ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 1, Spring 2000

A STRONG SIMILARITY PROPERTY OF NUCLEAR C^*-ALGEBRAS

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1. Introduction and main result. The aim of this note is to establish a new lifting property of the multiplication map on nuclear C^*-algebras, Theorem 1.2 below, and to apply it to two natural questions arising from Pisier's recent work on the similarity problem for operator algebras [**19**]. Let A be a C^*-algebra, H a Hilbert space, and let $u : A \to B(H)$ be a bounded homomorphism. An outstanding open problem going back to Kadison asks whether u is necessarily similar to a $*$-representation. By a result due to Paulsen [**14**], this is equivalent to the question: Is u automatically completely bounded? We refer to [**15, 17**] for wide information on completely bounded maps and Kadison's similarity problem.

When A is a nuclear C^*-algebra, Kadison's problem was solved positively by Bunce [**3**] and Christensen [**6**]. Moreover, in this situation we have the estimate $\|u\|_{\mathrm{cb}} \leq \|u\|^2$ for any bounded homomorphism u from A into $B(H)$ where $\|\cdot\|_{\mathrm{cb}}$ denotes the completely bounded norm. In [**19**], Pisier showed that this estimate is not far from characterizing nuclear C^*-algebras. Firstly he proved that if A is a C^*-algebra for which any bounded homomorphism $u : A \to B(H)$ is completely bounded, there exists a number $\alpha \geq 0$ and a constant $K > 0$ such that, for all u as above, $\|u\|_{\mathrm{cb}} \leq K\|u\|^{\alpha}$. Moreover, he showed that the infimum of the numbers $\alpha \geq 0$ for which this holds is attained and is an integer. This integer is denoted by $d(A)$ and called the similarity degree of A. With this terminology, we thus have $d(A) \leq 2$ when A is a nuclear C^*-algebra. Secondly it is shown in [**19**] that if A is a C^* algebra with $d(A) \leq 2$, then whenever a $*$-representation $\pi : A \to B(H)$ generates a semi-finite von Neumann algebra, that von Neumann algebra is injective.

Our first purpose is to show that the degree 2 property of nuclear C^*-algebras actually holds in a strong sense, as follows. Let A be a

Received by the editors on September 1, 1997, and in revised form on October 30, 1998.

C^*-algebra, and let $u : A \to B(H)$ be a bounded homomorphism. Let $[u(A)]'$ denote the commutant of the range $u(A)$ of u. We introduce a new homomorphism $\hat{u} : A \otimes [u(A)]' \to B(H)$ by letting

$$\hat{u}\Big(\sum a_i \otimes x_i\Big) = \sum u(a_i)x_i \tag{1.1}$$

for all finite families $(a_i)_i$ in A and $(x_i)_i$ in $[u(A)]'$. In general, $\hat{u}$ is not bounded when the algebraic tensor product $A \otimes [u(A)]'$ is equipped with the minimal (= spatial) tensor norm $\otimes_{\min}$. Indeed, this follows from the following elementary lemma which shows the relevance of nuclearity when considering boundedness for $\hat{u}$.

Lemma 1.1. *A C^*-algebra A is nuclear if and only if for any $*$-representation $u : A \to B(H)$ the homomorphism $\hat{u}$ defined by* (1.1) *extends to a bounded map from $A \otimes_{\min} [u(A)]'$ into $B(H)$.*

Proof. Following standard terminology, we denote by $\otimes_{\max}$ the maximal tensor product of C^*-algebras. Assume that A is nuclear, and let $u : A \to B(H)$ be a $*$-representation. Then $B = [u(A)]'$ is a C^*-algebra and $\hat{u}$ extends to a $*$-representation from $A \otimes_{\max} B$ into $B(H)$. However, $A \otimes_{\max} B = A \otimes_{\min} B$ under our hypothesis, whence $\|\hat{u} : A \otimes_{\min} B \to B(H)\| \leq 1$. Conversely, assume that $\hat{u}$ is bounded for any $*$-representation $u : A \to B(H)$. By a direct sum argument, we then obtain a constant $C > 0$ such that $\|\hat{u}\| \leq C$ for any such u. Now let B be any C^*-algebra, and let $(a_i)_i \subset A$ and $(b_i)_i \subset B$ be two finite families. We give ourselves two commuting $*$-representations $u : A \to B(H)$ and $v : B \to B(H)$. Then the $v(b_i)$'s belong to $[u(A)]'$ and we have $\sum u(a_i)v(b_i) = \hat{u}(\sum a_i \otimes v(b_i))$. Consequently,

$$\Big\|\sum u(a_i)v(b_i)\Big\| \leq \|\hat{u}\| \Big\|\sum a_i \otimes v(b_i)\Big\|_{\min} \leq C \Big\|\sum a_i \otimes b_i\Big\|_{\min}.$$

