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MIDDLE SEMICONTINUITY FOR UNBOUNDED OPERATORS

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ABSTRACT. Let A be a C^*-algebra and K_A its Pedersen's ideal. By making use of Mack's characterization of PCS-algebra and Phillips' new description of multipliers of K_A, [**14, 18**], we generalize the concept of middle semicontinuity [**6**] to the case of unbounded operators affiliated with A^{**}, the enveloping von Neumann algebra of A. Especially we obtain the unbounded version of a Dauns-Hofmann type theorem [**15**, Theorem 4.6] and a middle interpolation theorem [**6**, Theorem 3.40].

1. Introduction and preliminaries. Let A be a C^*-algebra and A^{**} its enveloping von Neumann algebra. The theory of semicontinuous operators in A^{**} was developed by Pedersen, Akemann and Brown [**2, 6, 15**]. This paper is a sequel to [**12**] which generalizes the theory of strong semicontinuity. We will adopt the same notations from it. In this paper the concept of middle semicontinuity is generalized for unbounded operators affiliated with A^{**}.

Let $M(A)$ denote the multiplier algebra of A and K_A the Pedersen's ideal (minimal dense ideal) of A. If A is commutative, that is, $A = C_0(X)$, the algebra of all complex valued continuous functions which vanish at infinity on some locally compact space X, then $M(A)$, respectively K_A, can be identified with $C_b(X)$, respectively $C_c(X)$, the algebra of all complex value bounded, respectively compactly supported, continuous functions on X. As a noncommutative generalization of the relation between $C_c(X)$ and its multiplier algebra $C(X)$, Lazar and Taylor [**13**] introduced $\Gamma(K_A)$, the multipliers (double centralizers) of Pedersen's ideal K_A and made an extensive study of it.

In [**18**], Phillips gave a new description of $\Gamma(K_A)$ as an inverse limit of C^*-algebras (pro C^*-algebra) and derived a number of the results of [**13**] directly from corresponding facts about inverse limits of C^*-algebras.

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Theorem 1.1 (Phillips [**18**, Theorem 4]). *Let A be a C^*-algebra. Then, for any approximate identity (e_λ) for A contained in K_A, we have*

$$\Gamma(K_A) \cong \varprojlim_{\alpha \in (K_A)_+} M(I_a) \cong \varprojlim_{\lambda} M(I_{e_\lambda})$$

where I_b is the closed two sided ideal generated by b.

Remark. This theorem enables us to consider the elements of $\Gamma(K_A)_{sa}$ as (unbounded) self-adjoint operators on the universal Hilbert space H_u of A. If h is in $\Gamma(K_A)_{sa}$, then there exists h_λ in $M(I_{e_\lambda})$ for all λ such that $h_\mu p_\lambda = h_\lambda$ for $\mu \geq \lambda$ where p_λ is the open central projection corresponding to I_{e_λ}; and h can be identified with the net (h_λ). For each λ, h_λ gives a projection valued measure $E_S^\lambda(h_\lambda)$ on $p_\lambda H_u$. Note that $(E_S^\lambda(h_\lambda))_\lambda$ is an increasing net of projections in A^{**} for every Borel set $S \subset \mathbf{R}$. Now we let $E(S)$ be the limit projection of $(E_S^\lambda(h_\lambda))_\lambda$ in A^{**}. Then $(E(S))$ forms a projection valued measure on H_u. Hence the operator that corresponds to $(E(S))$ is a densely defined self-adjoint operator on H_u and will be denoted again by h. Then $hp_\lambda = h_\lambda$ for all λ and ah, $hb \in K_A$ for all a and b in K_A.

A subset C of a topological space X, not necessarily Hausdorff, is called *relatively* (*quasi-*) *compact* if C is contained in a (quasi-) compact subset of X. Throughout this paper Λ will denote the set of all relatively compact open subsets of $\operatorname{Prim} A$, the primitive ideal space of A with hull-kernel topology. From [**13**, Lemma 5.39] it follows immediately that $\operatorname{Prim}(I_a)$ belongs to Λ for all a in $(K_A)_+$. Applying [**18**, Lemma 5], we see that $(C)_\Lambda$ forms an increasing cofinal net where Λ is ordered by set inclusion, and so we have the following:

Corollary 1.2. *Let $I(C)$ be the closed two sided ideal of A corresponding to $C \in \Lambda$. Then*

$$\Gamma(K_A) \cong \varprojlim_{C \in \Lambda} M(I(C)).$$

A topological space X is called *pseudocompact* if every continuous real valued function on X is bounded. When A is commutative and has pseudocompact spectrum the equality $\Gamma(K_A) = M(A)$ holds, i.e., every multiplier of Pedersen's ideal K_A is bounded. By Lazar and Taylor, A is called a *PCS-algebra* if $\Gamma(K_A) = M(A)$ holds. We refer the readers to [**13**] and [**14**] for the characterization of PCS-algebras. In [**13**], the authors asked whether or not it is true that $\Gamma(K_A) = M(A)$ if and only if $\operatorname{Prim} A$ is pseudocompact. Mack [**14**] has shown that the correct condition for A to be a PCS-algebra is that $\operatorname{Prim} A$ be weakly compact. Recall that X is called weakly compact if each infinite pairwise disjoint family of nonempty open subsets has an accumulation point or equivalently for each countable open cover $\{U_n\}$ of X, the associated closed cover $\{\overline{U}_n\}$ admits a finite subcover.

Lemma 1.3. *Let $\check{A}$ denote $\operatorname{Prim} A$. Then $A/I(\check{A} - \overline{C})$ is a PCS-algebra for all C in Λ.*

Proof. Note that $\operatorname{Prim}(A/I(\check{A} - \overline{C})) = \overline{C}$. Then it is easy to show that $\overline{C}$ is weakly compact since C is relatively compact. □

Theorem 1.4. *Let $\check{A}$ denote $\operatorname{Prim} A$. Then*

$$\Gamma(K_A) \cong \varprojlim_{C \in \Lambda} M(A/I(\check{A} - \overline{C}));$$

i.e., $\Gamma(K_A)$ is isomorphic to the inverse limit of the multiplier algebras $M(A/J)$ as J runs through all closed ideals of A such that the primitive ideal space $\operatorname{Prim}(A/J)$ is the closure of a relatively compact open subset of $\operatorname{Prim} A$.

