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BINARY FORMS, EQUIANGULAR POLYGONS AND HARMONIC MEASURE

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ABSTRACT. Consider binary forms F having complex coefficients and discriminant $D_F \neq 0$. In a sequence of previous papers, the first author studied the area A_F of the planar region $|F(x,y)| \leq 1$ defined by forms F of this type in connection with the enumeration of integer lattice points. In particular, the first author showed that the $GL_2(\mathbf{R})$-invariant quantity $|D_F|^{1/n(n-1)}A_F$ is uniformly bounded when the degree of F is at least three, and conjectured that this quantity is maximized over the forms of degree n by forms with a complete factorization over $\mathbf{R}$ and n equally spaced asymptotes. The first author obtained his results using a standard integral representation of A_F over the real line.

In this paper we establish $GL_2(\mathbf{C})$ invariance of $|D_F|^{1/n(n-1)}A_F$ with respect to Lagrangian planes in $\mathbf{C}^2$ (a fact not previously noticed by many earlier authors), and we subsequently give integral representations of A_F over every circle in the complex plane. In particular, we give a representation over the unit circle and we use this representation to give an explicit formula in terms of Beta functions for the conjectured maximum value of $|D_F|^{1/n(n-1)}A_F$. It turns out that integration over the unit circle is directly linked to binary forms in the complex indeterminates $\bar{z}$ and z.

In addition, we reformulate the maximization problem for binary forms in purely geometric and potential theoretic terms as a maximization problem for harmonic measures on the edges of equiangular polygons, with the inner harmonic radius of the polygon being normalized. In this context the conjectured extremal polygon is the regular n-gon.

We conclude the paper with tables that summarize the many equivalent formulas for A_F, D_F and related quantities.

1. Introduction. A *binary form* is a polynomial in two variables of

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homogeneous degree, i.e., a bivariate polynomial of the type $F(X,Y) = a_0X^n + a_1X^{n-1}Y + \cdots + a_nY^n$ where n is a positive integer and where the coefficients $a_0, a_1, \ldots, a_n$ belong to some ring K, usually $\mathbf{Z}$, $\mathbf{R}$ or $\mathbf{C}$, and are not all zero. The form F is said to have a *complete factorization* over K if $F(X,Y) = \prod_{j=1}^{n}(\alpha_j X - \beta_j Y)$ for some $\alpha_j, \beta_j \in K$. (If $K = \mathbf{C}$, then every form has a complete factorization over K; however, if $K = \mathbf{R}$, this is not generally the case.) In this paper we are primarily interested in forms with real coefficients which have a complete factorization over $\mathbf{R}$.

For any form $F \in \mathbf{C}[X,Y]$,[1] let A_F denote the area of the region $|F(x,y)| \le 1$ in the real affine plane, and let D_F denote the discriminant of F. Note that $A_F = (1/2)\int_{-\pi}^{\pi} |F(\cos\theta, \sin\theta)|^{-2/n}\, d\theta = \int_{-\infty}^{\infty} |F(1,v)|^{-2/n}\, dv$ and that $D_F = \prod_{j<k}(\alpha_j\beta_k - \alpha_k\beta_j)^2$ (facts which follow from the polar form $r \le |F(\cos\theta, \sin\theta)|^{-1/n}$ of the inequality $|F(x,y)| \le 1$ and the definition of the discriminant; see [**6**]). To avoid unnecessary complications, we will assume throughout the paper that all forms are of degree at least two, i.e., $n \ge 2$. Put

$$Q(F) = |D_F|^{1/n(n-1)} A_F$$

(whenever the multiplication makes sense, i.e., whenever either $D_F \ne 0$ or $A_F < \infty$), and define the sequence $\{M_n\}$ by

$$M_n = \max_F Q(F),$$

where the maximum is taken over all forms $F \in \mathbf{C}[X,Y]$ of degree n with $D_F \ne 0$ or $A_F < \infty$. Note that $M_2 = \infty$ since $A_F = \infty$ for the form $F(X,Y) = XY$.

In [**5**] and [**6**] the first author proved the following results about the sequence $\{M_n\}$:

(R1) $\{M_n\}$ is (strictly) decreasing for $n \ge 3$.

(R2) For each $n \ge 3$, M_n is attained by a form of degree n which has real coefficients and a complete factorization over $\mathbf{R}$; in fact, if the polynomial $F(1,v)$ has even one nonreal root, then $Q(F) < M_n$.

(R3) $M_3 = 3B[(1/3),(1/3)] \approx 15.90$ and $M_4 = 2^{7/6}B[(1/4),(1/2)] \approx 11.77$, where $B(\cdot,\cdot)$ denotes the Beta function.

These results revealed the surprising fact that $Q(F)$ is uniformly bounded (over the forms F with nonzero discriminant and degree at

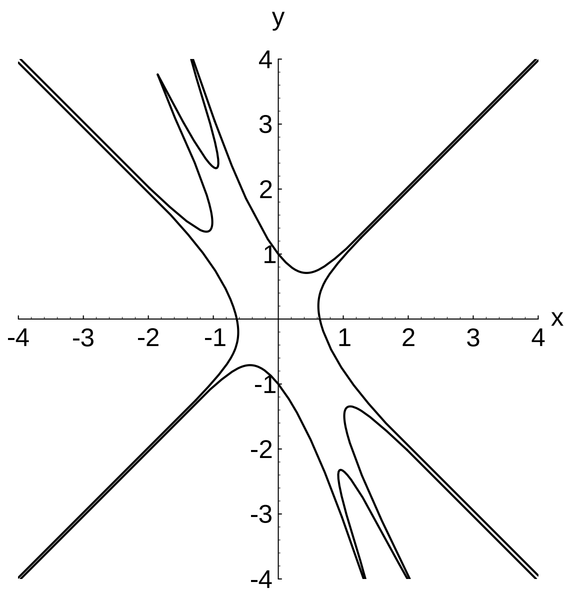

FIGURE 1. $|y^4 + 5xy^3 + 5x^2y^2 - 5x^3y - 6x^4| = 1$.

least three) by a relatively small number, $3B[(1/3), (1/3)] \approx 15.90$, even though the regions $|F(x, y)| \leq 1$ are, in general, unbounded and can contain as many as n asymptotes. (See Figure 1.) Note that the conditions on the discriminant and the degree cannot be relaxed since, for example, the forms X^n and $X^2 - Y^2$ give rise to infinite area. The fact that $Q(F)$ is uniformly bounded has important consequences for certain lattice point problems in the theory of numbers; for details, see [**6, 7**] and [**12**].

The first author's results raised several questions about the sequence $\{M_n\}$:

(Q1) Is there a formula, e.g., in terms of Beta functions, for the values of M_n?

(Q2) Are there canonical classes of forms F_n for which $M_n = Q(F_n)$?

(Q3) What is the limiting value of the sequence $\{M_n\}$?

Since each M_n is actually attained by a form with a complete factorization over $\mathbf{R}$, it is clear that these questions need only be considered over the restricted class of forms with a complete factorization over $\mathbf{R}$. Even so, these questions remain nontrivial since $Q(F)$, when considered over this restricted class, is a function of $n-3$ independent cross ratios [**5**, Theorem 3].

Note that $Q(F)$ is invariant under the action of $GL_2(\mathbf{R})$ in the following sense:

$$Q(F_T) = Q(F) \quad \text{for all} \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{R}),$$

where $F_T(X,Y) = F(aX+bY, cX+dY)$, see [**6**]. Hence, the value M_n is actually attained by a class of $GL_2(\mathbf{R})$ equivalent forms. This observation gives us insight into why it is possible to maximize $Q(F)$ but not A_F.[2] Indeed, the factor $|D_F|^{1/n(n-1)}$ "normalizes" the area A_F in such a way that when one of the quantities $|D_F|$ or A_F is large, the other must be small, because the inequality $Q(F) \le 3B[(1/3),(1/3)]$ must be maintained.

The first author was unable to answer questions (Q1), (Q2) and (Q3) for $n \ge 5$. However, he was able to formulate the following conjecture (and prove it for $n = 3, 4$):

Conjecture 1 [**5**]. *The maximum value M_n of $Q(F)$ over the forms F of degree n with discriminant $D_F \ne 0$ is attained precisely when F is a form which, up to multiplication by a complex number, is equivalent under $GL_2(\mathbf{R})$ to the form*

$$F_n^*(X,Y) = \prod_{k=1}^{n} \left(X \sin\left(\frac{k\pi}{n}\right) - Y \cos\left(\frac{k\pi}{n}\right)\right);$$

that is,

$$M_n = Q(F_n^*).$$

Moreover, the limit of the sequence $\{M_n\}$ is 2π.

The first author based this conjecture on a correspondence between binary forms and equiangular polygons, which he derived from the Schwarz-Christoffel mapping formula:

Schwarz-Christoffel correspondence [**5**, Theorem 2]. *There is a correspondence between the collection of forms F of degree $n \geq 3$ having nonzero discriminant and a complete factorization over $\mathbf{R}$ and the collection of n-sided equiangular polygons, such that the polygon $\mathcal{P}(F)$ corresponding to F has perimeter equal to A_F. Under this correspondence, $GL_2(\mathbf{R})$-equivalent forms are mapped to similar polygons, $SL_2(\mathbf{R})$-equivalent forms are mapped to congruent polygons, and forms equivalent to F_n^* are mapped to n-sided regular polygons. Moreover, the correspondence is bijective when considered as a map between equivalence classes of forms and polygons.*

More precisely, the first author conjectured that $M_n = Q(F_n^*)$ on the basis of the above correspondence, and that $\lim_{n\to\infty} Q(F_n^*) = 2\pi$ on the basis of numerical computations. (Note that the statement $\lim_{n\to\infty} Q(F_n^*) = 2\pi$ is consistent with the above correspondence since every circle is the limit of a sequence of regular polygons.) The first author actually proved his conjecture in the cases $n = 3$ and $n = 4$ using $GL_2(\mathbf{R})$-invariance and properties of hypergeometric functions; however, he was unable to adapt his methods to the case $n \geq 5$. Note that the forms F_n^* are natural candidates to maximize $Q(F)$ since their corresponding graphs have the most symmetries possible; indeed, the graph $|F_n^*(x, y)| = 1$ is invariant under every rotation which is an integer multiple of π/n radians and is symmetric about each of the asymptotes $x \sin(k\pi/n) - y \cos(k\pi/n) = 0$ for $k = 1, 2, \dots, n$. (See Figure 2 for the graph of F_5^*.) This observation, together with the Schwarz-Christoffel correspondence and numerical evidence, leads one to believe that Conjecture 1 is likely true.