Taking the supremum over all such pairs (u, v) yields $\|\sum a_i \otimes b_i\|_{\max} \leq C\|\sum a_i \otimes b_i\|_{\min}$. This shows that $A \otimes_{\min} B = A \otimes_{\max} B$ for any B, which means that A is nuclear. $\square$

Assume here that A is nuclear. Then it is not hard to deduce from Paulsen's theorem [**14**] and the Bunce-Christensen result quoted above that, for any bounded homomorphism $u : A \to B(H)$, the map $\hat{u}$ extends to a completely bounded homomorphism on $A \otimes_{\min} [u(A)]'$ with $\|\hat{u}\|_{\rm cb} \le \|u\|^4$. We will show, see Theorem 4.1 below, that we actually have the better estimate $\|\hat{u}\|_{\rm cb} \le \|u\|^2$. By Lemma 1.1 above, this strong degree 2 property characterizes nuclearity. That result is optimal since, if $\alpha \ge 0$ and $K > 0$ are two numbers such that $\|\hat{u}\|_{\rm cb} \le K\|u\|^\alpha$ for any nuclear C^*-algebra A and any bounded homomorphism $u : A \to B(H)$, then $K \ge 1$ and $\alpha \ge 2$.

We will make extensive use of the theory of operator spaces (= closed subspaces of $B(H)$ equipped with matrix norms) and especially duality and the Haagerup tensor product. We refer the reader to [**1, 2, 7, 11, 16, 18**] for the necessary background on operator spaces. We will use the following standard notation and terminology. By $\otimes_{\min}$ and $\otimes_h$, we will denote the minimal and the Haagerup tensor product, respectively. Given two operator spaces X and Y, we denote by $\mathrm{cb}\,(X, Y)$ the Banach space of all completely bounded maps $u : X \to Y$, equipped with the completely bounded norm $\|u\|_{\rm cb} = \sup_{n\ge1} \|u \otimes I_{M_n} : M_n(X) \to M_n(Y)\|$.

Let $u : X \to Y$ be any linear map. Then we say that u is a complete isometry when $u \otimes I_{M_n}$ is an isometry from $M_n(X)$ into $M_n(Y)$ for any integer $n \ge 1$ and that u is completely contractive when $\|u\|_{\rm cb} \le 1$. The map u is said to be a metric surjection when $\|u\| \le 1$ and, for any $y \in Y$ with $\|y\| < 1$, there exists $x \in X$ with $\|x\| < 1$ which satisfies $u(x) = y$. Furthermore, we say that u is a complete metric surjection if $u \otimes I_{M_n}$ is a metric surjection for any $n \ge 1$.

We recall that, given any Banach space E, there exists a greatest operator space structure on E, called $\max(E)$ [**1, 2**]. This is characterized by the following property. For any bounded linear map $u : E \to B(H)$, then u is completely bounded on $\max(E)$ and

$$\|u : \max(E) \longrightarrow B(H)\|_{\rm cb} = \|u\|. \tag{1.2}$$

Theorem 1.2. *Let A be a C^*-algebra. For any operator space X, there is a unique completely contractive linear map*

$$Q_X : \max(A) \otimes_h X \otimes_h \max(A) \longrightarrow A \otimes_{\min} X$$

satisfying $Q_X(a \otimes x \otimes b) = ab \otimes x$ for all $x \in X$ and $a, b \in A$.

Then Q_X is a complete metric surjection for all operator spaces X if, and only if, A is nuclear.

This result will be proved in Section 3. Its proof will consist of a reduction to the finite dimensional case which will be settled in Section 2 below. In Section 4 the strong degree 2 property of nuclear C^*-algebras will be deduced from Theorem 1.2. As a second application, see Corollary 4.4, we will derive a precise factorization property for elements in $M_n(A)$, when A is a nuclear C^*-algebra, as suggested in [**19**, Remark 4.6].

2. The finite-dimensional case. Here we shall consider the case when the C^*-algebra A is finite dimensional. In this situation the result stated in Theorem 1.2 can be strengthened as follows.

Proposition 2.1. *Let A be a finite-dimensional C^*-algebra, and let X be any operator space. We denote by $\mathcal{U}$ the unitary group in A and by dm the normalized Haar measure on $\mathcal{U}$. We define a linear map*

$$\varphi : A \otimes_{\min} X \longrightarrow \max(A) \otimes_h X \otimes_h \max(A)$$

by letting:

$$\varphi(a \otimes x) = \int_{\mathcal{U}} au \otimes x \otimes u^* \, dm(u), \quad a \in A, x \in X.$$

Then φ is a completely contractive map.