Remark. This is a corrected version of [**18**, Theorem 7], which may not be true when $\operatorname{Prim} A$ is not Hausdorff.

Proof. For an open subset D of $\operatorname{Prim} A$, we denote by p_D the open central projection corresponding to the ideal $I(D)$ and let $p_{\overline{D}} = 1 - p_{(\check{A} - \overline{D})}$. Note that the map $x \mapsto x p_{\overline{C}}$, $C \in \Lambda$, is a homomorphism from $\Gamma(K_A)$ into $\Gamma(K_{A/I(\check{A} - \overline{C})}) = M(A/I(\check{A} - \overline{C}))$. These maps obviously give an injective homomorphism

$$\Phi : \Gamma(K_A) \longrightarrow \varprojlim_{C \in \Lambda} M(A/I(\check{A} - \overline{C}))$$

since if $xp_{\overline{C}} = 0$ for all $C \in \Lambda$, then $x = 0$. To show the surjectivity of Φ, note that $I(C)$ can be regarded as an essential ideal of $A/I(\check{A} - \overline{C})$. Hence we have an isometric isomorphism $x \mapsto xp_C$ from $M(A/I(\check{A} - \overline{C}))$ into $M(I(C))$. By Corollary 1.2 above, this implies Φ surjective. $\square$

2. Definition of MLSC(A). The generalization of strong semicontinuity was quite smooth due to the cooperation of the quasi-state space $Q(A)$ and the theory of unbounded quadratic forms, see [**12**]. But for the concept of middle semicontinuity, there are some difficulties even though we have several candidates. In view of the theory of multipliers $\Gamma(K_A)$ of Pedersen's ideal K_A we can naturally expect $\Gamma(K_A)$ to be the set of unbounded middle continuous elements. There are several possibilities for a definition of middle semicontinuity, such as semicontinuous affine functions on $Q(A)$ or $S(A)$, q-semicontinuity, analogues of conditions for bounded middle semicontinuity, and Phillips' description of $\Gamma(K_A)$. Moreover, the commutative case is highly suggestive. In [**6**] Brown already pointed out that q-semicontinuity should not be considered as the basic notation. We consider the following list of five conditions on an unbounded self-adjoint operator h, possibly not densely defined, affiliated with A^{**}:

(M1) For all $C \in \Lambda$, there exists $\lambda_C > 0$ such that $(h + \lambda_C)p_C \in SLSC(I(C))_+$.

(M2) For all $C \in \Lambda$, there exists $\lambda_C > 0$ such that $(h + \lambda_C)p_{\overline{C}} \in SLSC(A/I(\check{A} - \overline{C}))_+$.

(M3) There exists (h_i) with h_i in $M(I(C_i))_{sa}$ such that $C_i \nearrow \operatorname{Prim} A$ in Λ; if $i \le i'$, then $h_i \le h_{i'}p_{C_i}$, and $h_i p_C \nearrow hp_C$ for all $C \in \Lambda$.

(M4) There exists (h_i) with h_i in $M(A/I(\check{A} - \overline{C}_i))_{sa}$ such that $C_i \nearrow \operatorname{Prim} A$ in Λ; if $i \le i'$ then $h_i \le h_{i'}p_{\overline{C}_i}$ and $h_i p_{\overline{C}} \nearrow hp_{\overline{C}}$, for all $C \in \Lambda$.

(M5) There exists x in $\Gamma(K_A)_+$ such that $h + x \in SLSC(A)_+$.

Proposition 2.1. (M5) $\Rightarrow$ (M4) $\Rightarrow$ (M3) $\Rightarrow$ (M2) $\Rightarrow$ (M1).

Proof. (M5) $\Rightarrow$ (M4). Let $(a_j + \lambda_j 1)_{j \in D}$ be a net in $\tilde{A}$ such that $a_j + \lambda_j 1 \nearrow h + x$ and $\lambda_j \nearrow 0$. Let $I = \Lambda \times D$. Then I is a directed set with the partial order defined by

$$(C,j) \leq (C',j') \Leftrightarrow C \subset C' \quad \text{and} \quad j \leq j'.$$

For $i = (C,j)$ in I, let $h_i = (a_j + \lambda_j 1 - x)p_{\overline{C}}$. Then h_i is in $M(A/I(\check{A} - \overline{C}))_{sa}$, by Lemma 1.3, and $h_i p_{\overline{C}} \nearrow h p_{\overline{C}}$ for all C in Λ.

(M4) $\Rightarrow$ (M3). Note that $I(C)$ can be regarded as an essential ideal of $A/(\check{A} - \overline{C})$. Hence the map $x \mapsto x p_C$ is an isometric isomorphism from $M(A/I(\check{A} - \overline{C}))$ into $M(I(C))$.