Now, in the work of the first author [**5, 6, 7, 8, 9, 10**], the quantity $Q(F)$ was analyzed from the perspective of the real line, i.e., using the representation

$$Q(F) = \prod_{j \neq k} |s_j - s_k|^{1/n(n-1)} \cdot \int_{-\infty}^{\infty} \prod_{j=1}^{n} |v - s_j|^{-2/n} \, dv$$

where $s_1 < s_2 < \cdots < s_n$ are the roots of $F(1, v)$ (assumed to lie on the real axis). In this paper we will analyze $Q(F)$ from the perspective

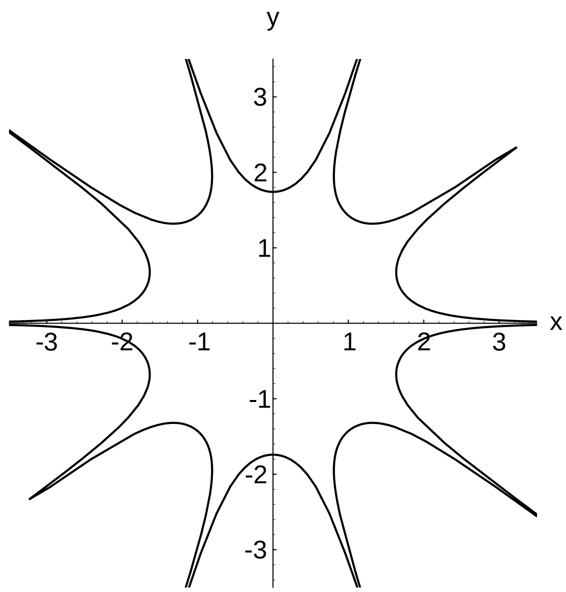

FIGURE 2. $|F_5^*(x,y) = 1$.

of the unit circle, i.e., using the representation

$$Q(F) = \prod_{j \neq k} |e^{i\theta_j} - e^{i\theta_k}|^{1/n(n-1)} \cdot \int_{-\pi}^{\pi} \prod_{j=1}^{n} |e^{i\theta} - e^{i\theta_j}|^{-2/n} \, d\theta$$

where $\theta_j = 2 \arctan s_j$. The latter representation will follow from a representation of $Q(F)$ over a general circle on the Riemann sphere which we will derive by considering binary forms in $\mathbf{C}^2$ and transformations in $GL_2(\mathbf{C})$; in fact, we will see that there is a direct link between integration over the unit circle and binary forms in z and $\bar{z}$.

This new approach will allow us to:

(a) give an explicit formula for the values $Q(F_n^*)$ in terms of Beta functions, and in so doing, reduce the proof of Conjecture 1 to verifying that $M_n = Q(F_n^*)$ (see Theorem 1 and Theorem 2.1 in Section 2 below);

(b) reformulate the maximization problem for binary forms, via the Schwarz-Christoffel correspondence, as a maximization problem for equiangular polygons which is expressed solely in terms of quantities intrinsic to the mapped polygon, e.g., harmonic measure and harmonic radius, and which does not refer to either the binary form F or the conformal map connecting F and the polygon $\mathcal{P}(F)$, see Theorem 3 and Theorem 4 in Section 2 below.

By considering appropriate generalizations of A_F and $Q(F)$, we will also establish the $GL_2(\mathbf{C})$ invariance of $Q(F)$ with respect to Lagrangian planes in $\mathbf{C}^2$, an important fact which has apparently not been noticed before, e.g., [**12**].

Unfortunately, the verification that $M_n = Q(F_n^*)$ for $n \geq 5$ has been elusive. Nevertheless, we believe that the new approach to proving maximality which we formulate in this paper holds promise. We suspect that the solution to this maximization problem may involve new ideas from (classical) mathematical physics, potential theory or probability. It is our hope that researchers in these and other areas will recognize something familiar in the reformulated maximization problem, stated at the end of Section 2 below, and will be able to provide the missing link needed to solve it. (Mathematical physicists might wish to regard the points $e^{i\theta}$ in formula $(*)$ on page 17 as point masses, potential theorists will already be familiar with the harmonic measure in formula $(**)$ on page 24, and probabilists might find a Brownian motion approach to the harmonic measure fruitful.)

2. Statement of results.

2.1. Formulas for calculating $Q(F_n^*)$. Let F_n^* be the form $F_n^*(X,Y) = \prod_{k=1}^{n}(X\sin(k\pi/n) - Y\cos(k\pi/n))$, the candidate to maximize $Q(F)$, and let $B(x,y)$ denote the Beta function of x and y. Note that the Beta function may be expressed either in terms of the Gamma function or as an integral:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

for $x > 0$ and $y > 0$, see [**1**, 6.2].

We will derive the following representations for $A_{F_n^*}$, $D_{F_n^*}$ and $Q(F_n^*)$ in Section 4.

Theorem 1. *For each $n \geq 2$, we have*

$$A_{F_n^*} = 4^{1-1/n} B\left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right) \tag{1}$$

and

$$D_{F_n^*}^{1/n(n-1)} = \frac{1}{2} n^{1/(n-1)}. \tag{2}$$

Consequently,

$$Q(F_n^*) = 2^{1-2/n} n^{1/(n-1)} B\left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right). \tag{3}$$

Notice that $Q(F_3^*) = 2^{1/3}3^{1/2}B(1/6, 1/2) = 3B(1/3, 1/3)$, the latter equality following from the identities $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)$ with $z = 1/3$ and $\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma[z + (1/2)]$, with $z = 1/6$ [**1**, formulas 6.1.8, 6.1.17, 6.1.18]; and that $Q(F_4^*) = 2^{7/6}B(1/4, 1/2)$. Hence the values of $Q(F_3^*)$ and $Q(F_4^*)$ provided by this theorem agree with the values of M_3 and M_4 previously determined in [**5**] and [**6**].

A further analysis in Section 4 of the above formulas for $A_{F_n^*}$, $D_{F_n^*}$ and $Q(F_n^*)$ will confirm that the general character of the sequence $\{Q(F_n^*)\}$ is in accord with the known and conjectured properties of the sequence $\{M_n\}$:

Corollary 1.1. *The sequences $\{A_{F_n^*}\}$ and $\{D_{F_n^*}^{1/n(n-1)}\}$ are both strictly decreasing for $n \geq 2$. Consequently, the sequence $\{Q(F_n^*)\}$ is also strictly decreasing.*

Corollary 1.2. *$A_{F_n^*} \to 4\pi$ and $D_{F_n^*}^{1/n(n-1)} \to 1/2$ as $n \to \infty$. Hence $Q(F_n^*) \to 2\pi$ as $n \to \infty$.*

Note that Corollary 1.2 provides a purely analytic basis for the conjecture that $\lim_{n\to\infty} M_n = 2\pi$ (as opposed to the numerical basis presented in [**5**]). Taken together, the two corollaries provide a nontrivial lower bound for the values M_n:

Corollary 1.3. *For each $n \geq 2$, we have*

$$M_n \geq Q(F_n^*) > 2\pi.$$

Consequently, $2\pi < M_n \leq 3B(1/3, 1/3) < 15.90$ for all $n \geq 3$.

2.2. Binary forms in $\mathbf{C}^2$. We will derive the formulas for $A_{F_n^*}$, $D_{F_n^*}$ and $Q(F_n^*)$ stated in Theorem 1 by using a representation for A_F of the type $(1/2)\int_{-\pi}^{\pi} |F((1+e^{i\theta})/2, i(1-e^{i\theta})/2)|^{-2/n}\, d\theta$ (integration over the unit circle) rather than the representation $\int_{-\infty}^{\infty} |F(1,v)|^{-2/n}\, dv$ (integration over the real line) employed in several papers. We will see in Section 3 that both these representations are actually special cases of the following general integral formula:

Theorem 2.1. *Let F be a binary form with complex coefficients, and let T be a transformation in $GL_2(\mathbf{C})$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the circles, on the Riemann sphere, defined by*

$$\mathcal{C}_1 = \left\{ \frac{\sigma}{\tau} : \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \in \mathcal{S} \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$
$$\mathcal{C}_2 = \left\{ \frac{\tau}{\sigma} : \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \in \mathcal{S} \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

where $\mathcal{S} = \left\{ T^{-1} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : x, y \in \mathbf{R} \right\}$. Then A_F has the representations

$$A_F = |\det T| \oint_{\mathcal{C}_1} |F_T(\sigma, 1)|^{-2/n} |d\sigma| \tag{4}$$

and

$$A_F = |\det T| \oint_{\mathcal{C}_2} |F_T(1, \tau)|^{-2/n} |d\tau|, \tag{5}$$

where the integrals are calculated in the complex plane. Moreover, for any pair of circles $\mathcal{C}_1, \mathcal{C}_2$ which are inverses of each other in the sense *that* $z \in \mathcal{C}_1 \Leftrightarrow z^{-1} \in \mathcal{C}_2$, *there is a transformation* $T \in GL_2(\mathbf{C})$ *such that formulas* (4) *and* (5) *hold.*

In particular, if $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, *then the formulas become*

$$A_F = \int_{-\infty}^{\infty} |F(u,1)|^{-2/n}\, du = \int_{-\infty}^{\infty} |F(1,v)|^{-2/n}\, dv, \tag{6}$$

while if $T = \left(\begin{smallmatrix} (1/2) & (1/2) \\ (i/2) & -(i/2) \end{smallmatrix}\right)$, *then the formulas become*

$$\begin{aligned} A_F &= \frac{1}{2}\int_{-\pi}^{\pi} \left| F\left(\frac{e^{i\phi}+1}{2}, \frac{i(e^{i\phi}-1)}{2} \right) \right|^{-2/n} d\phi \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \left| F\left(\frac{1+e^{i\theta}}{2}, \frac{i(1-e^{i\theta})}{2} \right) \right|^{-2/n} d\theta. \end{aligned} \tag{7}$$

Note that equations (6) and (7) are representations of A_F over the real line and the unit circle, respectively. Equation (7) is actually a disguised form of the usual polar integral formula for calculating A_F. Indeed, using the homogeneity of F and the relation $e^{i\xi} = \cos\xi + i\sin\xi$, we can certainly write

$$F\left(\frac{e^{i\phi}+1}{2}, \frac{i(e^{i\phi}-1)}{2} \right) = (e^{i\phi/2})^n F(\cos(\phi/2), -\sin(\phi/2))$$

and

$$F\left(\frac{1+e^{i\theta}}{2}, \frac{i(1-e^{i\theta})}{2} \right) = (e^{i\theta/2})^n F(\cos(\theta/2), \sin(\theta/2));$$

hence equation (7) can be written as

$$\begin{aligned} A_F &= \frac{1}{2}\int_{-\pi}^{\pi} |F(\cos(\phi/2), -\sin(\phi/2))|^{-2/n}\, d\phi \\ &= \frac{1}{2}\int_{-\pi}^{\pi} |F(\cos(\theta/2), \sin(\theta/2))|^{-2/n}\, d\theta \end{aligned}$$

which is the polar formula for the area of the region $|F(x,y)| \le 1$.

When $F(X,Y)$ has a complete factorization over $\mathbf{R}$, equations (6) and (7) can be written in the following more explicit form:

Corollary 2.1.1. *Let $F(X,Y) = \prod_{j=1}^{n}(\alpha_j X - \beta_j Y)$ be a binary form with real coefficients and a complete factorization over $\mathbf{R}$. Then A_F has the representations*

$$A_F = |a|^{-2/n} \int_{-\infty}^{\infty} \prod_{j=1}^{n} |v - s_j|^{-2/n}\, dv \tag{8}$$

and

$$A_F = \frac{1}{2}|\kappa|^{-2/n} \int_{-\pi}^{\pi} \prod_{j=1}^{n} |e^{i\theta} - e^{i\theta_j}|^{-2/n}\, d\theta \tag{9}$$

where $a = (-1)^n \beta_1 \cdots \beta_n$, $\kappa = \prod_{j=1}^{n}(\alpha_j + i\beta_j)/2$, $s_j = \alpha_j/\beta_j$ and $\theta_j = 2\arctan s_j$.

Note that if $\beta_k = 0$ and $s_k = \pm\infty$, then $\theta_k = \pi/2$ and, by convention, we replace β_k by α_k in the definition of a and we omit the kth factor of the product in (8).

We will find equations (8) and (9) to be the most useful when computing A_F for forms F with a complete factorization over $\mathbf{R}$.

