac{f((z + z_0)/(1 + \overline{z_0} z)) - f(z_0)}{(1 - |z_0|^2) f'(z_0)}.$$

Lemma 2.1 (Jones). *The Bloch function b has $\mathcal{M}(\varepsilon, R)$ for some $\varepsilon > 0$ and for some $R > 0$ if and only if there is no sequence $\{z_n\}$ in $\mathbf{D}$ such that $z_n \to \lambda$ and $\{F_{z_n}\}$ converges uniformly on compact subsets to*

$$F(z) = \frac{z}{1 + \lambda z}$$

for some $\lambda \in \mathbf{T}$.

This lemma tells us that if there is no sequence of conformal rescalings which blows up any piece of $\partial\Omega$ to a line, then any Bloch function which gives a Riemann map to Ω must have $\mathcal{M}(\varepsilon, R)$ for some $\varepsilon, R > 0$.

Proof of Lemma 2.1. It follows by integration that a Bloch function b fails to have $\mathcal{M}(\varepsilon, R)$ for all $\varepsilon, R > 0$ if and only if for each positive integer n there is a point $z_n \in \mathbf{D}$ such that

$$|b(z) - b(z_n)| < \frac{1}{n}, \quad \forall\, z \in D_n(z_n). \tag{2.1}$$

Let $b = \log f'$ for some univalent f, and suppose that we can find a sequence $\{z_n\}$ such that (2.1) holds. Then we have both

$$e^{-1/n} < \frac{|f'(z)|}{|f'(z_n)|} < e^{1/n}$$

and

$$-\frac{1}{n} < \arg\left(\frac{f'(z)}{f(z_n)}\right) < \frac{1}{n}.$$

Taking a subsequence, we may assume that $z_n \to \lambda$ for some $\lambda \in \mathbf{T}$. Then we have

$$F'_{z_n}(z) = \frac{f'((z+z_n)/(1+\overline{z_n}z))}{(1+\overline{z_n}z)^2 f'(z_n)} \longrightarrow \frac{1}{(1+\bar{\lambda}z)^2} \tag{2.2}$$

uniformly on compact subsets of $\mathbf{D}$. Therefore,

$$F_{z_n}(z) \longrightarrow \frac{z}{1+\bar{\lambda}z}$$

uniformly on compact subsets of $\mathbf{D}$. Conversely, we see that if $z_n \to \lambda \in \mathbf{T}$ and (2.2) holds uniformly on compact sets, then by taking a subsequence and relabeling we have (2.1). To make dimension estimates we will use

Lemma 2.2 (Hungerford). *Fix $0 < \varepsilon < c < 1$. Let $E_0 = \mathbf{T} = I_{0,0}$ and, for $n > 1$, $E_n = \cup I_{n,k}$ where $I_{n,k}$ are disjoint closed arcs such that, for each $I_{n,k}$, there is a unique $I_{n-1,j}$ with*

(i) $I_{n,k} \subset I_{n-1,j}$

(ii) $|I_{n,k}| \le \varepsilon |I_{n-1,j}|$

(iii) $\sum_{i(j)} |I_{n,i}| \ge c|I_{n-1,j}|$, *where $i(j)$ runs through all indices such that $I_{n,i} \subset I_{n-1,j}$.*

Let $E = \cap_n E_n$. Then with $\dim E$ *denoting the Hausdorff dimension of E, we have*

$$\dim E \ge 1 - \frac{\log c}{\log \varepsilon}.$$

Proofs appear in [**7**] and [**11**].

We also require a lemma from [**8**].

Lemma 2.3. *Let $b \in \mathcal{B}$ have $\mathcal{M}(\varepsilon, R)$ for some $\varepsilon, R > 0$, $I \subset \mathbf{T}$ an arc and $r_0 = 1 - |I|$. Then there exist $\alpha, \beta, \delta > 0$ depending only on ε and on R such that*

$$m\left(\left\{\zeta \in I : \operatorname{Re} b_r(\zeta) - \operatorname{Re} b_{r_0}(\zeta) > \alpha\left(\log\left(\frac{1-r_0}{1-r}\right)\right)^{1/2}\right\}\right) \geq \beta|I|$$

for all r with $(1-r) < \delta(1-r_0)$. Here m denotes Lebesgue measure.