(M3) $\Rightarrow$ (M2). Assume $(h_i)_I$ satisfies the conditions in (M3). For any given C in Λ, there exists i_0 such that $C \subset C_{i_0}$. Let $\lambda_C = \|h_{i_0}\|$, then $(h_i + \lambda_C)p_{\overline{C}} p_{C_i} \geq 0$ for all $i \geq i_0$ since $I(C)$ is essential in $A/I(\check{A} - \overline{C})$. Let $B = A/I(\check{A} - \overline{C})$, and let Λ_B denote the set of relatively compact open subsets of $\mathrm{Prim}\, B$. For any fixed D in Λ_B, there is a C_D in Λ such that $D = C_D \cap \overline{C}$. Let $J(D)$ be the ideal of B corresponding to D. Since $h_i p_{C_D} \nearrow h p_{C_D}$ and $(h_i + \lambda_C)p_{\overline{C}} p_{C_i} \geq 0$ for i sufficiently large, we have $(h_i + \lambda_C)p_{\overline{C}} p_{C_D} \nearrow (h + \lambda_C)p_{\overline{C}} p_{C_D}$ and $(h_i + \lambda_C)p_{\overline{C}} p_{C_D} \in M(J(D))_+$ for i sufficiently large by [**6**, Proposition 2.18]. Note that $M(J(D))_+ \subset J(D)_+^m \subset SLSC(J(D))_+$. Hence $(h + \lambda_C)p_{\overline{C}} p_{C_D} \in M(J(D))_+^M \subset SLSC(J(D))_+^M = SLSC(J(D))_+$ by [**12**, Corollary 3.9]. Therefore, $(h + \lambda_C)p_{\overline{C}} \in SLSC(B)_+$ by [**12**, Theorem 3.19].

(M2) $\Rightarrow$ (M1) follows from [**12**, Proposition 3.13]. $\square$

Proposition 2.2. (a) *For the commutative C^*-algebra $A = C_0(X)$,* (M1)–(M4) *are all equivalent. For h not necessarily densely defined, all describe the set of $(-\infty, \infty]$-valued lower semicontinuous functions on X. For densely defined h they describe the set of* **R** *-valued lower semicontinuous functions on X.*

(b) (M5) *describes the set of lower semicontinuous functions on X which are bounded below by an* **R***-valued continuous function.*

Proof. (a) Assume h satisfies (M1). Note that every bounded strongly lower semicontinuous element is determined completely by its atomic part by [**16**, Theorem 4.3.15]. Hence, by [**12**, Theorem 3.6], $(h + \lambda_C)p_C$ is determined by $z_{at}h$ for all $C \in \Lambda$, and so is h. Now it is easy to deduce from [**12**, Example 3.4A] that h corresponds to a $(-\infty, \infty]$-valued lower semicontinuous function on X. Consider the set I of all finite collections of elements in Λ. For any singleton, $i = \{C\}$ in I, let h_i be the constant function $x \mapsto \min_{y \in \overline{C}} h(y)$ on $\overline{C}$. For $i = \{C_1, C_2\}$, let $C_i = C_1 \cup C_2$ and $k_i(x) = \max\{h_m(x) \mid m = \{C_1\}$ or $\{C_2\}$ and $h_m(x)$ defined$\}$ on $\overline{C}_i$. Then $\overline{C}_i$ is compact Hausdorff, hence normal, k_i is upper semicontinuous on $\overline{C}_i$ and $k_i \le h|_{\overline{C}_i}$. Therefore, we can find a bounded continuous function h_i on $\overline{C}_i$ such that $k_i \le h_i \le h|_{\overline{C}_i}$. Assume that h_i has been selected for all i of order less than n. For $i = \{C_1, C_2, \dots, C_n\}$ in I, let $C_i = C_1 \cup \cdots \cup C_n$ and $k_i(x) = \max\{h_j(x) \mid j \subsetneqq i$ and $h_j(x)$ defined$\}$ on $\overline{C}_i$. Then $\overline{C}_i$ is normal, k_i is upper semicontinuous on $\overline{C}_i$ and $k_i \le h|_{\overline{C}_i}$. Choose a continuous function h_i on $\overline{C}_i$ such that $k_i \le h_i \le h|_{\overline{C}_i}$. By induction we can construct a net $(h_i)_{i \in I}$ where h_i is in $C_b(\overline{C}_i) \cong M(A/I(X - \overline{C}_i))$ such that $h_i \nearrow h$ pointwise. Therefore h satisfies (M4).

If h is densely defined as an operator on H_u, then $h(x) \in \mathbf{R}$ for all x in X. This shows the last statement.

(b) If f is a lower semicontinuous function on X such that $f \ge g$ for some continuous real valued g, then $f - (g \wedge 0)$ is positive and lower semicontinuous on X. Therefore f satisfies (M5). The converse is clear, since $\Gamma(K_A)$ is identified with $C(X)$. □

Example. We have an example that shows (M4) $\not\Rightarrow$ (M5), see [**9**, p. 97]. Let $X = \beta\mathbf{R} - (\beta\mathbf{N} - \mathbf{N})$ where β indicates the Stone Čech compactification. Then X is a locally compact Hausdorff space which is pseudocompact but not countably compact (and not normal). So there is a lower semicontinuous function f on X which is not bounded below. Since X is pseudocompact, f does not satisfy (M5). Note that X must be non-σ-compact, i.e., $C_0(X)$ non-σ-unital, for such an example.

Proposition 2.3. *If X is a normal, countably paracompact, locally compact Hausdorff space, then all of* (M1)–(M5) *are equivalent for $A = C_0(X)$.*

Proof. The given condition on X implies that if f_1, respectively f_2, is a lower, respectively upper, semicontinuous real function on X such that $f_1 > f_2$, then there exists a continuous real function g such that $f_1 > g > f_2$, see [**8**, Theorem 4].

Let h be a $(-\infty, \infty]$-valued lower semicontinuous function on X. Then the function f defined by $f(x) = \mathrm{Tan}^{-1}(h(x))$, for $h(x) < \infty$, and $f(x) = (\pi/2)$, for $h(x) = \infty$, is lower semicontinuous on X such that $-(\pi/2) < f \leq (\pi/2)$. Thus there exists a continuous function g such that $-(\pi/2) < g < f \leq (\pi/2)$. Then the function $\phi = \tan \circ g$ is continuous real valued such that $\phi < h$ on X. Therefore h satisfies (M5) and we are done. □

Proposition 2.4. *If A is σ-unital,* (M4) *is equivalent to* (M5).