Theorem 2.1 will follow from a consideration of binary forms over two-dimensional real vector spaces $\mathcal{S}$ in $\mathbf{C}^2$. Every such vector space is of the form $\mathcal{S} = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : x, y \in \mathbf{R} \right\}$ where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a fixed matrix in $GL_2(\mathbf{C})$. Recall from symplectic geometry [**13**, pp. 28–29] that $\mathcal{S}$ is said to be *Lagrangian* if $\mathcal{S}$ and $i\mathcal{S}$ are orthogonal with respect to the dot product on $\mathbf{R}^4$, where we use the usual identification of $\mathbf{C}^2$ with $\mathbf{R}^4$ via

$$\begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}.$$

Equivalently, $\mathcal{S}$ is Lagrangian precisely when

$$\begin{aligned} 0 &= \begin{pmatrix} iA \\ iC \end{pmatrix} \cdot \begin{pmatrix} B \\ D \end{pmatrix} \\ &= \Re\Big\langle \begin{pmatrix} iA \\ iC \end{pmatrix}, \begin{pmatrix} B \\ D \end{pmatrix} \Big\rangle \\ &= \Re(iA\overline{B} + iC\overline{D}) \\ &= -\Im(A\overline{B} + C\overline{D}), \end{aligned}$$

so that $\mathcal{S}$ is Lagrangian if and only if

$$A\overline{B} + C\overline{D} \in \mathbf{R}.$$

Obviously, $\mathbf{R}^2$ is Lagrangian (take $A = D = 1$, $B = C = 0$), but many other spaces $\mathcal{S}$ are Lagrangian too.

For any subspace $\mathcal{S}$, not necessarily Lagrangian, and any binary form F, let $A_F^{\mathcal{S}}$ be the two-dimensional area of the region $\left\{ \binom{\sigma}{\tau} \in \mathcal{S} : |F(\sigma,\tau)| \le 1 \right\}$ in $\mathcal{S}$. Extend the definition of $Q(F)$ by putting $Q(F,\mathcal{S}) = |D_F|^{1/n(n-1)} A_F^{\mathcal{S}}$, provided that either $D_F \neq 0$ or $A_F^{\mathcal{S}} < \infty$. Note that it is not necessary to extend the definition of the discriminant D_F since D_F only depends on the coefficients of F and has already been defined for all forms with complex coefficients.

With these new definitions, and observing that $T^{-1}(\mathcal{S}) = \left\{ T^{-1}\binom{\sigma}{\tau} : \binom{\sigma}{\tau} \in \mathcal{S} \right\}$ is a two-dimensional real vector subspace of $\mathbf{C}^2$, we will show in Section 3 that D_F, $A_F^{\mathcal{S}}$ and $Q(F,\mathcal{S})$ have the following $GL_2(\mathbf{C})$ invariance properties:

Theorem 2.2. *Let F be a binary form with complex coefficients, let T be a transformation in $GL_2(\mathbf{C})$, and let $\mathcal{S}$ be a two-dimensional real vector space in $\mathbf{C}^2$. Then*

$$D_F = (\det T)^{-n(n-1)} D_{F_T}.$$

Moreover, if $\mathcal{S}$ and $T^{-1}(\mathcal{S})$ are both Lagrangian planes in $\mathbf{C}^2$, then

$$A_F^{\mathcal{S}} = |\det T| A_{F_T}^{T^{-1}(\mathcal{S})},$$

and if also $D_F \neq 0$ *or* $A_F^{\mathcal{S}} < \infty$*, then*

$$Q(F, \mathcal{S}) = Q(F_T, T^{-1}(\mathcal{S})).$$

If $T \in GL_2(\mathbf{R})$ and $\mathcal{S} = \mathbf{R}^2$, then $T^{-1}(\mathcal{S}) = \mathbf{R}^2$ is again Lagrangian, and so Theorem 2.2 contains as a special case the $GL_2(\mathbf{R})$ invariance of $D_F, A_F, Q(F)$.

We remark that if $T^{-1}(\mathcal{S})$ is not Lagrangian, then $A_F^{\mathcal{S}}$ need not equal $|\det T| A_{F_T}^{T^{-1}(\mathcal{S})}$, as can be seen from the proof of Theorem 2.2 by considering $T^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & i \end{pmatrix}$ with $\mathcal{S} = \mathbf{R}^2$, for example.

The preceding two theorems allow us to express the area $A_F^{\mathcal{S}}$ in terms of formulas that directly parallel the known representation $\int_{-\infty}^{\infty} |F(1,v)|^{-2/n}\, dv$ for A_F:

Corollary 2.2.1. *Let $\mathcal{S}$ be a two-dimensional real vector space inside $\mathbf{C}^2$ that is Lagrangian. Then, for every binary form F,*

$$A_F^{\mathcal{S}} = \oint_{\mathcal{C}_1} |F(\sigma,1)|^{-2/n} |d\sigma| = \oint_{\mathcal{C}_2} |F(1,\tau)|^{-2/n} |d\tau| \tag{10}$$

where

$$\mathcal{C}_1 = \left\{ \frac{\sigma}{\tau} : \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \in \mathcal{S} \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

and

$$\mathcal{C}_2 = \left\{ \frac{\tau}{\sigma} : \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \in \mathcal{S} \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Now it is very important to be aware of the space $\mathcal{S}$ on which the area $A_F^{\mathcal{S}}$ is being calculated. Consider, for example, the form $F(X,Y) = XY(X+Y)$ on the two spaces $\mathbf{R}^2$ and

$$S = \left\{ \begin{pmatrix} \bar{z} \\ z \end{pmatrix} : z \in \mathbf{C} \right\} = \left\{ \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S} : x, y \in \mathbf{R} \right\}.$$

Obviously, $\mathbf{R}^2$ is Lagrangian and so is $\mathcal{S}$ since $A\overline{B} + C\overline{D} = (1)(\overline{-i}) + (1)(\overline{i}) = 0 \in \mathbf{R}$. From [**6**], or equation (32) in Section 3 below, it is clear that

$$A_F^{\mathbf{R}^2} = A_F = 3B(1/3, 1/3) = 2^{1/3}3^{1/2}B(1/6, 1/2).$$

On the other hand, from equation (10) above, we have

$$\begin{aligned} A_F^{\mathcal{S}} &= \int_{|\tau|=1} |\tau(1+\tau)|^{-2/3} |d\tau| \\ &= \int_{-\pi}^{\pi} |1 + e^{i\zeta}|^{-2/3} \, d\zeta \\ &= 2^{1/3} \int_0^{\pi} (\cos(\zeta/2))^{-2/3} \, d\zeta \\ &= 2^{1/3} \int_0^1 t^{-5/6}(1-t)^{-1/2} \, dt \qquad \text{using } t = \cos^2(\zeta/2) \\ &= 2^{1/3} B\left(\frac{1}{6}, \frac{1}{2}\right) \\ &< A_F^{\mathbf{R}^2}. \end{aligned}$$

Consequently, when referring to $A_F^{\mathcal{S}}$ and $Q(F, \mathcal{S})$, we must always pay attention to the space $\mathcal{S}$ on which the form F is to be considered.

Several authors, e.g., [**12**, p. 148, eq. (1.19)], have correctly remarked that $Q(F)$ is not $GL_2(\mathbf{C})$ invariant in the traditional sense. However, as Theorem 2.2 states, $Q(F, \mathcal{S})$ actually is $GL_2(\mathbf{C})$ invariant when a new perspective is adopted.

Now, when studying forms over $\mathcal{S} = \mathbf{R}^2$, the first author found two subclasses of binary forms with complex coefficients to be particularly relevant: (i) forms F with real coefficients; and (ii) forms F with a complete factorization over $\mathbf{R}$, i.e., forms F which factor as a product of linear forms with real coefficients. For a general space $\mathcal{S}$, the analogous subclasses are: (i) forms F which are real valued on $\mathcal{S}$ and (ii) forms F which factor completely as a product of linear forms that are real valued on $\mathcal{S}$. For convenience in the subsequent exposition, we will use the following notation:

$$\mathcal{F}_{\mathcal{S}} = \Big\{ \text{binary forms } F(X,Y) \in \mathbf{C}[X,Y] : F(\sigma,\tau) \in \mathbf{R} \text{ whenever } \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \in \mathcal{S} \Big\}, \tag{11}$$

(12)
$$\mathcal{F}_{\mathcal{S}}^{*} = \Big\{ \prod_{j=1}^{n} L_j(X,Y) : \text{ the } L_j \text{ are linear forms in } \mathcal{F}_{\mathcal{S}} \text{ and } n \geq 1\}.$$

While the consideration of all Lagrangian two-dimensional real vector spaces $\mathcal{S}$ in $\mathbf{C}^2$ is necessary for the correct formulation of the $GL_2(\mathbf{C})$ invariance, we will usually only work with the spaces

$$\mathcal{S}_c := \left\{ \begin{pmatrix} \bar{z} \\ z \end{pmatrix} : z \in \mathbf{C} \right\} \tag{13}$$

and

$$\mathcal{S}_l := \mathbf{R}^2, \tag{14}$$

for which the integrals in equation (10) are performed over the unit circle and the real line, respectively. (The subscripts c and l refer to the "circle" and the "line.") We remarked above that $\mathcal{S}_c$ and $\mathcal{S}_l$ are Lagrangian. The spaces $\mathcal{S}_c, \mathcal{S}_l$ and their corresponding sets of forms $\mathcal{F}_{\mathcal{S}_c}, \mathcal{F}_{\mathcal{S}_l}$ are connected in a natural way by the transformation

$$T = \begin{pmatrix} (1/2) & (1/2) \\ (i/2) & (-i/2) \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1}.$$

Indeed, $\mathcal{S}_l = T\mathcal{S}_c$ and

$$F(x,y) = F_T(\bar{z}, z)$$

for all $z = x + iy \in \mathbf{C}$, and $F \in \mathcal{F}_{\mathcal{S}_l} \Leftrightarrow F_T \in F \in \mathcal{F}_{\mathcal{S}_c}$. With this perspective, we see that moving from the unit circle to the real line in the complex plane amounts to putting $z = x + iy$ and $\bar{z} = x - iy$.