Proof. The finite dimensional C^*-algebra can be written, up to $*$-isomorphism, as a direct sum $A = \oplus_{k=1}^N M_{n_k}$ of matrix spaces M_{n_k}. For any $1 \leq k \leq N$, we denote by $(E_{ij}^k)_{1 \leq i,j \leq n_k}$ the canonical basis of M_{n_k}. The coordinates of an element $a \in A$ in the basis $(E_{ij}^k)_{k,i,j}$ will be denoted by (a_{ij}^k), that is, $a = \sum_{1 \leq k \leq N} \sum_{1 \leq i,j \leq n_k} a_{ij}^k E_{ij}^k$. Concerning the unitary group in A, we will need the following elementary fact, see, e.g., [**13**, Section 27]:

$$\int_{\mathcal{U}} u_{ij}^k \overline{u_{pq}^r} \, dm(u) = \frac{1}{n_k} \delta_{k,r} \delta_{i,p} \delta_{j,q}. \tag{2.1}$$

Let H be a Hilbert space, and let $\sigma_1 : \max(A) \to B(H)$, $\sigma : X \to B(H)$ and $\sigma_2 : \max(A) \to B(H)$ be three completely contractive maps. We denote by

$$\sigma_1 \cdot \sigma \cdot \sigma_2 : \max(A) \otimes_h X \otimes_h \max(A) \longrightarrow B(H)$$

the completely contractive linear map defined by $(\sigma_1 \cdot \sigma \cdot \sigma_2)(a \otimes x \otimes b) = \sigma_1(a)\sigma(x)\sigma_2(b)$. We shall show that:

$$(2.2)\quad (\sigma_1 \cdot \sigma \cdot \sigma_2)\varphi : A \otimes_{\min} X \longrightarrow B(H) \qquad \text{is completely contractive.}$$

From this it is easy to derive the result. Indeed, it follows from the Christensen-Sinclair factorization theorem [**7**] and its generalization [**16**] that one can find $\sigma_1, \sigma, \sigma_2$ as above such that $\sigma_1 \cdot \sigma \cdot \sigma_2$ is a complete isometry. In this situation $\|\varphi\|_{\rm cb} = \|(\sigma_1 \cdot \sigma \cdot \sigma_2)\varphi\|_{\rm cb}$, hence $\|\varphi\|_{\rm cb} \leq 1$.

For $1 \leq k \leq N$ and $1 \leq i,\ j \leq n_k$ we define $S^k_{ij} = \sigma_1(E^k_{ij})$ and $T^k_{ij} = \sigma_2(E^k_{ij})$ in $B(H)$. Given any $a \in A$ and $x \in X$ we have

$$\begin{aligned}
((\sigma_1 \cdot \sigma \cdot \sigma_2)\varphi)(a \otimes x) &= \int_{\mathcal{U}} \sigma_1(au)\sigma(x)\sigma_2(u^*)\, dm(u) \\
&= \int_{\mathcal{U}} \Bigg(\sum_{k,i,j} (au)^k_{ij} S^k_{ij} \Bigg) \sigma(x) \Bigg(\sum_{r,p,q} (u^*)^r_{pq} T^r_{pq} \Bigg)\, dm(u) \\
&= \int_{\mathcal{U}} \sum_{k,i,l,j} \sum_{r,p,q} a^k_{ij} u^k_{jl} S^k_{il} \sigma(x) \overline{u^r_{qp}} T^r_{pq}\, dm(u) \\
&= \sum_{k=1}^{N} \frac{1}{n_k} \sum_{1 \leq i,l,j \leq n_k} a^k_{ij} S^k_{il} \sigma(x) T^k_{lj}
\end{aligned}$$

by (2.1).

Let $m \geq 1$ be an integer. In view of the completely isometric identification $M_m(A \otimes_{\min} X) = A \otimes_{\min} M_m(X)$, we will write a generic element of $M_m(A \otimes_{\min} X)$ as $\theta = \sum_{1 \leq k \leq N} \theta^k$ with, for each k, $\theta^k = \sum_{1 \leq i,j \leq n_k} E^k_{ij} \otimes \theta^k_{ij}$, for some family $(\theta^k_{ij})_{i,j}$ in $M_m(X)$. With this notation it follows from the computation above that:

$$\begin{aligned}
&((\sigma_1 \cdot \sigma \cdot \sigma_2)\varphi \otimes I_{M_m})(\theta) \\
&\qquad = \sum_k \frac{1}{n_k} \sum_{i,l,j} (S^k_{il} \otimes I_{l^m_2})[(\sigma \otimes I_{M_m})(\theta^k_{ij})](T^k_{lj} \otimes I_{l^m_2}).
\end{aligned}$$