Proof. Assume h satisfies (M4), i.e., there is a net (h_i), $i \in I$, with h_i in $M(A/I(\check{A} - \overline{C}_i))_{sa}$ such that $C_i \nearrow \mathrm{Prim}\, A$; if $i \leq i'$, then $h_i \leq h_{i'} P_{\overline{C}_i}$ and $h_i p_{\overline{C}} \nearrow h p_{\overline{C}}$ for all $C \in \Lambda$. Since A is σ-unital, there exists an increasing sequence (C_n) in Λ such that $C_n \nearrow \mathrm{Prim}\, A$. Choose i_1 such that $C_{i_1} \supset C_1$, and let $h_1 = p_{\overline{C}_1} h_{i_1}$. Choose i_2 such that $i_2 \geq i_1$ and $C_{i_2} \supset C_2$ and let $h_2 = p_{\overline{C}_2} h_{i_2}$. By [**17**, Theorem 10], the canonical map $\rho : M(A/I(\check{A} - \overline{C}_2)) \to M(A/I(\check{A} - \overline{C}_1))$ is a surjective homomorphism. Choose m_1, m_2 in $\mathbf{R}$ such that $m_1 \leq h_1$, $m_2 \leq h_2$ and $m_2 \leq m_1 \leq 0$. Then we have

$$\rho(m_2) = m_2 \rho_{\overline{C}_1} \leq m_1 p_{\overline{C}_1} \leq h_1 \leq h_2 p_{\overline{C}_1} = \rho(h_2).$$

Applying [**16**, Proposition 1.5.10], we can find y_2 in $M(A/I(\check{A} - \overline{C}_2))_{sa}$ such that $\rho(y_2) = p_{\overline{C}_1} y_2 = h_1$ and $m_2 \leq y_2 \leq h_2 \leq h p_{\overline{C}_2}$. Continuing this process, we can find a sequence (y_n) where y_n is in $M(A/I(\check{A} - \overline{C}_n))_{sa}$ such that $y_n \leq h_n \leq h p_{\overline{C}_n}$ and $y_n p_{\overline{C}_{n-1}} = y_{n-1}$. By Theorem 1.4 above, (y_n) corresponds to a multiplier y in $\Gamma(K_A)_{sa}$ such that $y p_{\overline{C}_n} = y_n$. Let $x = y_-$. Then $x \in \Gamma(K_A)_+$, by [**13**, Lemma 5.14], and $-x p_{\overline{C}_n} \leq h_n \leq h p_{\overline{C}_n}$.

Now it is enough to show that $(h + x) p_{C_n} \in SLSC(I(C_n))_+$ for all $n \in \mathbf{N}$ by [**12**, Theorem 3.19]. Since $h_n \geq -x p_{\overline{C}_n}$,

$$(h_i + x) p_{\overline{C}_n} \in M(A/I(\check{A} - \overline{C}_n))_+ \subset (A/I(\check{A} - \overline{C}_n))_+^m,$$

for $i \geq i_n$ and $(h_i + x) p_{\overline{C}_n} \nearrow (h + x) p_{\overline{C}_n}$. Therefore, $(h + x) p_{\overline{C}_n} \in SLSC(A/I(\check{A} - \overline{C}_n))_+$ and hence $(h + x) p_{C_n} \in SLSC(I(C_n))_+$ by [**12**, Proposition 3.12]. □

If $\operatorname{Prim} A$ is Hausdorff, then it is easy to see that (M1) is equivalent to (M2) and (M3) is equivalent to (M4). But the following example shows that (M1) does not imply (M2) in general.

Example. Let $\mathcal{K}$ be the set of all compact operators on a separable infinite dimensional Hilbert space H and A an extension of $\mathcal{K}$ by c_0 in $B(H)$, i.e., $A = B + \mathcal{K}$ where $B = \{\sum_{n=1}^{\infty} \alpha_n p_n \mid (\alpha_n) \in c_0\}$ and (p_n) is a sequence of infinite dimensional mutually orthogonal projections such that $\sum p_n = 1$. Note that $A^{**} \cong \mathcal{K}^{**} \oplus c_0^{**} \cong B(H) \oplus l_\infty$ and $\operatorname{Prim} A$ is homeomorphic to the space $\{0\} \cup \mathbf{N}$ with the topology generated by $\{0\}$ and $\{0, n\}$, $n \in \mathbf{N}$. Let $h = 0 \oplus (-1, -2, -3, \dots)$. Then h satisfies (M1) but not (M2) since $\{\bar{0}\} = \{0\} \cup \mathbf{N} = \operatorname{Prim} A$ and h is not bounded below. Also note that $h \eta \mathcal{Z}$.

Proposition 2.5. *Let A be a σ-unital C^*-algebra with* $\operatorname{Prim} A$ *Hausdorff. Then the conditions* (M1)–(M5) *are all equivalent.*

Proof. We will show that (M2) $\Rightarrow$ (M5). Since $\operatorname{Prim} A$ is σ-compact and Hausdorff, we can find an increasing sequence (C_n) in Λ such that $C_n \nearrow \operatorname{Prim} A$ and $\overline{C}_n \subset C_{n+1}$ for all $n \in \mathbf{N}$. If h satisfies (M2), then there exists $\lambda_n > 0$ such that $(h + \lambda_n) p_{\overline{C}_n} \in SLSC(A/I(\check{A} - \overline{C}_n))_+$. We may assume that $\lambda_n \nearrow \infty$. Then the function

$$g = (-\lambda_1)\mathcal{X}_{C_1} + \sum_{n=2}^{\infty} (-\lambda_n) \mathcal{X}_{C_n - C_{n-1}}$$

is lower semicontinuous on $\operatorname{Prim} A$. Then we can find a continuous function f on $\operatorname{Prim} A$ such that $f < g < 0$ since $\operatorname{Prim} A$ is σ-compact, locally compact Hausdorff. Let x be the operator corresponding to $-f$. Then $x \eta \mathcal{Z}$ and $x \in \Gamma(K_A)_+$. Now we will show that $h + x \in SLSC(A)_+$, or equivalently $(h + x) p_{C_n} \in SLSC(I(C_n))_+$, for all n in $\mathbf{N}$. For $n = 1$,