Now every linear form which is real valued on $\mathcal{S}_c$ is of the type $\bar{\gamma}\bar{z} + \gamma z$ for some $\gamma \in \mathbf{C}$. Hence, every form in $\mathcal{F}_{\mathcal{S}_c}^{*}$ is of the type $\prod_{j=1}^{n}(\bar{\gamma}_j\sigma + \gamma_j\tau)$ for some $\gamma_j \in \mathbf{C}$. For such forms, the representations of $A_F^{\mathcal{S}_c}$ and D_F can be concisely written in the following way:

Corollary 2.2.2. *Let F be a form which factors completely as a product of linear forms that are real valued on $\mathcal{S}_c$. If F is given in the symmetric form $F(\sigma,\tau) = \prod_{j=1}^{n}(\bar{\gamma}_j\sigma + \gamma_j\tau)$, then*

$$A_F^{\mathcal{S}_c} = \int_{-\pi}^{\pi} \prod_{j=1}^{n} |\gamma_j e^{i\theta/2} + \bar{\gamma}_j e^{-i\theta/2}|^{-2/n}\, d\theta \tag{15}$$

and

$$|D_F|^{1/n(n-1)} = \prod_{j\neq k} |\gamma_j\bar{\gamma}_k - \bar{\gamma}_j\gamma_k|^{1/n(n-1)}. \tag{16}$$

On the other hand, if F *is in the asymmetric form* $F(\sigma,\tau) = \kappa\prod_{j=1}^n(\tau - e^{i\theta_j}\sigma)$*, then*

$$A_F^{\mathcal{S}_c} = |\kappa|^{-2/n}\int_{-\pi}^{\pi}\prod_{j=1}^n |e^{i\theta} - e^{i\theta_j}|^{-2/n}\,d\theta \tag{17}$$

and

$$|D_F|^{1/n(n-1)} = |\kappa|^{2/n}\prod_{j\neq k}|e^{i\theta_j} - e^{i\theta_k}|^{1/n(n-1)}. \tag{18}$$

Combining these representations of $A_F^{\mathcal{S}_c}$ and D_F for $F \in \mathcal{F}^*_{\mathcal{S}_c}$ with Theorem 2.2 and the fact that $\mathcal{S}_l = T\mathcal{S}_c$ gives us two more representations of $A_F(= A_F^{\mathcal{S}_l})$ and D_F when $F \in \mathcal{F}^*_{\mathcal{S}_l}$:

Corollary 2.2.3. *Let* F *be a form which factors completely over* $\mathbf{R}$ *as* $\prod_{j=1}^n(\alpha_j X - \beta_j Y)$ *where* $\alpha_j, \beta_j \in \mathbf{R}$*. Then*

$$\begin{aligned} A_F &= \frac{1}{2}\int_{-\pi}^{\pi}\prod_{j=1}^n|\gamma_j e^{i\theta/2} + \bar{\gamma}_j e^{-i\theta/2}|^{-2/n}\,d\theta \\ &= \frac{1}{2}|\kappa|^{-2/n}\int_{-\pi}^{\pi}\prod_{j=1}^n|e^{i\theta} - e^{i\theta_j}|^{-2/n}\,d\theta \end{aligned} \tag{19}$$

and

$$\begin{aligned} D_F^{1/n(n-1)} &= 2\prod_{j\neq k}|\gamma_j\bar{\gamma}_k - \bar{\gamma}_j\gamma_k|^{1/n(n-1)} \\ &= 2|\kappa|^{2/n}\prod_{j\neq k}|e^{i\theta_j} - e^{i\theta_k}|^{1/n(n-1)} \end{aligned} \tag{20}$$

where $\gamma_j = (\alpha_j + i\beta_j)/2$*,* $\kappa = \prod_{j=1}^n \gamma_j$ *and* $\theta_j = 2\arctan(\alpha_j/\beta_j)$*.*

Note that the second representation of A_F in equation (19) above is identical to formula (9) in Corollary 2.1.1.

The reader may be puzzled by the apparent incompatibility of formulas (16) and (18) with formula (20). In fact, these formulas are completely consistent once one takes into account the spaces $\mathcal{S}_c$ and $\mathcal{S}_l$. More precisely, if $F \in \mathcal{F}^*_{\mathcal{S}_l}$ and if $T = \begin{pmatrix} (1/2) & (1/2) \\ (i/2) & (-i/2) \end{pmatrix}$, then the formulas stated in Corollary 2.2.2 actually apply to the form $F_T \in \mathcal{F}^*_{\mathcal{S}_c}$, not to F; moreover, the factors of $1/2$ and 2 in equations (19) and (20), respectively, come from the fact that $|\det T| = (1/2)$.

Corollary 2.2.3 reveals that the maximization conjecture for $Q(F)$, Conjecture 1 in Section 1, can be interpreted as claiming that

$$(*) \qquad Q(F) = \prod_{j \neq k} |e^{i\theta_j} - e^{i\theta_k}|^{1/n(n-1)} \cdot \int_{-\pi}^{\pi} \prod_{j=1}^{n} |e^{i\theta} - e^{i\theta_j}|^{-2/n}\, d\theta$$

is maximal when the points $e^{i\theta_j}$ are evenly distributed around the unit circle. This is a nontrivial problem, because while Polya and Schur [**18**, p. 385] long ago showed that the discriminant term $\prod_{j \neq k} |e^{i\theta_j} - e^{i\theta_k}|$ is indeed maximal when the points $e^{i\theta_j}$ are equidistributed, it follows from work of Arestov [**2**, Theorem 4] that the area term $\int_{-\pi}^{\pi} \prod_{j=1}^{n} |e^{i\theta} - e^{i\theta_j}|^{-2/n}\, d\theta$ is actually *minimal* when the $e^{i\theta_j}$ are equidistributed. (See Baernstein's paper [**3**, p. 144] for a proof by a different method and a discussion of Arestov's result.) Conjecture 1 therefore asserts, roughly speaking, that the maximum of the discriminant overwhelms the minimum of the area.

2.3. Binary forms and equiangular polygons. The Schwarz-Christoffel transformations provide us with yet another representation of A_F, namely as the perimeter of an equiangular polygon. An appealing feature of this representation is that it enables us to translate the analytic problem of maximizing $Q(F)$ into the geometric problem of maximizing the perimeter of an equiangular polygon of a given "size" and in so doing, it provides us with another plausible approach to answering questions (Q1), (Q2), (Q3) of the introduction. We will shortly give a precise reformulation of the maximization problem in geometric and potential theoretic terms. However, before we can do this, we will need the following explicit version of the Schwarz-Christoffel correspondence:

Theorem 3. *Let F be a binary form of degree $n \geq 3$ with $D_F \neq 0$ and with a complete factorization over $\mathbf{R}$, and let $s_1, \ldots, s_n$ be the roots of $F(1, v)$ ordered such that $s_1 < \cdots < s_n$, where possibly $s_n = \infty$. Let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a transformation in $GL_2(\mathbf{C})$, and let V_T be the associated fractional linear transformation defined by $V_T(w) = (c + dw)/(a + bw)$ for $w \in \mathbf{C}$. Let $\mathcal{D}$ be the image of the upper half plane under V_T^{-1}, and let $z_0 = V_T^{-1}(i) \in \mathcal{D}$.*

Then the map g_F^T defined by

$$g_F^T(z) = (\det T) \int_{z_0}^{z} F_T(1, \tau)^{-2/n} \, d\tau \tag{21}$$

is a conformal map of $\mathcal{D}$ onto an n-sided equiangular polygon $\mathcal{P}$ with interior angles $(n-2)\pi/n$ such that

(i) $g_F^T(z_0) = 0 \in \mathcal{P}$;

(ii) *the vertices $v_1, \ldots, v_n$ of the polygon $\mathcal{P}$, in consecutive order, are given by $v_j = g_F^T(V_T^{-1}(s_j))$;*

(iii) *the length of the edge $v_j v_{j+1}$ on $\partial\mathcal{P}$ is*

$$|\det T| \int_{V_T^{-1}(s_j)}^{V_T^{-1}(s_{j+1})} |F_T(1, \tau)|^{-2/n} |d\tau|.$$

Moreover, the mapped polygon $\mathcal{P}$ is the same for all $T \in GL_2(\mathbf{C})$, provided that we choose the correct branch of the nth root in (21), *and has perimeter $|\partial\mathcal{P}| = A_F$.*

Conversely, for any n-sided equiangular polygon $\mathcal{P}$ containing the origin and for any domain $\mathcal{D}$ whose boundary is a circle on the Riemann sphere, there is a binary form F of degree n with nonzero discriminant and a complete factorization over $\mathbf{R}$, and there is a transformation $T \in GL_2(\mathbf{C})$ such that the function g_F^T defined by (21) *maps $\mathcal{D}$ conformally onto $\mathcal{P}$ and has properties* (i), (ii) *and* (iii).

From this it follows, by taking $T = I$ and using Section 5 of [**5**], that for forms F and G with a complete factorization over $\mathbf{R}$:

• F and G are equivalent under $GL_2(\mathbf{R})$, up to multiplication by -1, if and only if $\mathcal{P}(F)$ and $\mathcal{P}(G)$ are similar polygons;

• F and G are equivalent under $SL_2(\mathbf{R})$, up to multiplication by -1, if and only if $\mathcal{P}(F)$ and $\mathcal{P}(G)$ are congruent polygons;

• $\mathcal{P}(F_n^*)$ is an n-sided regular polygon.

Note that there is no closed polygon for $n = 2$ which gives a concrete reason for A_F to be infinite when F is an indefinite binary quadratic form, i.e., equivalent under $GL_2(\mathbf{R})$ to $X^2 - Y^2$.

The maps g_F^T with $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} (1/2) & (1/2) \\ (i/2) & (-i/2) \end{smallmatrix}\right)$ are of particular interest and will be denoted by g_F and h_F, respectively, throughout the paper. Note that $g_F : \mathbf{H} \to \mathcal{P}(F)$ and that $h_F : \mathbf{D} \to \mathcal{P}(F)$, where $\mathbf{H} := \{z \in \mathbf{C} : \Im z > 0\}$ and $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$. Closely related to g_F and h_F are the fractional linear transformations $U^* : \mathbf{H} \to \mathbf{D}$ and $V^* : \mathbf{D} \to \mathbf{H}$ defined by

$$U^*(z) = \frac{i - z}{i + z}, \tag{22}$$

$$V^*(w) = i\frac{1 - w}{1 + w}, \tag{23}$$

or equivalently by $U^* = V_T^{-1}$ and $V^* = V_T$ where V_T is the fractional linear transformation associated to $T = \left(\begin{smallmatrix} (1/2) & (1/2) \\ (i/2) & (-i/2) \end{smallmatrix}\right)$. The explicit form of g_F, h_F and the connections among g_F, h_F, U^* and V^* are given in the following corollary:

Corollary 3.1. *Let $F(X,Y) = \prod_{j=1}^n (\alpha_j X - \beta_j Y)$ be a binary form of degree $n \geq 3$ with $D_F \neq 0$ and with a complete factorization over $\mathbf{R}$, and let $s_1 = \alpha_1/\beta_1, \ldots, s_n = \alpha_n/\beta_n$ be the roots of $F(1,v)$ ordered such that $s_1 < \cdots < s_n$, where possibly $s_n = \infty$. Let $\mathcal{P}$ be the polygon corresponding to F under the family of conformal maps g_F^T defined by* (21).

If $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then g_F^T is a conformal map from $\mathbf{H}$ to $\mathcal{P}$ with defining equation

$$g_F(z) = a^{-2/n} \int_i^z \prod_{j=1}^n (v - s_j)^{-2/n} \, dv \tag{24}$$

where $a = (-1)^n \beta_1 \cdots \beta_n$*, while if* $T = \begin{pmatrix} (1/2) & (1/2) \\ (i/2) & (-i/2) \end{pmatrix}$*, then* g_F^T *is a conformal map from* $\mathbf{D}$ *to* $\mathcal{P}$ *with defining equation*

$$h_F(w) = \frac{-i}{2}\kappa^{-2/n} \int_0^w \prod_{j=1}^n (\tau - e^{i\theta_j})^{-2/n}\, d\tau \tag{25}$$

where $\kappa = \prod_{j=1}^n (\alpha_j + i\beta_j)/2$ *and* $\theta_j = 2\arctan(\alpha_j/\beta_j)$*. The vertices of* $\mathcal{P}$ *in counterclockwise order are* $v_1, \dots, v_n$*, where* $v_j = g_F(s_j) = h_F(e^{i\theta_j})$*. Moreover,* $s_j = V^*(e^{i\theta_j}), e^{i\theta_j} = U^*(s_j)$ *and* $g_F \circ V^* = h_F$*,* $h_F \circ U^* = g_F$*.*

Note that if $\beta_n = 0$ and $s_n = \infty$, then $\theta_n = \pi$ and, by convention, we replace β_n by α_n in the definition of a and we omit the nth factor of the product in (24).