By a classical use of the Cauchy-Schwarz inequality we deduce

$$\begin{aligned}\|((\sigma_1\cdot\sigma\cdot\sigma_2)\varphi\otimes I_{M_m})(\theta)\| \leq & \Big\|\sum_{k,i,l}\frac{S^k_{il}(S^k_{il})^*}{n_k}\Big\|^{1/2}\\ & \times\sup_k\|(\sigma\otimes I_{M_m}\otimes I_{M_{n_k}})(\theta^k)\|\\ & \times\Big\|\sum_{k,l,j}\frac{(T^k_{lj})^*T^k_{lj}}{n_k}\Big\|^{1/2}.\end{aligned}$$

Since $\|\theta\| = \sup_k\|\theta^k\|$ and σ is completely contractive, we infer

$$\|(\sigma_1\cdot\sigma\cdot\sigma_2)\varphi\|_{\mathrm{cb}} \leq \Big\|\sum_{k,i,l}\frac{S^k_{il}(S^k_{il})^*}{n_k}\Big\|^{1/2}\Big\|\sum_{k,l,j}\frac{(T^k_{lj})^*T^k_{lj}}{n_k}\Big\|^{1/2}. \tag{2.3}$$

Now observe that, by (2.1),

$$\begin{aligned}\sum_{k,i,l}\frac{S^k_{il}(S^k_{il})^*}{n_k} &= \int_{\mathcal{U}}\Big(\sum_{k,i,l}S^k_{il}u^k_{il}\Big)\Big(\sum_{k,i,l}S^k_{il}u^k_{il}\Big)^*\,dm(u)\\ &= \int_{\mathcal{U}}\sigma_1(u)\sigma_1(u)^*\,dm(u).\end{aligned}$$

Since $\|\sigma_1\|\leq 1$ we thus obtain $\|\sum_{k,i,l}(S^k_{il}(S^k_{il})^*/n_k)\|\leq 1$. Similarly it follows from the fact that σ_2 is a contraction that $\|\sum_{k,l,j}((T^k_{lj})^*T^k_{lj}/n_k)\| \leq 1$. Hence the desired property (2.2) follows from the inequality (2.3). This completes the proof. □

3. Proof of Theorem 1.2. In this section we give ourselves a C^*-algebra A and an operator space X. We first define $Q_X : A\otimes X\otimes A\to A\otimes X$ on the algebraic tensor product by letting $Q_X(a\otimes x\otimes b) = ab\otimes x$ for any $x\in X$ and $a,b\in A$. Let H be a Hilbert space that $X\subset B(H)$ completely isometrically, and let $C = A\otimes_{\min} B(H)$. We denote by $p : C\otimes C\to C$ the multiplication mapping on the C^*-algebra C. Then p extends to a completely contractive map from $C\otimes_h C$ into C. Since the Haagerup tensor norm dominates the minimal one, we obtain, using the associativity of $\otimes_h$, that

$$\|p : A\otimes_h B(H)\otimes_h A\otimes_h B(H)\longrightarrow A\otimes_{\min} B(H)\|_{\mathrm{cb}}\leq 1.$$

Now let $J : A \otimes_h B(H) \otimes_h A \to A \otimes_h B(H) \otimes_h A \otimes_h B(H)$ be the complete isometry defined by letting $J(z) = z \otimes I_H$ for any $z \in A \otimes_h B(H) \otimes_h A$. We obtain from above that $\|pJ\|_{\rm cb} \leq 1$. Moreover, the restriction of pJ to the algebraic tensor product $A \otimes X \otimes A$ coincides with Q_X. Since $A \otimes_{\min} X \subset A \otimes_{\min} B(H)$ completely isometrically, we deduce that Q_X extends to a completely contractive map from $A \otimes_h X \otimes_h A$ into $A \otimes_{\min} X$. A fortiori, Q_X extends to a completely contractive map from $\max(A) \otimes_h X \otimes_h \max(A)$ into $A \otimes_{\min} X$.

The "only if" part of Theorem 1.2 follows from Pisier's work in [**18**, Section 6.3] on the so-called delta norm. However, for the sake of completeness, we provide a simple direct proof in Remark 4.3 below.