Proof. The letters $C, C_1, C_2, \ldots$ denote absolute constants, and the constant $C(\varepsilon, R)$ may change from line to line. By a standard computation with Green's theorem,

$$\|\operatorname{Re}(b_r - b(0))\|_2^2 = \frac{1}{2}\|b_r - b(0)\|_2^2 \sim \iint_{\mathbf{D}} |b_r'|^2(1-|z|)\,dx\,dy$$

where $a \sim b$ means that a/b is bounded above and below by two positive numerical constants. See, for instance, [**5**, p. 237].

We claim that the integral on the right is bounded below by

$$C(\varepsilon, R)\log\left(\frac{1}{1-r}\right).$$

With $1-r$ sufficiently small, break the disk $\{|z| < r\}$ into annuli

$$A_j = \{1 - 2^{-C_1 R_j} < |z| < 1 - 2^{-C_1 R(j+1)}\}$$

where the numerical constant C_1 is chosen so that a radius of A_j has hyperbolic length, say, $> 3R$.

By integration of b, there is a $\rho > 0$ such that at each point w where $(1-|w|^2)|b'(w)| > \varepsilon$ we have $(1-|w'|^2)|b'(w')| > (\varepsilon/2)$ for each w' in the hyperbolic disk $D_\rho(w)$. By the condition $\mathcal{M}(\varepsilon, R)$, there are at least $C_2 2^{C_1 R_J}$ such disjoint disks in each of the annuli A_j. For each j, then, let $\mathcal{U}_j$ denote a union of at least $C_2 2^{C_1 R_j}$ disjoint disks contained in A_j such that $(1-|w'|^2)|b'(w')| > (\varepsilon/2)$ for each $w' \in \mathcal{U}_j$. Then

$$\begin{aligned}\iint_{\mathbf{D}} |b_r'|^2(1-|z|)\,dx\,dy &\geq \left(\frac{\varepsilon}{2}\right)^2 \sum_j \iint_{\mathcal{U}_j} \frac{1}{1-|z|}\,dx\,dy \\ &\geq \left(\frac{\varepsilon}{2}\right)^2 \sum_j 2^{-C_1 R(j+1)} \iint_{\mathcal{U}_j} \frac{1}{(1-|z|)^2}\,dx\,dy \\ &\geq C(\varepsilon, R)\sum_j 1 \\ &\geq C(\varepsilon, R)\log\left(\frac{1}{1-r}\right).\end{aligned}$$

Suppose now that, for whatever choice of $\alpha_0', \beta_0' > 0$, there exists b with the property $\mathcal{M}(\varepsilon, R)$ such that

$$m\left(\left\{|\mathrm{Re}\,(b_r - b(0))|^2 \geq \alpha_0' \log\frac{1}{1-r}\right\}\right) < \beta_0'.$$

Then, since

$$\int |\mathrm{Re}\,(b_r - b(0))|^2 \frac{d\theta}{2\pi} \leq C'\beta_0' \log\frac{1}{1-r} + (1-\beta_0')\alpha_0' \log\frac{1}{1-r}$$

we violate the above claim for some b by choosing α_0' and β_0' sufficiently small. So

$$m\left(\left\{|\mathrm{Re}\,(b_r - b(0))| \geq \alpha_0\left(\log\frac{1}{1-r}\right)^{1/2}\right\}\right) > \beta_0' > 0$$

for some α_0 and β_0' depending only on ε and R, for all b with $\mathcal{M}(\varepsilon, R)$.

We claim now that there are $0 < \alpha(\varepsilon, R) \leq \alpha_0$ and $0 < \beta_0(\varepsilon, R) \leq \beta_0'$ such that

$$m\left(\left\{\mathrm{Re}\,(b_r - b(0)) \geq \alpha\left(\log\frac{1}{1-r}\right)^{1/2}\right\}\right) > \beta_0 > 0$$

for all r sufficiently close to one. To prove the claim, we may assume that

$$m\left(\left\{\mathrm{Re}\,(b_r - b(0)) \leq -\alpha_0\left(\log\frac{1}{1-r}\right)^{1/2}\right\}\right) > \frac{\beta_0'}{2} > 0$$

since otherwise the claim is immediate. The function $\operatorname{Re}(b_r - b(0))$ is harmonic and has mean value zero. But, if