Notice the similarity between the formulas for g_F, h_F and the formulas for A_F in Corollary 2.1.1. Figure 3 gives a visual representation of the composition properties for the maps g_F, h_F, U^* and V^* which will be helpful later in the paper.

The proof of Theorem 3 and its corollary will be given in Section 5.

2.4. Equiangular polygons and harmonic measure. The correspondence between binary forms and equiangular polygons described in Theorem 3 and its corollary gives us a potentially important tool with which to attack the problem of maximizing $Q(F)$. Indeed, from the above correspondence, it is clear that maximizing $Q(F)$ over all binary forms with nonzero discriminant and degree at least three is equivalent to maximizing the perimeter of an n-sided equiangular polygon of a given "size" (as measured by the discriminant). The hope here is that one might solve the problem using geometry alone. However, before we can approach the problem in this way, we need to formulate the notion of "size" of a polygon—as measured by the discriminant—solely in terms of (geometric) quantities intrinsic to the polygon and without reference to the discriminant of the associated binary form.

To achieve this formulation, we first consider the meaning of the discriminant of a form F, with a complete factorization over $\mathbf{R}$, in the context of the geometry of the curve $|F(x, y)| = 1$. Hence, suppose that $F(X, Y) = \prod_{j=1}^n (\alpha_j X - \beta_j Y)$ with $\alpha_j, \beta_j \in \mathbf{R}$. Then the discriminant

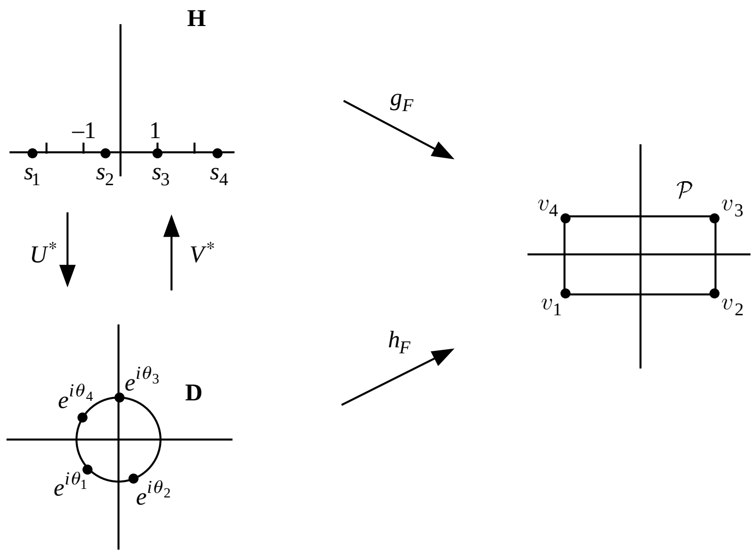

FIGURE 3.

of F is $D_F = \prod_{j<k}(\alpha_j\beta_k - \alpha_k\beta_j)^2$. Viewed from a purely algebraic perspective, the discriminant is an indicator of degeneracy in the form, in the sense that $D_F = 0$ if and only if F has two factors $\alpha_j X - \beta_j Y$ and $\alpha_k X - \beta_k Y$ that are proportional. However, from the perspective of the curve $|F(x, y)| = 1$ in the real affine plane, $D_F^{1/n(n-1)}$ can be considered a measure of two quantities: the size of the region $|F(x, y)| \leq 1$ and the relative separation of the asymptotic lines L_j defined by $\alpha_j x - \beta_j y = 0$.

To see how this is possible, put $\rho_j = \sqrt{\alpha_j^2 + \beta_j^2} > 0$ and $\psi_j = \arctan(\alpha_j/\beta_j) \in (-(\pi/2), (\pi/2)]$, so that ψ_j gives the direction of the asymptote L_j. Notice that either $\sin\psi_j = \alpha_j/\rho_j$ and $\cos\psi_j = \beta_j/\rho_j$ or else $\sin\psi_j = -\alpha_j/\rho_j$ and $\cos\psi_j = -\beta_j/\rho_j$. Then

$$\begin{aligned} D_F &= (\rho_1 \cdots \rho_n)^{2(n-1)} \prod_{j<k} \left(\frac{\alpha_j}{\rho_j}\frac{\beta_k}{\rho_k} - \frac{\alpha_k}{\rho_k}\frac{\beta_j}{\rho_j} \right)^2 \\ &= (\rho_1 \cdots \rho_n)^{2(n-1)} \prod_{j<k} (\sin\psi_j \cos\psi_k - \sin\psi_k \cos\psi_j)^2 \\ &= (\rho_1 \cdots \rho_n)^{2(n-1)} \prod_{j<k} (\sin(\psi_k - \psi_j))^2. \end{aligned}$$

Hence,

$$D_F^{1/n(n-1)} = (\rho_1 \cdots \rho_n)^{2/n} \prod_{j \neq k} |\sin(\psi_k - \psi_j)|^{1/n(n-1)}. \tag{26}$$

We claim that $\prod_{j\neq k} |\sin(\psi_k - \psi_j)|^{1/n(n-1)}$ is a measure of average separation of the asymptotes and $(\rho_1 \cdots \rho_n)^{-1/n}$ is a measure of the average distance of the graph $|F(x,y)| = 1$ from the origin. Indeed, $|\sin(\psi_k - \psi_j)|$ is a measure of the separation of the lines L_j and L_k, normalized on a scale from 0 to 1, and $\prod_{j\neq k} |\sin(\psi_k-\psi_j)|^{1/n(n-1)}$ is the geometric mean of these separations. On the other hand, $(\rho_1 \cdots \rho_n)^{-1/n}$ can be considered a measure of the "average distance" of the graph $|F(x,y)| = 1$ from the origin since replacement of any ρ_j by $c\rho_j$, with the other ρ_k fixed, has the effect of magnifying the graph by a factor of $c^{-1/n}$. Hence, putting

$$d(L_j, L_k) = |\sin(\psi_k - \psi_j)| \tag{27}$$

and

$$r_F = (\rho_1 \cdots \rho_n)^{-1/n}, \tag{28}$$

we see that

$$D_F^{1/n(n-1)} = \frac{1}{r_F^2} \prod_{j\neq k} d(L_j, L_k)^{1/n(n-1)}. \tag{29}$$

Consequently, $D_F^{1/n(n-1)}$ encapsulates information about both the relative separation of the asymptotes and also the size of the region $|F(x,y)| \leq 1$ as measured by the "area estimator" r_F^2. (Of course, there is no general functional relationship between A_F and r_F^2, though r_F^2 will have the same order of magnitude as A_F.)

Now, in light of the Schwarz-Christoffel correspondence and the above discussion, it is reasonable to expect $D_F^{1/n(n-1)}$, which measures the size of the polygon corresponding to F, to be expressible in terms of quantities which measure the relative separation of the vertices v_j and also the average "spread" of the polygon. Unfortunately, the most natural measure of separation and spread—Euclidean distance—will not work. Indeed, if $\mathcal{P}$ is an equiangular triangle, for example, with side length s, then the geometric mean of the distances between vertices is $\prod_{j\neq k}|v_j-v_k|^{1/6}=s$, while the perimeter of $\mathcal{P}$ is $3s$. However, for any cubic form F, with $D_F>0$, the equiangular triangle corresponding to F under the Schwarz-Christoffel correspondence has side length $B[(1/3),(1/3)]/|D_F|^{1/6}$, see (32) below, and so it cannot be the case that $|D_F|^{1/6}=\prod_{j\neq k}|v_j-v_k|^{1/6}=s$. Similar remarks can be made when $n>3$. Hence, a different measure of separation and spread on the polygon is required.

It turns out that the appropriate measures of separation and spread for the mapped polygon are *harmonic measure* and *harmonic radius.* In the context of the complex plane, these notions are actually quite natural. However, the definitions are somewhat technical to state, and so we will postpone them until Section 6, where we will prove the following result:

Theorem 4. *Let F be a binary form of degree $n\geq 3$ with $D_F\neq 0$ and with a complete factorization over $\mathbf{R}$, and let $\mathcal{P}$ be the polygon, containing the origin, corresponding to F under the family of maps g_F^T of Theorem 3. Let $v_1,\dots,v_n$ be the vertices of $\mathcal{P}$ in consecutive order. Then*

$$A_F=|\partial\mathcal{P}|$$

and

$$D_F^{1/n(n-1)}=\frac{2}{R_{\mathcal{P}}}\left\{\prod_{j\neq k}\sin(\pi\mu_{\mathcal{P}}(\widehat{v_jv_k}))\right\}^{1/n(n-1)} \tag{30}$$

where $R_{\mathcal{P}}$ is the harmonic radius of $\mathcal{P}$ with respect to the origin, and $\mu_{\mathcal{P}}(\widehat{v_jv_k})$ is the harmonic measure of the arc $\widehat{v_jv_k}$ on $\partial\mathcal{P}$ with respect to the origin. Consequently,

$$Q(F)=\frac{2|\partial\mathcal{P}|}{R_{\mathcal{P}}}\left\{\prod_{j\neq k}\sin(\pi\mu_{\mathcal{P}}(\widehat{v_jv_k}))\right\}^{1/n(n-1)}. \tag{31}$$

We will actually show in Section 6 that $r_F=(R_{\mathcal{P}}/2)^{1/2}$ and $d(L_j,L_k)=\sin(\pi\mu_{\mathcal{P}}(\widehat{v_jv_k}))$. We will also show that $R_{\mathcal{P}}$ and $\mu_{\mathcal{P}}$ can be replaced in Theorem 4 by the harmonic radius and the harmonic measure with respect to any fixed point ζ in $\mathcal{P}$.

Using Theorems 1, 3 and 4, we can now reformulate the maximization problem (Q2) in Section 1 and its conjectured solution in purely geometric and potential theoretic terms:

Reformulated maximization problem. *To determine, among all n-sided equiangular polygons $\mathcal{P}$ containing the origin, the maximum value of*

$$Q(\mathcal{P}) = \frac{2|\partial\mathcal{P}|}{R_{\mathcal{P}}}\left\{\prod_{j\neq k}\sin(\pi\mu_{\mathcal{P}}(\widehat{v_j v_k}))\right\}^{1/n(n-1)} \tag{$**$}$$

and to characterize the equiangular polygons for which this maximum is achieved.

Reformulated conjecture. *The maximum value M_n of $Q(\mathcal{P})$, taken over all n-sided equiangular polygons $\mathcal{P}$ containing the origin, is equal to*

$$2^{1-2/n}n^{1/(n-1)}B\left(\frac{1}{2}-\frac{1}{n},\frac{1}{2}\right)$$

and is achieved precisely when $\mathcal{P}$ is an n-sided regular polygon.

3. Binary forms in $\mathbf{C}^2$—Proof of Theorems 2.1 and 2.2 and their corollaries. One of the key ideas in the work of the first author [**5, 6, 7, 8, 9, 10**] was the use of transformations in $GL_2(\mathbf{R})$ to simplify the computation of A_F and $Q(F)$. Indeed, the first author defined the action of $GL_2(\mathbf{R})$ on a binary form $F(X,Y) \in \mathbf{C}[X,Y]$ by

$$F_T(X,Y) = F(aX+bY, cX+dY), \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{R})$$

and then used the resulting invariance properties

(P1) $A_F = |\det T| A_{F_T}$

(P2) $D_F = (\det T)^{-n(n-1)} D_{F_T}$

(P3) $Q(F) = Q(F_T)$

(along with several other ideas) to prove the results (R1), (R2) and (R3) stated in the introduction. In this section we will explain how transformations in $GL_2(\mathbf{C})$ can be used to further simplify the calculation of A_F and, as a consequence, we will obtain the representations of A_F stated in Theorem 2.1 and Corollary 2.1.1. In the next section we will use a special case of these representations to derive the formula for $A_{F_n^*}$ stated in Theorem 1.