We now assume that A is nuclear and shall prove that Q_X is a complete metric surjection. First note that it suffices to show that Q_X is a metric surjection. Indeed, let us assume that this, a priori, weaker result holds for all operator spaces. Then we obtain that, for any $n \geq 1$, $Q_{M_n(X)}$ is a metric surjection. Moreover, the identity map from $\max(A) \otimes_h M_n(X) \otimes_h \max(A)$ onto $M_n(\max(A) \otimes_h X \otimes_h \max(A))$ is a contraction. Since the identification $A \otimes_{\min} M_n(X) = M_n(A \otimes_{\min} X)$ is isometric, we deduce that

$$Q_X \otimes I_{M_n} : M_n(\max(A) \otimes_h X \otimes_h \max(A)) \longrightarrow M_n(A \otimes_{\min} X)$$

is a metric surjection. This yields the full result.

By definition, the algebraic tensor product $A \otimes X$ is dense in $A \otimes_{\min} X$, hence in the proof to come we may and do assume that X is finite dimensional. Applying [**5**, Proposition 5], we can also assume that A is separable.

To show that Q_X is a metric surjection we shall prove the equivalent property that the biadjoint map Q_X^{**} is a metric surjection. We then introduce the von Neumann algebra $M = A^{**}$. Note that, since A is nuclear, it is locally reflexive, hence we have, see [**9**],

$$(A \otimes_{\min} X)^{**} = M \otimes_{\min} X$$

isometrically. To go further, we need some information on the bidual of a Haagerup tensor product. We shall here consider triples of operator spaces although the results we present hold true as well for the tensor product of an arbitrary number of operator spaces. Thus we let

V_1, V_2, V_3 be three operator spaces. Our aim is to describe a natural embedding of $V_1^{**} \otimes_h V_2^{**} \otimes_h V_3^{**}$ into the bidual of $V_1 \otimes_h V_2 \otimes_h V_3$. This description goes back to [**12**] where this bidual is identified with the so-called normal Haagerup tensor product of the spaces V_i^{**}, $i = 1, 2, 3$.

We denote by $(V_1^{**} \otimes_h V_2^{**} \otimes_h V_3^{**})^*_\sigma$ the set of all elements in $(V_1^{**} \otimes_h V_2^{**} \otimes_h V_3^{**})^*$ for which the associated trilinear form on $V_1^{**} \times V_2^{**} \times V_3^{**}$ is separately w^*-continuous. Given any F in the space $(V_1^{**} \otimes_h V_2^{**} \otimes_h V_3^{**})^*_\sigma$, we let $\tilde{F} \in (V_1 \otimes_h V_2 \otimes_h V_3)^*$ be the restriction of F to $V_1 \otimes_h V_2 \otimes_h V_3$. By definition, $F \mapsto \tilde{F}$ is completely contractive. We claim that actually the map $F \mapsto \tilde{F}$ is a surjective complete isometry, i.e.,

$$(V_1 \otimes_h V_2 \otimes_h V_3)^* = (V_1^{**} \otimes_h V_2^{**} \otimes_h V_3^{**})^*_\sigma. \tag{3.1}$$

To check this, we let $G \in M_n((V_1 \otimes_h V_2 \otimes_h V_3)^*) = \mathrm{cb}\,(V_1 \otimes_h V_2 \otimes_h V_3, M_n)$ with $\|G\| \leq 1$. By the factorization theorem for multilinear completely bounded maps [**7, 16**], there exist a Hilbert space H as well as three completely contractive maps

$$\sigma_1 : V_1 \longrightarrow B(H, l_2^n), \sigma_2 : V_2 \longrightarrow B(H), \sigma_3 : V_3 \longrightarrow B(l_2^n, H)$$

such that $G = \sigma_1 \cdot \sigma_2 \cdot \sigma_3$, that is, $G(x_1 \otimes x_2 \otimes x_3) = \sigma_1(x_1)\sigma_2(x_2)\sigma_3(x_3)$ for any $x_i \in V_i$, $i = 1, 2, 3$. Let $\hat{\sigma}_2 : V_2^{**} \to B(H)$ be defined by $\hat{\sigma}_2 = J^* \sigma_2^{**}$, where J is the canonical embedding of $B(H)_*$ into $B(H)^*$. Similarly, we set $\hat{\sigma}_1 = \sigma_1^{**}$, $\hat{\sigma}_3 = \sigma_3^{**}$. We can now define $F \in (V_1^{**} \otimes_h V_2^{**} \otimes_h V_3^{**})^*$ by letting $F = \hat{\sigma}_1 \cdot \hat{\sigma}_2 \cdot \hat{\sigma}_3$. For any $i \in \{1, 2, 3\}$, $\|\hat{\sigma}_i\|_{\mathrm{cb}} = \|\sigma_i\|_{\mathrm{cb}} \leq 1$, hence $\|F\| \leq 1$. It is clear that F is separately w^*-continuous and that $\tilde{F} = G$. This shows that $F \mapsto \tilde{F}$ is a complete metric surjection. Since $F \mapsto \tilde{F}$ is obviously one-to-one, we finally obtain the surjective complete isometry (3.1).