$$m\left(\left\{\operatorname{Re}(b_r - b(0)) \geq \alpha\left(\log\frac{1}{1-r}\right)^{1/2}\right\}\right) < \beta_0,$$

then

$$\int \operatorname{Re}(b_r - b(0)) < -\alpha_0\frac{\beta_0'}{2}\left(\log\frac{1}{1-r}\right)^{1/2} + (1-\beta_0)\alpha\left(\log\frac{1}{1-r}\right)^{1/2} + \int_{\alpha(\log(1/(1-r)))^{1/2}}^{\infty} m(\{\operatorname{Re}(b_r - b(0)) > \lambda\})\,d\lambda.$$

By Exercise 3 [**11**, p. 188], this is less than

$$-\alpha_0\frac{\beta_0'}{2}\left(\log\frac{1}{1-r}\right)^{1/2} + (1-\beta_0)\alpha\left(\log\frac{1}{1-r}\right)^{1/2} + C_3\int_{\alpha}^{\infty} ue^{-u^2}\,du$$

which gives a contradiction for sufficiently small α and for r sufficiently close to one. Therefore, there exists β_0 and $O < r_1 < 1$ such that

$$m\left(\left\{\operatorname{Re}(b_r - b(0)) \geq \alpha\left(\log\frac{1}{1-r}\right)^{1/2}\right\}\right) > \beta_0 > 0$$

for each $r > r_1$. Since Bloch functions are Lipschitz from the hyperbolic metric to the Euclidean metric in the plane, we may, by taking α_0 slightly smaller and increasing r if necessary, assume that the above set is a union of arcs with disjoint interiors of length $\sim (1-r)$. Fix $r' > 0$. Let τ be the conformal self mapping of $\mathbf{D}$ which maps the arc J, complementary to $[e^{-i(\beta_0/20)}, e^{i(\beta_0/20)}]$, onto I, and let Q_I denote the Carleson square determined by I. Notice that the hyperbolic distance from $\tau(0)$ to any point in $Q_I \cap \{|z| = r_0\}$ is uniformly bounded with a bound only depending on β_0, hence on ε and R. We have

$$m\left(\left\{\operatorname{Re}((b\circ\tau)_{\tau'} - (b\circ\tau)(0)) \geq \alpha\left(\log\frac{1}{1-r'}\right)^{1/2}\right\}\right) \geq \beta_0,$$

and the above set is the radial projection onto $\mathbf{T}$ of a certain set of arcs on the circle $|z| = r'$. Denote the union of these arcs by $E \subset \{|z| = r'\}$. Let r be determined by

$$\log\frac{1+r}{1-r} = \log\frac{1+|\tau(0)|}{1-|\tau(0)|} + \log\frac{1+r'}{1-r'}.$$

We project the set $\tau(E) \cap Q_I$ outward from the ball

$$\left\{ \left| \frac{w - \tau(0)}{1 - \overline{\tau(0)} w} \right| \le r' \right\}$$

along geodesic rays through $\tau(0)$ onto the circular arc $\{|w| = r\} \cap Q_I$. Let E' denote the image on $\{|w| = r\} \cap Q_I$. Each arc of $\tau(E)$ is projected through a hyperbolic distance which is less than

$$\gamma = \gamma(\beta_0) = \gamma(\varepsilon, R) = 2 \cdot d(\tau(0), \{|z| = r_0\}) + 1.$$

Using again the Lipschitz property of Bloch functions and letting $E'' \subset I$ denote the radial projection of E', there exists an $\alpha > 0$ such that

$$\operatorname{Re} b_r(\zeta) - \operatorname{Re} b_{r_0}(\zeta) > \alpha \left(\log \left(\frac{1 - r_0}{1 - r} \right) \right)^{1/2}, \quad \forall \zeta \in E''$$

if, say,

$$\log \frac{1 + r'}{1 - r'} \ge 100\gamma.$$

By the choice of τ we also have $\beta > 0$ such that

$$|E''| \ge \beta |I|$$

and β depends only on ε and R. The requirement on the size of $\log((1 + r')/(1 - r'))$ is met if $(1 - r) < \delta(1 - r_0)$ for sufficiently small $\delta = \delta(\gamma) = \delta(\varepsilon, R) > 0$. Shrinking δ further if necessary to meet the earlier demands on r completes the proof.