Now the natural way to define the action of $GL_2(\mathbf{C})$ on a binary form $F(X,Y) \in \mathbf{C}[X,Y]$ is by:

$$F_T(X,Y) = F(aX+bY, cX+dY) \quad \text{for } T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{C}).$$

Unfortunately, properties (P1) and (P3) do not hold when the action of $GL_2(\mathbf{C})$ is defined in this way.[3] Indeed, it is well known [**11**, p. 17] that all the (nondegenerate) cubic forms in $\mathbf{C}[X,Y]$ are equivalent to one another under $GL_2(\mathbf{C})$, but $Q(F)$ is not constant over the class of cubic forms with complex coefficients [**8**, p. 1978]. In fact, even over the restricted class of cubic forms with real coefficients, $Q(F)$ assumes more than one value:

$$Q(F) = \begin{cases} 3B[(1/3),(1/3)] & \text{if } D_F > 0, \\ \sqrt{3}B[(1/3),(1/3)] & \text{if } D_F < 0. \end{cases} \tag{32}$$

At first glance, then, it would seem that properties (P1) and (P3) cannot be preserved. However, a little reflection reveals that the problem is not with our definition of the action of $GL_2(\mathbf{C})$; rather, it is with our failure to transform the variables X,Y when calculating the area. Indeed, the area A_{F_T} on the right hand side of the equation in (P1) should not, in general, be calculated in $\mathbf{R}^2$; rather, it should be calculated in the *image* of $\mathbf{R}^2$ under T^{-1}. Hence, we will need to keep track of the two-dimensional real vector space inside $\mathbf{C}^2$ on which the area is to be calculated.

Now every two-dimensional real vector space $\mathcal{S}$ inside $\mathbf{C}^2$ is of the type

$$\mathcal{S} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} : x,y \in \mathbf{R} \right\}$$

for some fixed matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GL_2(\mathbf{C})$. Moreover, the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ defining $\mathcal{S}$ is unique up to right multiplication by an element of $GL_2(\mathbf{R})$.

Hence, with a slight abuse of notation, we can write

$$\mathcal{S} = \left\{ W_{\mathcal{S}} \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbf{R} \right\}$$

where $W_{\mathcal{S}}$ is the "unique" matrix in $GL_2(\mathbf{C})$ which defines $\mathcal{S}$.

From the preceding discussion, it is clear that if the variables of a form $F(X,Y)$ are transformed to the space $\mathcal{S}$ under the action of $T \in GL_2(\mathbf{C})$, then the appropriate area to calculate is the area of the region $\left\{ \binom{\sigma}{\tau} \in \mathcal{S} : |F(\sigma,\tau)| \le 1 \right\}$. Hence, for any space $\mathcal{S}$ and any form F, write $A_F^{\mathcal{S}}$ for the area of the two real dimensional region $\{ \binom{\sigma}{\tau} \in \mathcal{S} : |F(\sigma,\tau)| \le 1 \}$ in $\mathbf{C}^2 \simeq \mathbf{R}^4$. Further, put $Q(F,\mathcal{S}) = |D_F|^{1/n(n-1)} A_F^{\mathcal{S}}$. Then, as we will see before the end of the section, the modified invariance properties

(P1′) $A_F^{\mathcal{S}} = |\det T| A_{F_T}^{T^{-1}(\mathcal{S})}$

(P3′) $Q(F,\mathcal{S}) = Q(F_T, T^{-1}(\mathcal{S}))$

do hold for all suitable transformations T and Lagrangian spaces $\mathcal{S}$.

Note that, for a general space $\mathcal{S}$, the classes of forms

$$\mathcal{F}_{\mathcal{S}} = \Big\{ \text{binary forms } F(X,Y) \in \mathbf{C}[X,Y] : \\ F(\sigma,\tau) \in \mathbf{R} \text{ whenever } \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \in \mathcal{S} \Big\},$$

$$\mathcal{F}_{\mathcal{S}}^* = \left\{ \prod_{j=1}^{n} L_j(X,Y) : \text{ the } L_j \text{ are linear forms in } \mathcal{F}_{\mathcal{S}} \text{ and } n \ge 1 \right\}$$

are generalizations, respectively, of the classes of forms with real coefficients and with complete factorizations over $\mathbf{R}$. Indeed, $\mathcal{F}_{\mathcal{S}}$ is the collection of binary forms which are real valued on $\mathcal{S}$, while $\mathcal{F}_{\mathcal{S}}^*$ is the collection of forms which factor as a product of real valued linear forms. It is fairly straightforward to show that

$$F \in \mathcal{F}_{\mathcal{S}} \iff F_T \in \mathcal{F}_{T^{-1}(\mathcal{S})}$$

and that

$$F \in \mathcal{F}_{\mathcal{S}}^* \iff F_T \in \mathcal{F}_{T^{-1}(\mathcal{S})}^*.$$

Hence the action of $GL_2(\mathbf{C})$ preserves the properties of being real valued and of having a complete factorization. It is also straightforward to show that every linear form in $\mathcal{F}_{\mathcal{S}}$ is of the type $L(\sigma,\tau) = \gamma\sigma + \delta\tau$ where $\left(\begin{smallmatrix}\gamma & \delta\end{smallmatrix}\right) \in \mathcal{C}_{\mathcal{S}} := \{\left(\begin{smallmatrix}\alpha & \beta\end{smallmatrix}\right) W_{\mathcal{S}}^{-1} : \alpha, \beta \in \mathbf{R}\}$. Hence, the coefficient vector of a linear form which is real valued on $\mathcal{S}$ need not (and generally will not) be in the variable space $\mathcal{S}$.

Before proceeding to the proof of Theorem 2, it is worth remarking that binary forms are really just matrix products of the type

$$F(X,Y) = \prod_{j=1}^{n} \begin{pmatrix}\alpha_j & -\beta_j\end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

With this perspective, the action of $T \in GL_2(\mathbf{C})$ on F can be written as

$$F_T(X,Y) = \prod_{j=1}^{n} \begin{pmatrix}\alpha_j & -\beta_j\end{pmatrix} T \begin{pmatrix} X \\ Y \end{pmatrix},$$

and it is a straightforward matter to determine the coefficient vectors of the linear factors of F_T: if $F(X,Y) = \prod_{j=1}^{n}(\alpha_j X - \beta_j Y)$, then $F_T(X,Y) = \prod_{j=1}^{n}(\hat{\alpha}_j X - \hat{\beta}_j Y)$ where $\begin{pmatrix}\hat{\alpha}_j & -\hat{\beta}_j\end{pmatrix} = \left(\begin{smallmatrix}\alpha_j & -\beta_j\end{smallmatrix}\right) T$.

3.1. *Proof of Theorem* 2.1 *and Corollary* 2.1.1. We now proceed to derive the representations of A_F stated in Theorem 2.1. Hence, let $F(X,Y) = \prod_{j=1}^{n}(\alpha_j X - \beta_j Y)$ be a binary form, and let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a transformation in $GL_2(\mathbf{C})$. Further, put

$$\mathcal{S} = \left\{ T^{-1} \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbf{R} \right\}$$

and let

$$\mathcal{C}_1 = \left\{ \frac{\sigma}{\tau} : \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \in \mathcal{S} \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$
$$\mathcal{C}_2 = \left\{ \frac{\tau}{\sigma} : \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \in \mathcal{S} \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are circles on the Riemann sphere, i.e., they are either straight lines or circles in the complex plane. Indeed, $z \mapsto (az+b)/(cz+d)$ is a fractional linear transformation whose inverse maps

$\mathbf{R}$ to $\mathcal{C}_1$, $w \mapsto (c+dw)/(a+bw)$ is a fractional linear transformation whose inverse maps $\mathbf{R}$ to $\mathcal{C}_2$ and (as is well known) fractional linear transformations map circles and lines to circles and lines.

Suppose that we apply the substitution

$$u = \frac{a\sigma + b}{c\sigma + d}$$

to the integral $\int_{-\infty}^{\infty} |F(u,1)|^{-2/n}\,du$, which is a known representation of A_F; see [**6**]. Then, by the remarks just made, this integral will be transformed into a line integral over the circle $\mathcal{C}_1$. Under this substitution, the differential du becomes

$$du = |du| = \left| \frac{ad-bc}{(c\sigma+d)^2}\,d\sigma \right| = \frac{|\det T|}{|c\sigma+d|^2}\,|d\sigma|$$

and each linear factor $L(u,1) = \alpha u - \beta$ of $F(u,1) = \prod_{j=1}^{n}(\alpha_j u - \beta_j)$ becomes

$$\begin{aligned} L(u,1) &= (\alpha - \beta)\begin{pmatrix} u \\ 1 \end{pmatrix} \\ &= (\alpha - \beta)\begin{pmatrix} \frac{a\sigma+b}{c\sigma+d} \\ 1 \end{pmatrix} \\ &= (\alpha - \beta)\begin{pmatrix} a\sigma+b \\ c\sigma+d \end{pmatrix}(c\sigma+d)^{-1} \\ &= (\alpha - \beta)T\begin{pmatrix} \sigma \\ 1 \end{pmatrix}(c\sigma+d)^{-1} \\ &= L_T(\sigma,1)(c\sigma+d)^{-1}. \end{aligned}$$

Hence,

$$|F(u,1)|^{-2/n}\,du = |F_T(\sigma,1)|^{-2/n}|\det T|\,|d\sigma|$$

and so

$$A_F = \int_{-\infty}^{\infty} |F(u,1)|^{-2/n}\,du = |\det T| \oint_{\mathcal{C}_1} |F_T(\sigma,1)|^{-2/n}\,|d\sigma|,$$

which is formula (4) in the statement of Theorem 2.1.

On the other hand, suppose that we apply the substitution

$$v = \frac{c + d\tau}{a + b\tau}$$

to the integral $\int_{-\infty}^{\infty} |F(1,v)|^{-2/n}\,dv$. Then, arguing as above, we find that

$$A_F = \int_{-\infty}^{\infty} |F(1,v)|^{-2/n}\,dv = |\det T| \oint_{\mathcal{C}_2} |F_T(1,\tau)|^{-2/n}\,|d\tau|,$$

which is formula (5) in the statement of Theorem 2.1.

Now suppose that $F(X,Y) \in \mathbf{C}[X,Y]$ is a binary form and that we are given circles $\mathcal{C}_1, \mathcal{C}_2$ which are inverses of each other in the sense that $z \in \mathcal{C}_1 \Leftrightarrow z^{-1} \in \mathcal{C}_2$. Then there is a fractional linear transformation $z \mapsto (az+b)/(cz+d)$, $a,b,c,d \in \mathbf{C}$, $ad-bc \neq 0$, which maps $\mathcal{C}_1$ to $\mathbf{R}$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are inverses, the transformation $w \mapsto (c+dw)/(a+bw)$ is also a fractional linear map which takes $\mathcal{C}_2$ to $\mathbf{R}$. Put $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then clearly $T \in GL_2(\mathbf{C})$ and, by the previous discussion, formulas (4) and (5) in the statement of Theorem 2.1 hold with these T, $\mathcal{C}_1$ and $\mathcal{C}_2$.