$$(h + x) p_{C_1} = (h + \lambda_1) p_{C_1} + (x - \lambda_1) p_{C_1} \in SLSC(I(C_1))_+$$

since $(x - \lambda_1) p_{C_1} \in M(I(C))_+ \subset SLSC(I(C))_+$. Assume that $(h + x) p_{C_{n-1}}$ belongs to $SLSC(I(C_{n-1}))_+$. By the choice of x, $\check{x}^{-1}[0, \lambda_n]$ is a closed subset of C_{n-1}. Let $D_n = C_n \backslash \check{x}^{-1}[0, \lambda_n]$. Then D_n is open in C_n and $D_n \cup C_{n-1} = C_n$. Hence we have $I(C_{n-1}) +$

$I(D_n) = I(C_n)$. Since $(h+x)p_{D_n} = (h+\lambda_n)p_{D_n} + (x-\lambda_n)p_{D_n}$ and $(x-\lambda_n)p_{D_n} \in M(I(D_n))_+ \subset SLSC(I(D_n))_+$, $(h+x)p_{D_n} \in SLSC(I(D_n))_+$. Applying [**12**, Proposition 3.13], we get $(h+x)p_{C_n} \in SLSC(I(C_n))_+$. So we are done. □

We have seen that (M5) is strictly stronger than (M1)–(M4) even in the commutative case. So far, it is not completely clear which one is the best choice, but our preference is for (M5). One thing we should note is that only (M5) gives an affirmative answer, see Proposition 2.7 below, to the first question (Q1) in [**6**]: Is every lower semicontinuous element the limit of a monotone increasing net of continuous elements? (It was trivially yes for the bounded case.) Also considering the relation with q-semicontinuity, Theorem 3.4 below, we like to choose (M5) at least in the σ-unital case. Furthermore it is pleasing that it is described globally. For a bounded operator h, even (M5) does not imply $h \in \tilde{A}^m_{sa}$, but we believe that (M5) is a better concept of middle semicontinuity than "$h \in \tilde{A}^m_{sa}$." For example, when $\oplus_i A_i$, a c_0-direct sum, the new concept behaves better, see [**6**, Proposition 2.11].

Definition 2.6. Let h be a selfadjoint operator, not necessarily densely defined, such that $h\eta A^{**}$. Then h is called *unbounded middle lower semicontinuous*, $h \in MLSC(A)$, if h satisfies (M5), i.e., there exists x in $\Gamma(K_A)_+$ such that $h + x$ is in $SLSC(A)_+$. Also h is called *unbounded middle upper semicontinuous* ($h \in MUSC(A)$) if $-h$ is in $MLSC(A)$. As a special case, we denote by $MLSC^d(A)$, respectively $MUSC^d(A)$, the set of all densely defined h in $MLSC(A)$, respectively $MUSC(A)$.

Remark. If A is a PCS-algebra, then

$$h \in MLSC(A) \Longrightarrow h \text{ is bounded below}$$

by the definition of PCS-algebra. In other words, if we assume boundedness for continuous elements, then we would have semiboundedness for semicontinuous elements. (This would be false if we used (M4) instead of (M5) by the example after Proposition 2.2 above.)

Notation. For self-adjoint operators h and k affiliated with A^{**}, we write $h \geq k$ if $hp_C \geq kp_C$ for all C in Λ. For a net (h_i), write $h_i \nearrow h$

if $h_i p_C \nearrow h p_C$ for all C in Λ. These agree with the same notations in [**12**] if all the operators are bounded below.

Proposition 2.7. *$h \in MLSC(A)$ if and only if there exists a net (h_i) in $\Gamma(K_A)_{sa}$ such that $h_i \nearrow h$.*

Proof. Let $x \in \Gamma(K_A)_+$ be such that $h + x \in SLSC(A)_+$. Then there exists a net $(a_i + \lambda_i 1)$ in $\tilde{A}_{sa}$ such that $a_i + \lambda_i 1 \nearrow h + x$. Let $h_i = a_i + \lambda_i 1 - x$. Then $h_i \in \Gamma(K_A)_{sa}$ and $h_i \nearrow h$.

For the converse, let $h_i \nearrow h$ for a net (h_i) in $\Gamma(K_A)_{sa}$. Fix i_0. Since $h_i - h_{i_0} \in \Gamma(K_A)_+ \subset SLSC(A)_+$ for $i \geq i_0$ and $h_i - h_{i_0} \nearrow h - h_{i_0}$, we have $h - h_{i_0} \in SLSC(A)_+$. $\square$

Proposition 2.8. $\Gamma(K_A)_{sa} = MLSC^d(A) \cap MUSC^d(A)$.

Proof. If $h \in MLSC^d(A) \cap MUSC^d(A)$, then hp_C is bounded for all C in Λ, and so

$$\begin{aligned} hp_C &\in M(I(C))_{sa}^m \cap (M(I(C))_{sa})_m \\ &= \widetilde{I(C)}_{sa}^m \cap ((\widetilde{I(C)}_{sa}))_m \\ &= M(I(C))_{sa} \end{aligned}$$

by [**6**, Proposition 2.8] and [**15**, Theorem 2.5]. This means that $h \in \Gamma(K_A)_{sa}$ by Corollary 1.2 above.

The other direction is obvious. $\square$

Remark. This means that the middle continuous elements are just the multipliers of Pedersen's ideal as we expected. The conditions (M1)–(M4) have this property too.

Proposition 2.9. *Let I be an ideal of A with open central projection z.*

(a) $h \in MLSC(A) \Rightarrow zh \in MLSC(I)$.