Now let T be in $V_1^{**} \otimes_h V_2^{**} \otimes_h V_3^{**}$. We may define $\theta(T) \in (V_1 \otimes_h V_2 \otimes_h V_3)^{**}$ as follows. For any $F \in (V_1^{**} \otimes_h V_2^{**} \otimes_h V_3^{**})^*_\sigma$, we let $\tilde{F} \in (V_1 \otimes_h V_2 \otimes_h V_3)^*$ be given by (3.1) and we set

$$\langle \theta(T), \tilde{F} \rangle = \langle F, T \rangle. \tag{3.2}$$

By construction, the linear map θ is a complete contraction. (It is easy to check that it is actually a complete isometry, see [**12**].)

We shall now use the construction above with

$$V_1 = \max(A), \qquad V_2 = X, \qquad V_3 = \max(A).$$

Since $\max(M) = \max(A)^{**}$ completely isometrically, see [**1**], the formula (3.2) defines a completely contractive map

$$\theta : \max(M) \otimes_h X \otimes_h \max(M) \longrightarrow (\max(A) \otimes_h X \otimes_h \max(A))^{**}.$$

Let $\pi_X : \max(M) \otimes_h X \otimes_h \max(M) \to M \otimes_{\min} X$ be the completely contractive extension of the tensor product of the multiplication map on M with the identity map on X. Then the following key relation holds:

$$Q_X^{**}\theta = \pi_X. \tag{3.3}$$

Indeed, take $f \in M_* = A^*$ and $\xi \in X^*$, and let $F = \pi_X^*(f \otimes \xi)$. Then we have

$$F(m_1 \otimes x \otimes m_2) = \xi(x) f(m_1 m_2), \quad x \in X; m_1, m_2 \in M.$$

Hence we see that F is separately w^*-continuous and that its restriction $\tilde{F}$ belonging to $(\max(A) \otimes_h X \otimes_h \max(A))^*$ is given by $\tilde{F} = Q_X^*(f \otimes \xi)$. Therefore, for any $x \in X$ and any $m_1, m_2 \in M$, we have

$$\begin{aligned} \langle Q_X^{**}\theta(m_1 \otimes x \otimes m_2), f \otimes \xi\rangle &= \langle \theta(m_1 \otimes x \otimes m_2), \tilde{F}\rangle \\ &= \langle F, m_1 \otimes x \otimes m_2\rangle \qquad \text{by (3.2)} \\ &= \langle \pi_X(m_1 \otimes x \otimes m_2), f \otimes \xi\rangle, \end{aligned}$$

whence (3.3).

Since A is nuclear, the von Neumann algebra M is injective [**10**, Theorem 6.4]. Furthermore, M is countably generated because we assumed A separable. Consequently, M can be written as a direct sum of injective von Neumann algebras with separable predual, see [**4**, Section 3]. It therefore follows from Connes's theorem [**8**] that M is hyperfinite. Namely, there exists an upward directed net $(N_\lambda)_\lambda$ of finite dimensional C^*-subalgebras of M such that

$$M = \overline{\bigcup_\lambda N_\lambda}^{w^*}, \quad w^* = \sigma(M, M_*).$$

In fact, a more precise approximation property expressed by Lemma 3.1 below turns out to be true. Note that, in the next statement, X is finite dimensional, hence the $\sigma(M, M_*)$-topology on $M \otimes_{\min} X$ simply means the (finite) product topology induced by the $\sigma(M, M_*)$-topology of M.

Lemma 3.1. *The union over λ of the closed unit balls of the spaces $N_\lambda \otimes_{\min} X$ is $\sigma(M, M_*)$-dense in the closed unit ball of $M \otimes_{\min} X$.*

Proof. In this proof we denote by $(Z)_1$ the closed unit ball of any Banach space Z. We let C be the norm closure of the union of the C^*-algebras N_λ. Then it clearly suffices to show that $(C \otimes_{\min} X)_1$ is $\sigma(M, M_*)$-dense in $(M \otimes_{\min} X)_1$. We regard C as a subspace of its bidual C^{**}, in the canonical way, and we let $j : C \to M$ be the inclusion map. By the universal property of C^{**}, there exists a normal $*$-representation $\pi : C^{**} \to M$ such that $j = \pi_{/C}$. Since the kernel of π is a direct summand of C^{**}, we find a completely isometric map $\Gamma : M \to C^{**}$ such that $\pi\Gamma = I_M$.