It remains to consider the two special cases:

- $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
- $T = \left(\begin{smallmatrix} (1/2) & (1/2) \\ (i/2) & -(i/2) \end{smallmatrix}\right)$.

In the first case the integration takes place on the real line, while in the second case (the more interesting one) it takes place on the unit circle.

Case 1. $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Here the variable space is

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbf{R} \right\} = \mathbf{R}^2 = \mathcal{S}_l$$

and the "circles" of integration $\mathcal{C}_1, \mathcal{C}_2$ are both equal to the real line. Hence formulas (4) and (5) reduce to

$$A_F = \int_{-\infty}^{\infty} |F(u,1)|^{-2/n}\,du = \int_{-\infty}^{\infty} |F(1,v)|^{-2/n}\,dv,$$

which is equation (6) in the statement of Theorem 2.1.

If $F(X, Y)$ has a complete factorization over $\mathbf{R}$ of the type $\prod_{j=1}^{n}(\alpha_j X - \beta_j Y)$ and if either $\alpha_j \neq 0$ for all j or else $\beta_j \neq 0$ for all j, then we can write the expressions (6) for A_F as

$$A_F = |b|^{-2/n} \int_{-\infty}^{\infty} \prod_{j=1}^{n} |u - t_j|^{-2/n} \, du$$

and

$$A_F = |a|^{-2/n} \int_{-\infty}^{\infty} \prod_{j=1}^{n} |v - s_j|^{-2/n} \, dv,$$

respectively, where $s_j = \alpha_j/\beta_j = 1/t_j$, $a = (-1)^n \beta_1 \cdots \beta_n$ and $b = \alpha_1 \cdots \alpha_n$.[4] That is, we have shown that formula (8) in the statement of Corollary 2.1.1 is true. If $\beta_k = 0$ for some k, then (6) again gives (8), but with β_k replaced by α_k in the definition of a and with the kth factor of the product in (8) omitted.

Note that we can impose an ordering on either the s_j or the t_j, but not both. To maintain consistency throughout the paper, we will always assume that $s_1 \leq \cdots \leq s_n$ when $F(X, Y)$ has a complete factorization over $\mathbf{R}$; if also $D_F \neq 0$ then we may assume that $s_1 < \cdots < s_n$.

Case 2. $T = \begin{pmatrix} (1/2) & (1/2) \\ (i/2) & -(i/2) \end{pmatrix}$.

Here the variable space is

$$\begin{aligned} \mathcal{S} &= \left\{ T^{-1} \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbf{R} \right\} \\ &= \left\{ \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbf{R} \right\} \\ &= \left\{ \begin{pmatrix} \bar{z} \\ z \end{pmatrix} : z \in \mathbf{C} \right\} \\ &= \mathcal{S}_c \end{aligned}$$

and so the circles of integration $\mathcal{C}_1, \mathcal{C}_2$ are both equal to the unit circle $\{z \in \mathbf{C} : |z| = 1\}$. Hence formulas (4) and (5) reduce to

$$A_F = \frac{1}{2} \int_{-\pi}^{\pi} \left| F\left(\frac{e^{i\phi} + 1}{2}, \frac{i(e^{i\phi} - 1)}{2} \right) \right|^{-2/n} d\phi$$

and

$$A_F = \frac{1}{2}\int_{-\pi}^{\pi} \left| F\left(\frac{1+e^{i\theta}}{2}, \frac{i(1-e^{i\theta})}{2}\right)\right|^{-2/n} d\theta,$$

which is equation (7) in the statement of Theorem 2.1.

Suppose that $F(X,Y)$ has a complete factorization over $\mathbf{R}$ of the type $F(X,Y) = \prod_{j=1}^{n}(\alpha_j X - \beta_j Y)$ with $\alpha_j, \beta_j \in \mathbf{R}$. We proceed to show that formulas (4) and (5) can actually be reduced to the much simpler expressions

$$A_F = \frac{1}{2}|\bar{\kappa}|^{-2/n} \int_{-\pi}^{\pi} \prod_{j=1}^{n} |e^{i\phi} - e^{-i\theta_j}|^{-2/n}\, d\phi \tag{33}$$

and

$$A_F = \frac{1}{2}|\kappa|^{-2/n} \int_{-\pi}^{\pi} \prod_{j=1}^{n} |e^{i\theta} - e^{i\theta_j}|^{-2/n}\, d\theta, \tag{34}$$

where

$$\kappa = \prod_{j=1}^{n} \frac{\alpha_j + i\beta_j}{2}$$

and

$$\theta_j = 2\arctan(\alpha_j/\beta_j).$$

For this, first notice that the linear form $L(X,Y) = \alpha X - \beta Y \in \mathcal{F}_{\mathcal{S}_l}$ is transformed (under T) to the linear form

$$\begin{aligned} L_T(\sigma,\tau) &= (\,\alpha \quad -\beta\,)\, T \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\alpha - i\beta}{2} & \dfrac{\alpha + i\beta}{2} \end{pmatrix} \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \\ &= \frac{\alpha - i\beta}{2}\sigma + \frac{\alpha + i\beta}{2}\tau \in \mathcal{F}^*_{\mathcal{S}_c}. \end{aligned}$$

Hence the linear factor $\alpha_j X - \beta_j Y$ of $F(X,Y)$ is transformed to the linear factor $\bar{\gamma}_j\sigma + \gamma_j\tau$ of $F_T(\sigma,\tau)$ where $\gamma_j = (\alpha_j + i\beta_j)/2$, and so $F_T(\sigma,\tau) = \prod_{j=1}^{n}(\bar{\gamma}_j\sigma + \gamma_j\tau)$.

We consider the integrations over $\mathcal{C}_1$ and $\mathcal{C}_2$ separately to obtain (33) and (34).

(i) *Integration over* $\mathcal{C}_1$. Notice that

$$\begin{aligned} F_T(\sigma,\tau) &= \bar{\kappa}\prod_{j=1}^{n}\left(\sigma + \frac{\alpha_j + i\beta_j}{\alpha_j - i\beta_j}\tau\right) \\ &= \bar{\kappa}\prod_{j=1}^{n}\left(\sigma - \frac{i + \alpha_j/\beta_j}{i - \alpha_j/\beta_j}\tau\right) \\ &= \bar{\kappa}\prod_{j=1}^{n}(\sigma - U^*(\alpha_j/\beta_j)^{-1}\tau) \end{aligned}$$

where U^* is the fractional linear map defined by equation (22) of Section 2. Now

$$U^*(\alpha_j/\beta_j) = \frac{1 + i\alpha_j/\beta_j}{1 - i\alpha_j/\beta_j} = \frac{\cos(\theta_j/2) + i\sin(\theta_j/2)}{\cos(\theta_j/2) - i\sin(\theta_j/2)} = e^{i\theta_j}.$$

Hence

$$F_T(\sigma,\tau) = \bar{\kappa}\prod_{j=1}^{n}(\sigma - e^{-i\theta_j}\tau)$$

and so, by (4) of Theorem 2.1,

$$\begin{aligned} A_F &= |\det T| \oint_{\mathcal{C}_1} |F_T(\sigma,1)|^{-2/n}\,|d\sigma| \\ &= \frac{1}{2}|\bar{\kappa}|^{-2/n}\int_{-\pi}^{\pi}\prod_{j=1}^{n}|e^{i\phi} - e^{-i\theta_j}|^{-2/n}\,d\phi \end{aligned}$$

as claimed in (33).

(ii) *Integration over* $\mathcal{C}_2$. Here we can write

$$F_T(\sigma,\tau) = \kappa\prod_{j=1}^{n}(\tau - U^*(\alpha_j/\beta_j)\sigma) = \kappa\prod_{j=1}^{n}(\tau - e^{i\theta_j}\sigma).$$

Consequently,

$$\begin{aligned} A_F &= |\det T| \oint_{\mathcal{C}_2} |F_T(1,\tau)|^{-2/n}\,|d\tau| \\ &= \frac{1}{2}|\kappa|^{-2/n}\int_{-\pi}^{\pi}\prod_{j=1}^{n}|e^{i\theta} - e^{i\theta_j}|^{-2/n}\,d\theta \end{aligned}$$

as claimed in (34).

Hence we have shown that formula (9) in the statement of Corollary 2.1.1 is true. This completes the proof of Theorem 2.1 and Corollary 2.1.1.

3.2. *Proof of Theorem* 2.2. Let $\mathcal{S}$ be a two-dimensional real vector space in $\mathbf{C}^2$, and let $T \in GL_2(\mathbf{C})$. Suppose that $F(\sigma, \tau)$ is a binary form on $\mathbf{C}^2$. Then $D_F = (\det T)^{-n(n-1)} D_{F_T}$, by exactly the same direct calculation that works when $T \in GL_2(\mathbf{R})$. Now suppose also that $\mathcal{S}$ and $T^{-1}(\mathcal{S})$ are Lagrangian. We aim to show that

$$A_F^{\mathcal{S}} = |\det T| A_{F_T}^{T^{-1}(\mathcal{S})}.$$

Actually, since

$$T\left(\left\{ \begin{pmatrix} \sigma^* \\ \tau^* \end{pmatrix} \in T^{-1}(\mathcal{S}) : |F_T(\sigma^*, \tau^*)| \le 1 \right\}\right) \\ = \left\{ \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \in \mathcal{S} : |F(\sigma, \tau)| \le 1 \right\},$$

we need only show that the linear transformation $T : T^{-1}(\mathcal{S}) \to \mathcal{S}$ magnifies area on $T^{-1}(\mathcal{S})$ by a factor of $|\det T|$.

To determine this area magnification, first recall that T magnifies volume in $\mathbf{C}^2 \simeq \mathbf{R}^4$ by a factor of $|\det T|^2$; see [**17**, p. 19] for a proof of this by linear algebra or [**16**, p. 51] for a proof using differential forms. Next the hypothesis that $\mathcal{S}$ and $T^{-1}(\mathcal{S})$ are Lagrangian tells us that $\mathcal{S}$ and $i\mathcal{S}$ are orthogonal in $\mathbf{R}^4$ and that $T^{-1}(\mathcal{S})$ and $iT^{-1}(\mathcal{S}) = T^{-1}(i\mathcal{S})$ are orthogonal in $\mathbf{R}^4$. Hence,

$$\mathbf{R}^4 = \mathcal{S} \oplus i\mathcal{S} = T^{-1}(\mathcal{S}) \oplus T^{-1}(i\mathcal{S}).$$

Thus the volume magnification $|\det T|^2$ of T on $\mathbf{R}^4$ equals the product of the area magnifications of T on $T^{-1}(\mathcal{S})$ and on $iT^{-1}(\mathcal{S})$, and these area magnifications are of course equal. Therefore, $|\det T|^2$ equals the square of the area magnification of T on $T^{-1}(\mathcal{S})$, as we wished to show.

Finally, if $D_F \neq 0$ or $A_F^{\mathcal{S}} < \infty$, then $Q(F, \mathcal{S})$ is well defined, and from the invariance relations proved above for D_F and $A_F^{\mathcal{S}}$ we deduce that $Q(F, \mathcal{S}) = Q(F_T, T^{-1}(\mathcal{S}))$, completing the proof of Theorem 2.2.