(b) $h \in MLSC(A)_+ \Rightarrow zh \in MLSC(A)_+$ *and* $zh \in MLSC(I)_+$.

Proof. (a) Let $x \in \Gamma(K_A)_+$ such that $h + x \in SLSC(A)_+$. By [**12**, Proposition 3.12], $z(x+h) \in SLSC(I)_+$. Since zx is in $\Gamma(K_I)_+$, we have $zh \in MLSC(I)$.

(b) Let $h \in MLSC(A)_+$ such that $h + x \in SLSC(A)_+$ for some x in $\Gamma(K_A)_+$. It is enough to show that $zh + x \in SLSC(A)_+$. To see this, let $\varphi_\alpha \to \varphi$ in $Q(A)$. Passing to a subnet, we may assume $z\varphi_\alpha \to \theta$ and $(1-z)\varphi_\alpha \to \psi$, where $\theta + \psi = \varphi$. Since $(1-z)\psi = \psi$,

$$\begin{aligned}
\varphi(zh+x) &= \theta(zh+x) + \psi(x) \le \theta(h+x) + \psi(x) \\
&\le \underline{\lim}(z\varphi_\alpha)(h+x) + \underline{\lim}(1-z)\varphi_\alpha(x) \\
&\quad \text{(since } x \in SLSC(A)_+) \\
&\le \underline{\lim}[z\varphi_\alpha(h+x) + (1-z)\varphi_\alpha(x)] \\
&= \underline{\lim}[\varphi_\alpha(zh+zx) + \varphi_\alpha((1-z)x)] \\
&= \underline{\lim}\,\varphi_\alpha(zh+x). \qquad \square
\end{aligned}$$

3. A Dauns-Hoffmann type theorem and middle interpolation. Recall that h is called q-LSC if $E_{(t,\infty)}(h) + (1-p_h)$ is open for all $t \in \mathbf{R}$ where p_h is the projection on $\overline{D(h)}$ and $E_S(h)$ stands for the spectral projection of h corresponding to $S \subset \mathbf{R}$. h is called q-USC if $-h$ is q-LSC, and h is called q-continuous if h is densely defined, q-LSC and q-USC. For h bounded, we use q-lsc or q-usc instead. We refer to [**6**, p. 905] for the history of q-semicontinuity. Unlike the bounded case, q-continuity does not imply middle continuity in general. Let A be the algebra $\mathcal{K}$ of compact operators on a separable Hilbert space H, and let h be an arbitrary unbounded self-adjoint densely defined operator on H. Since every projection is open, h is q-continuous. Note that A is a PCS-algebra and hence $\Gamma(K_A) = M(A) = B(H)$. Therefore, h cannot be in $\Gamma(K_A)$. However, if we assume local boundedness of h, then we still have a similar result.

Proposition 3.1. *$h \in \Gamma(K_A)_{sa} \Leftrightarrow h$ is q-continuous and hp_C, respectively $hp_{\overline{C}}$, is bounded for every C in Λ.*

Proof. Assume $h \in \Gamma(K_A)_{sa}$ and let $g(t) = \mathrm{Tan}^{-1}(t)$. Then $g(h)$ is in $\Gamma(K_A)_{sa}$, by [**13**, Lemma 5.14], and bounded. Therefore, $g(h)$ is in $M(A)_{sa}$ and q-continuous by Akemann [**1**]. Since g is continuous and monotone increasing, this implies that h is q-continuous. Since $hp_C \in M(I(C))_{sa}$ and $I(C)$ is essential in $A/I(\check{A}-\overline{C})$, $hp_{\overline{C}}$ has to be bounded.

Now assume that h is q-continuous and hp_C is bounded for every C in Λ. Then hp_C is q-continuous with respect to $I(C)$, and hence $hp_C \in M(I(C))_{sa}$ by [**3**, Theorem 2.2]. Therefore, $h \in \Gamma(K_A)_{sa}$ by Corollary 1.2. □

Definition 3.2. For a self-adjoint operator h such that $h\eta A^{**}$ and hp_C is bounded below for all $C \in \Lambda$, consider the set $\mathcal{Z}_h$ of all central projections z such that zh is densely defined on zH_u and bounded above. Each zh has a central cover $c(zh)$ in $\mathcal{Z}$ by Pedersen [**15**]. Then there exists a self-adjoint operator $c(h)$ on the Hilbert space z_hH_u where $z_h = \vee_{z\in\mathcal{Z}_h} z$ such that $c(h)\eta\mathcal{Z}$ and $zc(h) = c(zh)$ for all $z \in \mathcal{Z}_h$, and it will be called *central cover* of h. We think of $c(h)$ as being $+\infty$ on $(1-z_h)H_u$. Clearly the map $[\pi] \mapsto \pi^{**}(c(h))$ defines a function $\check{h} : \hat{A} \to (-\infty, \infty]$. Note that if $h_i \nearrow h$, then $c(h_i) \nearrow c(h)$ and hence $\check{h}_i \nearrow \check{h}$ pointwise on $\hat{A}$, cf. [**16**, Lemma 2.6.5].

Proposition 3.3. *If h satisfies* (M1) *and $h\eta\mathcal{Z}$, then $\check{h}$ is lower semicontinuous on $\hat{A}$, or* Prim A*, and h is q*-LSC.