Let z be in $(M \otimes_{\min} X)_1$. Then $y = (\Gamma \otimes I_X)(z)$ belongs to $(C^{**} \otimes_{\min} X)_1$. By construction, C is a nuclear, hence locally reflexive C^*-algebra [**9**]. Therefore, $(C \otimes_{\min} X)_1$ is $\sigma(C^{**}, C^*)$-dense in $(C^{**} \otimes_{\min} X)_1$. Hence we can find a net $(y_\alpha)_\alpha$ in $(C \otimes_{\min} X)_1$ such that $y_\alpha \to y$ in the $\sigma(C^{**}, C^*)$-topology. Since π is normal we deduce that $(\pi \otimes I_X)(y_\alpha) \to (\pi \otimes I_X)(y)$ in the $\sigma(M, M_*)$-topology. We thus finally obtain that $(j \otimes I_X)(y_\alpha) \to z$ in the $\sigma(M, M_*)$-topology, whence the result. □

We can now finish the proof of Theorem 1.2, "if part." We wish to show that Q_X^{**} is a metric surjection. Let λ be given, and let z be in $N_\lambda \otimes_{\min} X$ with $\|z\| < 1$. Since the embedding

$$\max(N_\lambda) \otimes_h X \otimes_h \max(N_\lambda) \longrightarrow \max(M) \otimes_h X \otimes_h \max(M)$$

is completely contractive, it readily follows from Proposition 2.1 that we can find z' in $\max(M) \otimes_h X \otimes_h \max(M)$ such that $\|z'\| < 1$ and $\pi_X(z') = z$. Then $z'' = \theta(z')$ satisfies $\|z''\| < 1$ and $Q_X^{**}(z'') = z$ by (3.3). The metric surjection property of Q_X^{**} now simply follows from Lemma 3.1 above. □

4. Applications. In this last section we shall give two applications of Theorem 1.2 to nuclear C^*-algebras. First we establish the strong degree 2 property announced in Section 1 and deduce a joint similarity property for commuting homomorphisms on nuclear C^*-algebras.

Theorem 4.1. *Let A be a nuclear C^*-algebra, let H be a Hilbert space, and let $u : A \to B(H)$ be a bounded homomorphism. Let $\hat{u} : A \otimes_{\min} [u(A)]' \to B(H)$ be the homomorphism defined by* (1.1).

(i) *Then $\hat{u}$ is completely bounded and $\|u\|_{\rm cb} \leq \|\hat{u}\|_{\rm cb} \leq \|u\|^2$.*

(ii) *There exists an isomorphism $S \in B(H)$ such that $\|S\|\,\|S^{-1}\| \leq \|u\|^2$ and the homomorphism $w \mapsto S^{-1}\hat{u}(w)S$ is completely contractive on $A \otimes_{\min} [u(A)]'$.*

Proof. The assertion (ii) follows from (i) by Paulsen's theorem [**14**] so we only have to prove (i). Obviously, $\|u\|_{\rm cb} \leq \|\hat{u}\|_{\rm cb}$, hence we only need to establish the inequality $\|\hat{u}\|_{\rm cb} \leq \|u\|^2$. For this purpose we apply Theorem 1.2 with the operator space $X = [u(A)]'$. We let $\varepsilon > 0$ and $n \geq 1$ and we give ourselves $\tau \in M_n(\max(A) \otimes_h X \otimes_h \max(A))$. By the definition of the Haagerup tensor norm, we may write

$$\tau = \sum_{\substack{1\leq i,j\leq n\\ 1\leq k,l\leq m}} a_{ik} \otimes x_{kl} \otimes b_{lj} \otimes E_{ij}$$

with $m \geq 1$, $a_{ik}, b_{lj} \in A$, $x_{kl} \in X$ and

$$\|[a_{ik}]\|_{M_{n,m}(\max(A))}\|[x_{kl}]\|_{M_m(X)}\|[b_{lj}]\|_{M_{m,n}(\max(A))} \leq \|\tau\| + \varepsilon. \tag{4.1}$$

Then we have

$$\begin{aligned}(\hat{u}Q_X \otimes I_{M_n})(\tau) &= (\hat{u} \otimes I_{M_n})\Bigg(\sum_{i,j,k,l} a_{ik}b_{lj} \otimes x_{kl} \otimes E_{ij}\Bigg)\\ &= \sum_{i,j,k,l} u(a_{ik}b_{lj})x_{kl} \otimes E_{ij}.\end{aligned}$$

Since u is a homomorphism and the x_{kl}'s commute with $[u(A)]$, we deduce

$$(\hat{u}Q_X \otimes I_{M_n})(\tau) = \sum_{i,j,k,l} u(a_{ik})x_{kl}u(b_{lj}) \otimes E_{ij}.$$

