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FUNDAMENTAL GROUPS OF MANIFOLDS WITH LITTLE NEGATIVE CURVATURE

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ABSTRACT. We prove that, for a class of manifolds with diameter bounded above, systol bounded below and Ricci curvature almost nonnegative except for a well of small diameter, the fundamental group is of polynomial growth.

Let M be a compact Riemannian manifold. Given a finite set of generators for a finitely generated group G, the growth function $\gamma(s)$ is defined to be the number of distinct words in the generators and their inverses of length at most s. G is said to have polynomial growth of degree $\leq k$ if there exists $C > 0$ such that for all integers $s \geq 1$,

$$\gamma(s) \leq Cs^k.$$

The systol $\operatorname{sys}(M)$ is the lower bound of the lengths of noncontractible closed geodesics in M. Note that if the injectivity radius of M is bounded below by i_0, then the $\operatorname{sys}(M)$ has to be bounded below by $2i_0$.

The following theorem was proved in [**1**]:

Theorem 1. *Given $n, L > 0$, there exists a constant $\varepsilon = \varepsilon(n, L) > 0$, depending only on n and L such that if a closed n-manifold M admits a metric satisfying the conditions* $\operatorname{diam}(M) = 1$*,* $\operatorname{sys}(M) \geq L$ *and* $\operatorname{Ric}(M) \geq -\varepsilon$*, then the fundamental group of M is of polynomial growth with degree $\leq n$.*

It is clear that the theorem also holds if one replaces the lower bound in $\operatorname{sys}(M)$ by lower bound in volume. The aim of this paper is to show that the condition of almost nonnegative curvature in the above

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theorem can indeed by relaxed to allow for little negative curvature on a well in M. We shall use the idea developed in [**3**].

Theorem 2. *Given $n, K, D, i_0 > 0$, there exist positive numbers $\varepsilon = \varepsilon(n, i_0, D)$ and $\varepsilon^* = \varepsilon^*(n, K, D, i_0)$ such that if a complete manifold M admits a metric satisfying the conditions diameter $d(M) \leq D$, systol* $\operatorname{sys}(M) \geq i_0$, *and* Ric $\geq -\varepsilon$ *except a set of diameter smaller than ε^* on which* Ric $\geq -K$, *then $\pi_1(M)$ has a polynomial growth of degree $\leq n$.*

Proof. In the class of compact n-dimensional Riemannian manifolds satisfying

$$\operatorname{Ric} M \geq -(n-1)K^2, \ \operatorname{sys}(M) \geq i_0, \ \operatorname{diam}(M) \leq D,$$

there are only finitely many isomorphism classes for $\pi_1(M)$. Indeed, this is a theorem in [**1**]. Suppose that $\widetilde{M}$ is the universal covering of M with base point $\tilde{p} : \rho : \widetilde{M} \to M$, and let $p = \rho(\tilde{p})$. It is also well known that we can indeed find a set of generators $\{g_i\}_{i=1}^N$ for $\pi_1(M)$ such that

$$d(g_i\tilde{p}, \tilde{p}) \leq 3D$$

for all i and every relation is of the form

$$g_{i_1} g_{i_2} = g_{i_3}.$$

We shall prove our theorem by contradiction. Suppose that the theorem is false. For arbitrary fixed $\varepsilon > 0$, there exists a sequence of Riemannian manifolds M_j such that $\operatorname{diam}(M_j) \leq D$, $\operatorname{sys}(M_j) \geq i_0$ and $\operatorname{Ric} M_j \geq -\varepsilon$ except on a set A_j of diameter $1/j$ and $\pi_1(M_j)$ is not of polynomial growth with degree $\leq n$.

Let $s \geq 1$ be an integer. Let $\Gamma_j(s) = \{g \mid g \in \pi_1(M_j)$ with length $\leq s\}$ and $\gamma_j(s) = \#\Gamma_j(s)$. Then $d(\tilde{p}_j, g\tilde{p}_j) \leq 3sD$ for $g \in \Gamma_j(s)$. Furthermore, it is clear that

$$B_{g_\alpha \tilde{p}_j}(i_0/2) \cap B_{g_\beta \tilde{p}_j}(i_0/2) = \varnothing \quad \text{for } \alpha \neq \beta.$$

Thus,

$$\bigcup_{g \in \Gamma_j(s)} g(B_{\tilde{p}_j}(i_0/2)) \subset B_{\tilde{p}_j}(3sD + i_0/2).$$