Proof. Assume that h satisfies (M1) and $h\eta\mathcal{Z}$. Then for all $C \in \Lambda$, there exists λ_C such that $(h+\lambda_C)p_C \in SLSC(I(C))_+$. Therefore there exists a net $(a_i + \lambda_i p_C)$ in $\widetilde{I(C)}$ such that $a_i + \lambda_i p_C \nearrow (h+\lambda_C)p_C$, $a_i \in I(C)_+$ and $\lambda_i \nearrow 0$. By Pedersen [**16**, Proposition 4.4.4], $\check{a}_i$ is lower semicontinuous on C $(= \operatorname{Prim}(I(C)))$. So $[(h+\lambda_C)p_C]^\vee$ is the limit of the increasing net $(\check{a}_i + \lambda_i)$ of lower semicontinuous functions, and hence a lower semicontinuous function on C. This implies that $(hp_C)^\vee$ is lower semicontinuous as well. Since $\check{h}|_C = (hp_C)^\vee$ for all $C \in \Lambda$, and each C is open in Prim A, this shows that $\check{h}$ is lower semicontinuous on Prim A, or $\hat{A}$. Applying [**12**, Theorem 3.23] to $(h+\lambda_C)p_C$, we see that hp_C is q-LSC with respect to $I(C)$. Clearly hp_C q-LSC for all C in Λ implies h q-LSC. □

Recall that we write $h \overset{q}{\geq} k$ if and only if $E_{(-\infty,s]}(h) \cdot E_{[t,\infty)}(k) = 0$

for all $s, t \in \mathbf{R}$ such that $s < t$. Note $h \overset{q}{\geq} k$ implies $h \geq k$ if k is locally bounded above, cf. [**6**, Definition 3.39].

Theorem 3.4. *Let A be a σ-unital C^*-algebra, and let h be q-*LSC *such that $hp_{\overline{C}}$ is bounded below for every C in Λ. Then h is in $MLSC(A)$.*

Proof. Assume h is q-LSC such that $hp_{\overline{C}}$ is bounded below for every C in Λ. By [**12**, Theorem 3.22], h_+ is in $SLSC(A)$. Hence we may assume $h \leq 0$. Let

$$g_{-n} = -1 \vee [(id + (n-1)) \wedge 0]$$

and

$$h^{(n)} = g_{-n}(h) \quad \text{for } n \in \mathbf{N}.$$

Then $h = \sum_{n=1}^{\infty} h^{(n)}$ and g_{-n} is a bounded, continuous, monotone increasing function such that $-1 \leq g_{-n} \leq 0$. Therefore, $h^{(n)}$ is q-lsc for all $n \in \mathbf{N}$.

Since A is σ-unital, there exists a sequence (C_n) in Λ such that $C_n \nearrow \operatorname{Prim} A$. Let m_n be an integer lower bound of $hp_{\overline{C}_n}$ for all $n \in \mathbf{N}$. Then, without loss of generality, we may assume that $0 = m_1 > m_2 > \cdots$. For n with $m_s > -n \geq m_{s+1}$, let $g^{(n)} = \mathcal{X}_{\overline{C}_s} - 1$. Then $g^{(n)}$ is upper semicontinuous on $\operatorname{Prim} A$. By the proof of [**16**, Theorem 4.4.6], there is $k^{(n)}$ with $-k^{(n)} \in \overline{A_+^m} \cap \mathcal{Z}$ such that $(k^{(n)})^\vee = g^{(n)}$. Note that $h^{(n)}$ is q-lsc, $k^{(n)}$ is q-usc and $h^{(n)} \overset{q}{\geq} k^{(n)}$. By the middle interpolation theorem [**6**, Theorem 3.40], there is $x^{(n)}$ in $M(A)_{sa}$ such that $k^{(n)} \leq x^{(n)} \leq h^{(n)}$ and $h^{(n)} - x^{(n)} \in \overline{A_+^m}$. Here note that $-1 \leq x^{(n)} \leq 0$ and $p_{\overline{C}_s} x^{(n)} = 0$ if $-n < m_s$. Let $x = -\sum_{n=1}^{\infty} x^{(n)}$. Then for each s there are only finitely many terms nonzero on $p_{C_s} H_u$. Thus x is bounded on $p_{C_s} H_u$ and $xp_{C_s} \in M(I(C_s))_+$ for all $s \in \mathbf{N}$. This implies $x \in \Gamma(K_A)_+$.

Now we will show that $h + x \in SLSC(A)_+$. Note that $h + x = \sum_{n=1}^{\infty}(h^{(n)} - x^{(n)})$. Since $h^{(n)} - x^{(n)} \in \overline{A_+^m}$ for all $n \in \mathbf{N}$, we have $\sum_{n=1}^{l}(h^{(n)} - x^{(n)}) \in \overline{A_+^m}$ for all $l \in \mathbf{N}$, and

$$\sum_{n=1}^{l}(h^{(n)} - x^{(n)}) \nearrow \sum_{n=1}^{\infty}(h^{(n)} - x^{(n)}) \quad \text{as } l \nearrow \infty.$$

Therefore $h + x \in SLSC(A)_+$. □

Remark. The example before Proposition 3.1 shows that the hypothesis that $hp_{\overline{C}}$ be bounded below cannot be dropped. Also the assumption of A being σ-unital cannot be dropped either by the example $A = C_0(X)$ with $X = \beta\mathbf{R} - (\beta\mathbf{N} - \mathbf{N})$ since X is pseudocompact but not countably compact.

Theorem 3.5. *Let A be a σ-unital C^*-algebra. Then the map $h \mapsto \check{h}$ is an isomorphism from $\{h \in MLSC(A) \mid h\eta\mathcal{Z}\}$ onto the set of $(-\infty, \infty]$-valued lower semicontinuous functions on* $\operatorname{Prim} A$*, or $\hat{A}$, which are bounded below on $\overline{C}$ for all C in Λ.*

Proof. Proposition 3.3 above shows that the map $h \mapsto \check{h}$ is well defined. For injectivity, let h_1 and h_2 be in $\{h \in MLSC(A) \mid h\eta\mathcal{Z}\}$ such that $\check{h}_1 = \check{h}_2$. Then, for any C in Λ, there is $\lambda_C > 0$ such that $(h_i + \lambda_C)p_C \in SLSC(I(C))_+$ for $i = 1, 2$, and $(h_1 p_C)^\vee = (h_2 p_C)^\vee$. Applying [**15**, Theorem 4.6] to $f_\delta[(h_i + \lambda_C)p_C]$ we can see that $h_1 p_C = h_2 p_C$ for all C in Λ. Therefore $h_1 = h_2$.