Proof of Theorem 2.1. Assume that $\|b\|_{\mathcal{B}} \le 1$, and let $r_j = 1 - 2^{-j}$ for all $j \ge 0$. Choose a large j_0 so that $2^{-j_0} < \delta$. By Lemma 2.3 there are $\alpha, \beta > 0$ and there is a set

$$C_1 \subset \{\operatorname{Re} b_{r_{j_0}}(\zeta) > \alpha \sqrt{j_0}\}$$

which has $|C_1| > \beta$ and is the union of arcs of length 2^{-j_0}. In each of these arcs we again apply Lemma 2.3 to obtain a set

$$C_2 \subset \{\operatorname{Re} b_{r_{2j_0}}(\zeta) - \operatorname{Re} b_{r_{j_0}}(\zeta) > \alpha \sqrt{j_0}\}$$

which is the union of arcs of length 2^{-2j_0} and has the property that if $I \subset C_1$ is an arc of length 2^{-j_0} then $|C_2 \cap I| \geq \beta|I|$. We continue in this way, at the lth step obtaining

$$C_l \subset \{\operatorname{Re} b_{r_{lj_0}}(\zeta) - \operatorname{Re} b_{r_{(l-1)j_0}}(\zeta) > \alpha\sqrt{j_0}\}$$

the union of arcs of length 2^{-lj_0} such that if $I \subset C_{l-1}$ is an arc of length $2^{-(l-1)j_0}$ then $|C_l \cap I| \geq \beta|I|$. We are in the situation of Lemma 2.2, and the set $E = \cap_l C_l$ has

$$\dim E \geq 1 - \frac{\log \beta}{\log 2^{-j_0}}.$$

Let $\zeta \in E$. Choose a large j, and let m satisfy

$$mj_0 \leq j < (m+1)j_0.$$

We have

$$\begin{aligned}
\text{(2.3)} \qquad \operatorname{Re} b_{r_j}(\zeta) &\geq \operatorname{Re} b_{r_{mj_0}}(\zeta) - cj_0 \geq \alpha m\sqrt{j_0} - cj_0 \\
&\geq c\left(\frac{\alpha}{\sqrt{j_0}} \int_0^{r_{mj_0}} |b'(\rho\zeta)|\, d\rho - j_0\right) \\
\text{(2.4)} \qquad &\geq c\left(\frac{\alpha}{\sqrt{j_0}} \int_0^{r_j} |b'(\rho\zeta)|\, d\rho - j_0 - \alpha\sqrt{j_0}\right).
\end{aligned}$$

By (2.3), we have

$$\operatorname{Re} b_{r_j}(\zeta) \longrightarrow +\infty, \quad j \to +\infty$$

for all $\zeta \in E$. Therefore, if

$$\int_0^1 |b'(\rho\zeta)|\, d\rho < +\infty,$$

we have

$$\liminf_{r\to 1} \frac{\operatorname{Re} b(r\zeta)}{\int_0^r |b'(\rho\zeta)|\, d\rho} = +\infty.$$

Otherwise, we have by (4) that

$$\liminf_{r\to 1} \frac{\operatorname{Re} b(r\zeta)}{\int_0^r |b'(\rho\zeta)|\, d\rho} \geq c\frac{\alpha}{\sqrt{j_0}}.$$

Noting that $\dim E \to 1$ as $j_0 \to +\infty$, the proof is complete. $\square$

We remark that the Anderson conjecture with lower bound dimension estimates is known for the case of Bloch functions with lacunary power series [**6**]. Notice also that Anderson's conjecture holds at any point where the radial variation of the Bloch function b is finite. So if b is a bounded function then, by the result of Bourgain [**4**], Anderson's conjecture holds on a set with dimension one. As the functions in Jones's class obey a lower bound law of the iterated logarithm at almost every point, these two cases are, in the sense of boundedness of the Bloch function, at the opposite extremes.

Note added in proof. In November of 1999 Paul Müller informed the author that the ideas in [**9**] lead to a proof that Anderson's conjecture holds on a set with full Hausdorff dimension. Because this article contains a complete proof of the lemma of Jones announced in [**8**] and because of the simplification of the proof of Anderson's conjecture in the case of fractal boundaries, both Prof. Müller and the Editor have suggested that it should appear here.

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