Therefore,

$$\sum_{g\in\Gamma(s)} \operatorname{vol}(g(B_{\tilde{p}_j}(i_0/2))) \leq \operatorname{vol}(B_{\tilde{p}_j}(3sD+i_0/2)) \leq \operatorname{vol}(B_{\tilde{p}_j}(4sD)).$$

Let $\tilde{A}_j = \rho_j^{-1}(A_j)$ and $\tilde{A}_j(r) = \tilde{A}_j \cap B_{\tilde{p}_j}(r)$. There in fact exists a uniform bound on the number of bad sets $\tilde{A}_j(4sD)$. This can be seen from the following lemma which is a generalization to Theorem 2 in [**4**].

Lemma 1. *Given positive numbers $K, D, i_0, \delta < D$ and an integer n, there exists an explicit number $m = m(K, D, i_0, \delta, n)$ such that if M is a compact n-dimensional manifold satisfying the bounds*

$$\operatorname{Ric} M \geq -(n-1)K^2,\ \operatorname{diam}(M) \leq D,\ \operatorname{sys}(M) \geq i_0,$$

then, for any $\tilde{p} \in \widetilde{M}$, the set $G = \{g \in \pi_1(M) \mid d(\tilde{p}, g\tilde{p}) < \delta)$ has at most m elements.

Proof of Lemma 1. Let $p = \pi(\tilde{p})$. Given $r \leq D$,

$$\frac{\operatorname{vol}(B_{\tilde{p}}(D))}{\operatorname{vol}(B_{\tilde{p}}(r))} \leq \frac{\operatorname{vol}(B(D))}{\operatorname{vol}(B(r))}$$

where $B(D)$ and $B(r)$ are the balls of radius D and r, respectively, in the space form of constant curvature $-K^2$. It is clear that

$$B_{g_\alpha\tilde{p}}(i_0/2) \cap B_{g_\beta\tilde{p}}(i_0/2) = \varnothing \quad \text{for } \alpha \neq \beta,$$

and hence,

$$\bigcup_{g\in G} g(B_{\tilde{p}}(i_0/2)) \subset B_{\tilde{p}}(2\delta).$$

Therefore,

$$\sum_{g\in G} \operatorname{vol}(g(B_{\tilde{p}}(i_0/2))) \leq \operatorname{vol}(B_{\tilde{p}}(2\delta)).$$

One has

$$\#G \leq \frac{\operatorname{vol}(B_{\tilde{p}}(2\delta))}{\operatorname{vol}(B_{\tilde{p}}(i_0/2))} \leq \frac{\operatorname{vol}(B(2\delta))}{\operatorname{vol}(B(i_0/2))} = m(K, D, i_0, \delta, n).$$

This completes the proof of the lemma.

The lemma implies that the bad part $\tilde{A}_j(4sD)$ in $B_{\tilde{p}_j}(4sD)$ in $\tilde{M}_j$ can be covered with at most m ε_j^*-balls in $\widetilde{M}_j$ where m depends only on K, D, i_0, n and s. It is then clear that

$$\lim_{j\to\infty} \mathrm{vol}\,(B_{\tilde{p}_j}(4sD)) \leq \mathrm{vol}\,(B(4sD)),$$

where $B(4sD)$ is the $4sD$-ball in the space form of sectional curvature $-\varepsilon/(n-1)$. By Bishop volume comparison theorem,

$$\gamma(s) \leq \frac{\mathrm{vol}\,(B_{\tilde{p}_j}(4sD))}{\mathrm{vol}\,(B_{g\tilde{p}_j}(i_0/2))} \leq \frac{\mathrm{vol}\,(B(4sD))}{\mathrm{vol}\,(B(i_0/2))}.$$

From Theorem 3 of [**1**], we know that ε can be chosen so that the last term is of the order cs^n where c depends only on n, i_0 and D. This contradicts the previous assumption and hence completes our proof. □

REFERENCES

1. Z. Shen and G. Wei, *On Riemannian manifolds of almost nonnegative curvature*, Indiana Univ. Math. J. **40** (1991), 551–565.

2. G. Wei, *On the fundamental groups of manifolds with almost nonnegative Ricci curvature*, Proc. Amer. Math. Soc. **110** (1990), 197–199.

3. J. Wu, *Complete manifolds with a little negative curvature* II, Amer. J. Math. **114** (1992), 649–656.

4. ———, *Complete manifolds with a little negative curvature*, Amer. J. Math. **113** (1991), 567–572.

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