Consequently,

$$\begin{aligned}\|(\hat{u}Q_X \otimes I_{M_n})(\tau)\| &\leq \|[u(a_{ik})]\|_{M_{n,m}(B(H))} \\ &\quad \times \|[x_{kl}]\|_{M_m(X)}\|[u(b_{lj})]\|_{M_{m,n}(B(H))} \\ &\leq \|u\|^2(\|\tau\| + \varepsilon)\end{aligned}$$

by (4.1) and (1.2). This shows that $\|\hat{u}Q_X\|_{\mathrm{cb}} \leq \|u\|^2$, whence the result by Theorem 1.2. □

Corollary 4.2. *Let $A_1, \ldots, A_N$ be nuclear C^*-algebras, let H be a Hilbert space and, for any $1 \leq j \leq N$, let $u_j : A_j \to B(H)$ be a bounded homomorphism. Assume that the ranges of the u_j's commute.*

(i) *Let $u : A_1 \otimes \cdots \otimes A_N \to B(H)$ be the homomorphism defined by*

$$u(a_1 \otimes \cdots \otimes a_N) = u_1(a_1) \cdots u_N(a_N), \quad a_j \in A_j.$$

Then u extends to a completely bounded homomorphism from $A_1 \otimes_{\min} \cdots \otimes_{\min} A_N$ into $B(H)$ with $\|u\|_{\mathrm{cb}} \leq \|u_1\|^2 \cdots \|u_N\|^2$.

(ii) *There exists an isomorphism $S \in B(H)$ such that $\|S\|\,\|S^{-1}\| \leq \|u_1\|^2 \cdots \|u_N\|^2$ and, for any $1 \leq j \leq N$, the homomorphism $a \mapsto S^{-1}u_j(a)S$ is completely contractive on A_j.*

Proof. Once more (ii) follows from (i) by [**14**]. To prove (i), we assume for simplicity that $N = 2$, the reader will easily check that the general case follows by induction. By assumption $u_2(A_2) \subset [u_1(A_1)]'$ hence we may write

$$u = \widehat{u_1} \circ (I_{A_1} \otimes u_2).$$

Since $\|I_{A_1} \otimes u_2 : A_1 \otimes_{\min} A_2 \to A_1 \otimes_{\min} [u_1(A_1)]'\| \leq \|u_2\|_{\mathrm{cb}}$ we obtain $\|u\|_{\mathrm{cb}} \leq \|\widehat{u_1}\|_{\mathrm{cb}}\|u_2\|_{\mathrm{cb}}$, whence the result by Theorem 4.1. □

Remark 4.3. At this point it is easy to give a direct proof of the "only if" part of Theorem 1.2. Indeed, let A be a C^*-algebra and assume that the map Q_X defined in Theorem 1.2 is a complete metric surjection for all operator spaces X. Then it follows from the proof of Theorem 4.1 that $\hat{u}$ is bounded on $A \otimes_{\min} [u(A)]'$ for any bounded homomorphism u. It therefore follows from Lemma 1.1 that A is nuclear.

We now come to our second application of Theorem 1.2, which is a factorization result for A-valued matrices, when A is a nuclear C^*-algebra. This result had been established by Pisier [**19**, Remark 4.6] in the case when A is the space of compact operators on Hilbert space and left open in the general case.

Corollary 4.4. *Let A be a nuclear C^*-algebra. For any $n \geq 1$ and any $a \in M_n(A)$ with $\|a\| < 1$ there exist, for some integer $N \geq 1$, three scalar valued matrices $\alpha_0 \in M_{n,N}$, $\alpha_1 \in M_N$, $\alpha_2 \in M_{N,n}$ and two A-valued* diagonal *matrices $D_1, D_2 \in M_N(A)$ such that*

$$a = \alpha_0 D_1 \alpha_1 D_2 \alpha_2,$$
$$\|\alpha_0\| \, \|D_1\| \, \|\alpha_1\| \, \|D_2\| \, \|\alpha_2\| < 1.$$

Proof. Applying Theorem 1.2 with $X = \mathbf{C}$, we obtain that the multiplication mapping on A induces a complete metric surjection from $\max(A) \otimes_h \max(A)$ onto A. The result therefore follows from the implication (v) $\Rightarrow$ (vi) of [**19**, Theorem 4.2] and its proof. $\square$

Acknowledgments. This work was deeply influenced by Gilles Pisier through conversations on the contents of his paper [**19**]. In particular, I am grateful for his explanations about the result presented in Remark 4.6 of [**19**], which led to Proposition 2.1.

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