Now it suffices to show that the map $h \mapsto \check{h}$ is surjective. Let f be a $(-\infty, \infty]$-valued lower semicontinuous function on $\operatorname{Prim} A$ such that $f|_{\overline{C}}$ is bounded below for all $C \in \Lambda$. Then the function g defined by $g(x) = \operatorname{Tan}^{-1}(f(x))$ for $f(x) < \infty$ and $g(x) = (\pi/2)$ for $f(x) = \infty$ is bounded lower semicontinuous on $\operatorname{Prim} A$. By Pedersen [**15**, Theorem 4.6], there is an h' in $\mathcal{Z} \cap \tilde{A}^m_{sa}$ such that $\check{h}' = g$. Let $h = \tan(h') \oplus (+\infty)z$ where z is the spectral projection of h' corresponding to $\{\pi/2\}$. Then $h\eta\mathcal{Z}$ and $\check{h} = f$ on $\operatorname{Prim} A$ and hence h is q-LSC and $hp_{\overline{C}}$ is bounded below for all $C \in \Lambda$. Applying Theorem 3.4, we have $h \in MLSC(A)$, and we are done. □

The above is a generalization of Pederson [**15**, Theorem 4.6], which gives a new proof of the Dauns-Hofmann theorem in the unbounded case for σ-unital C^*-algebras. Using a similar idea we generalize the middle interpolation theorem, [**6**, Theorem 3.40], as follows:

Theorem 3.6. *Assume that A is a σ-unital C^*-algebra, h is q-LSC, k is q-USC such that for all $C \in \Lambda$, $hp_{\overline{C}}$ is bounded below and $kp_{\overline{C}}$ is* *bounded above, and $h \overset{q}{\geq} k$. Then there exists $x \in \Gamma(K_A)_{sa}$ such that $k \leq x \leq h$. Moreover, if h is bounded below and k is bounded above, then there exists $x \in M(A)_{sa}$ such that $k \leq x \leq h$.*

Proof. Since A is σ-unital, there is an increasing sequence (C_n) in Λ such that $C_n \nearrow \operatorname{Prim} A$. Let M_n (respectively m_n) be an integer upper (respectively lower) bound of $kp_{\overline{C}_n}$ (respectively $hp_{\overline{C}_n}$), such that $0 < M_1 \leq M_2 \leq \cdots$, and $0 > m_1 \geq m_2 \geq \cdots$. Let

$$g_n = [0 \vee (id - (n-1))] \wedge 1$$

and

$$g_{-n} = -1 \vee [(id + (n-1)) \wedge 0] \quad \text{for } n \in \mathbf{N}.$$

Let $\tilde{g}_{\pm n}(h)$, $n \in \mathbf{N}$, denote $g_n(h) \oplus 1(1-p_h)$ and $g_{-n}(h) \oplus 0(1-p_h)$ respectively, and let $\tilde{g}_{\pm n}(k)$ denote $g_n(k) \oplus 0(1-p_k)$ and $g_{-n}(k) \oplus (-1)(1-p_k)$, respectively. Then $\tilde{g}_{\pm n}(h)$ is q-lsc, $\tilde{g}_{\pm n}(k)$ is q-usc, $\tilde{g}_{\pm n}(h) \overset{q}{\geq} \tilde{g}_{\pm n}(k)$,

$$h = \sum_{n=1}^{\infty} (\tilde{g}_n(h) + \tilde{g}_{-n}(h)) \quad \text{and} \quad k = \sum_{n=1}^{\infty} (\tilde{g}_n(k) + \tilde{g}_{-n}(k)).$$

Note that if $M_n < t \in \mathbf{N}$, then $\tilde{g}_t(k)p_{\overline{C}_n} = 0$, and if $t < m_n$ then $\tilde{g}_t(h)p_{\overline{C}_n} = 0$. For $M_n < t \leq M_{n+1}$, let $q_t = 1 - p_{\overline{C}_n}$. Then q_t is an open central projection such that $\tilde{g}_t(k) = \tilde{g}_t(k)q_t \leq \tilde{g}_t(h)q_t \leq \tilde{g}_t(h)$. This implies that $\tilde{g}_t(h)q_t$ is q-lsc and $\tilde{g}_t(k) \overset{q}{\leq} \tilde{g}_t(h)q_t$. By Brown [**6**, Theorem 3.40], we can find x_t in $M(A)_{sa}$ for $t > M_1$ such that

$$\tilde{g}_t(k) \leq x_t \leq \tilde{g}_t(h)q_t \leq \tilde{g}_t(h).$$

Similarly, we can find x_{-t} in $M(A)_{sa}$ for $-t < m_1$ such that

$$\tilde{g}_{-t}(k) \leq \tilde{g}_{-t}(k)q_{-t} \leq x_{-t} \leq \tilde{g}_{-t}(h)q_{-t} = \tilde{g}_{-t}(h).$$

For $m_1 \leq t \leq M_1$, choose $x_t \in M(A)_{sa}$ such that $\tilde{g}_t(k) \leq x_t \leq \tilde{g}_t(h)$. Let $x = \sum_{n=1}^{\infty}(x_n + x_{-n})$. Then there are only finitely many terms nonzero on $p_{C_n}H_u$ for each $n \in \mathbf{N}$ since $x_t p_{C_n} = 0$ for sufficiently large $|t|$. Therefore, $xp_{C_n} \in M(I(C_n))_{sa}$, $n \in \mathbf{N}$, and hence x belongs to

$\Gamma(K_A)_{sa}$. By the choice of x_t, the inequality $kp_C \leq xp_C \leq hp_C$ holds for all $C \in \Lambda$. Hence $k \leq x \leq h$.

If h is bounded below and k is bounded above, then x_t can be taken to be zero for $|t|$ large. This proves the last statement. □

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