3.3. *Proof of Corollaries* 2.2.1, 2.2.2, 2.2.3. We first prove Corollary 2.2.1. Let $\mathcal{S} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbf{R} \right\}$ be Lagrangian, and write $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_2(\mathbf{C})$ so that $T^{-1}(\mathcal{S}) = \mathbf{R}^2$. From Theorem 2.2 we know that

$$A_F^{\mathcal{S}} = |\det T| A_{F_T}$$

while by Theorem 2.1 applied with T^{-1} in place of T and with F_T in place of F we have

$$\begin{aligned} A_{F_T} &= |\det T^{-1}| \oint_{\mathcal{C}_1} |F(\sigma, 1)|^{-2/n} \, |d\sigma| \\ &= |\det T^{-1}| \oint_{\mathcal{C}_2} |F(1, \tau)|^{-2/n} \, |d\tau|. \end{aligned}$$

Corollary 2.2.1 follows immediately.

Formulas (15) and (17) of Corollary 2.2.2 are direct consequences of formula (10) of Corollary 2.2.1, while formulas (16) and (18) of Corollary 2.2.2 follow directly from the definition of the discriminant.

Corollary 2.2.3 follows from Theorem 2.2 and Corollary 2.2.2 using the transformation $T = \begin{pmatrix} (1/2) & (1/2) \\ (i/2) & -(i/2) \end{pmatrix}$ and the fact that $\mathcal{S}_l = T\mathcal{S}_c$, along with the earlier observation that if F factors completely over $\mathbf{R}$ then F_T factors completely as a product of linear forms that are real valued on $\mathcal{S}_c$.

4. Formulas for calculating $A_{F_n^*}$, $D_{F_n^*}$ and $Q(F_n^*)$—Proof of Theorem 1 and its corollaries. In this section we will show that

$$A_{F_n^*} = 4^{1-1/n} B\left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right),$$

$$D_{F_n^*}^{1/n(n-1)} = \frac{1}{2} n^{1/(n-1)}$$

and that the sequences $\{A_{F_n^*}\}$, $\{D_{F_n^*}^{1/n(n-1)}\}$ decrease to limiting values 4π, $1/2$ respectively. Theorem 1 and its corollaries will then follow.

In the argument which follows, we will find it convenient to work with the form $\hat{F}_n^*$ defined by

$$\hat{F}_n^*(X, Y) = \prod_{j=1}^{n} (X \sin\psi_j - Y \cos\psi_j)$$

where

$$\psi_j = \frac{j\pi}{n} - \frac{\pi}{2},$$

rather than with the form F_n^*. (The advantage is that all ψ_j are in the interval $(-\pi/2, \pi/2]$, which is the same as the range of the arctangent function used in our representations of A_F on the unit circle.) The form $\hat{F}_n^*$ is actually equal to $(-1)^{n/2} F_n^*$ when n is even and is $SL_2(\mathbf{R})$-equivalent to it for all n. (Apply the transformation $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to F_n^*.) Hence the formulas which we derive for $\hat{F}_n^*$ below will automatically hold for F_n^* as well, since A_F and D_F are both invariant under $SL_2(\mathbf{R})$.

We begin by deriving the formula for $A_{\hat{F}_n^*}$. Our point of departure is formula (9) in Corollary 2.1.1 which, in our present context, has the form

$$A_{\hat{F}_n^*} = 2\int_{-\pi}^{\pi} \prod_{j=1}^{n} |e^{i\theta} - e^{2i\psi_j}|^{-2/n}\, d\theta \tag{35}$$

(since $\kappa = (i/2)^n e^{-i\psi_1} \cdots e^{-i\psi_n}$ in this case). We claim that this formula can be written as

$$A_{\hat{F}_n^*} = 2\int_{-\pi}^{\pi} |e^{i\zeta} + 1|^{-2/n}\, d\zeta.$$

To see why this is so, consider the polynomial $p(z) = \prod_{j=1}^{n}(z - e^{2i\psi_j})$. With this notation, we can write equation (35) as $A_{\hat{F}_n^*} = 2\int_{-\pi}^{\pi} |p(e^{i\theta})|^{-2/n}\, d\theta$. From the definition of ψ_j, we clearly have $e^{2i\psi_j} = -e^{2ij\pi/n}$, and so $p(z) = \prod_{j=1}^{n}(z + e^{2\pi ij/n}) = z^n + (-1)^{n+1}$ and $A_{\hat{F}_n^*} = 2\int_{-\pi}^{\pi} |e^{in\theta} + (-1)^{n+1}|^{-2/n}\, d\theta$. Hence, using the substitution $\theta = (\zeta + (n+1)\pi)/n$ and the periodicity of $e^{i\zeta}$, we obtain $A_{\hat{F}_n^*} = 2\int_{-\pi}^{\pi} |e^{i\zeta} + 1|^{-2/n}\, d\zeta$ as claimed.

Now $e^{i\zeta} + 1 = e^{i\zeta/2}(e^{i\zeta/2} + e^{-i\zeta/2}) = 2e^{i\zeta/2}\cos(\zeta/2)$, and so, by symmetry, we have $A_{\hat{F}_n^*} = 2^{2-2/n}\int_0^{\pi} (\cos(\zeta/2))^{-2/n}\, d\zeta$. Applying the transformation $t = \cos^2(\zeta/2)$ to this latter expression, we then obtain $A_{\hat{F}_n^*} = 2^{2-2/n}\int_0^1 t^{-1/2-1/n}(1-t)^{-1/2}\, dt$, which we recognize to be

$2^{2-2/n}B(1/2-1/n,1/2)$. Hence,

$$A_{F_n^*}=4^{1-1/n}B\left(\frac{1}{2}-\frac{1}{n},\frac{1}{2}\right),$$

as required.

We now proceed to derive the formula for $D_{\hat{F}_n^*}$. Here our point of departure is formula (20) in the statement of Corollary 2.2.3 which, in our present context has the form

$$D_{\hat{F}_n^*}^{1/n(n-1)}=2(2^{-n})^{2/n}\prod_{j\neq k}|e^{2i\psi_j}-e^{2i\psi_k}|^{1/n(n-1)} \tag{36}$$

(since $\kappa=(i/2)^n e^{-i\psi_1}\cdots e^{-i\psi_n}$ in this case). As in the derivation of $A_{\hat{F}_n^*}$, we can make profitable use of the polynomial $p(z)=z^n+(-1)^{n+1}$. Indeed, we can write the product $\prod_{j\neq k}|e^{2i\psi_j}-e^{2i\psi_k}|$ as $\prod_{j=1}^n|p'(e^{2i\psi_j})|$ and then use the simple formula $p'(z)=nz^{n-1}$ to conclude that

$$D_{\hat{F}_n^*}^{1/n(n-1)}=\frac{1}{2}\prod_{j=1}^n n^{1/n(n-1)}=\frac{1}{2}n^{1/(n-1)}$$

as required.

Combining the formulas for $A_{F_n^*}$ and $D_{\hat{F}_n^*}^{1/n(n-1)}$ just derived, we obtain

$$Q(F_n^*)=2^{1-2/n}n^{1/(n-1)}B\left(\frac{1}{2}-\frac{1}{n},\frac{1}{2}\right)$$

for $n\geq 2$. This completes the proof of Theorem 1.

It remains to show that sequences $\{A_{F_n^*}\}$ and $\{D_{F_n^*}^{1/n(n-1)}\}$ are decreasing and have respective limits 4π and $1/2$. By direct differentiation, it is straightforward to show that $\{D_{F_n^*}^{1/n(n-1)}\}$ is strictly decreasing. Moreover, we clearly have $\lim_{n\to\infty}D_{F_n^*}^{1/n(n-1)}=1/2$ and $\lim_{n\to\infty}A_{F_n^*}=4\pi$ since $n^{1/(n-1)}\to 1$ and $B(1/2-1/n,1/2)\to B(1/2,1/2)=\Gamma(1/2)^2=\pi$. Hence, we need only show that $\{A_{F_n^*}\}$ is decreasing. For this purpose, consider the function

$$f(x)=4^{-x}B\left(\frac{1}{2}-x,\frac{1}{2}\right)$$

on the interval $x \in [0, 1/2)$. We will show that $f'(x) > 0$ for $x > 0$ and the result will follow.

The function f is actually strictly convex since it is the integral of strictly convex functions of x:

$$f(x) = \int_0^1 (4t)^{-x} t^{-1/2} (1-t)^{-1/2}\, dt,$$

as follows from the integral representation of the Beta function. Since f is strictly convex, it suffices to show that $f'(0) \geq 0$ because then $f'(x) > 0$ for all $x \in (0, 1/2)$.

The identity $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ gives us the representation

$$f(x) = 4^{-x} \frac{\Gamma((1/2) - x)\Gamma(1/2)}{\Gamma(1-x)},$$

and by applying the duplication formula for the gamma function [**1**, 6.1.18] with $z = (1/2) - x$ we obtain the simple formula

$$f(x) = \sqrt{\pi}\, \frac{\Gamma(1-2x)}{\Gamma(1-x)^2}\, \Gamma\left(\frac{1}{2}\right).$$

Differentiating this gives $f'(0) = 0$. Consequently, $f'(x) > 0$ for all $x > 0$ and so f is increasing for $x > 0$.

Therefore, the sequence $\{A_{F_n^*}\}$ is strictly decreasing and has limit 4π. Combining this with the fact that the sequence $\{D_{F_n^*}^{1/n(n-1)}\}$ strictly decreases to $1/2$, we see that the sequence $\{Q(F_n^*)\}$ strictly decreases to 2π.

This completes the proof of Corollary 1.1 and Corollary 1.2. Corollary 1.3 follows immediately. $\square$

5. Binary forms and equiangular polygons—Proof of Theorem 3. In this section we will briefly explain why Theorem 3 is true. Let F be a binary form of degree $n \geq 3$ with $D_F \neq 0$ and with a complete factorization over $\mathbf{R}$, let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a transformation in $GL_2(\mathbf{C})$, let V_T be the fractional linear transformation defined by $V_T(w) = (c+dw)/(a+bw)$, let $\mathcal{D} = V_T^{-1}(\mathbf{H})$ (where $\mathbf{H} = \{z \in \mathbf{C} : \Im z > 0\}$), let $z_0 = V_T^{-1}(i)$, and let g_F^T be the map

$$g_F^T(z) = (\det T) \int_{z_0}^{z} F_T(1,\tau)^{-2/n}\, d\tau.$$

Now the case $T = I$ of Theorem 3 was established by the first author in [**5**, pp. 4974–4977]. The general case of Theorem 3 is clearly an immediate consequence of the case $T = I$ and the fact that $g_F^T = g_F^I \circ V_T$, with the latter fact being true because

$$(g_F^I \circ V_T)(z_0) = g_F^I(i) = 0 = g_F^T(z_0)$$

and for all $z \in \mathcal{D}$,

$$\begin{aligned}
(g_F^I \circ V_T)'(z) &= (g_F^I)'(V_T(z))V_T'(z) \\
&= F(1, V_T(z))^{-2/n} V_T'(z) \\
&= F\left(1, \frac{c+dz}{a+bz}\right)^{-2/n} \frac{ad-bc}{(a+bz)^2} \\
&= (\det T) F(a+bz, c+dz)^{-2/n} \\
&= (\det T) F_T(1,z)^{-2/n} \\
&= (g_F^T)'(z).
\end{aligned}$$

Lastly, Corollary 3.1 follows easily from Theorem 3. Note here that if $T = \left(\begin{smallmatrix} (1/2) & (1/2) \\ (i/2) & -(i/2) \end{smallmatrix}\right)$, then $\det T = -i/2$ and $F_T(1,\tau) = \kappa \prod_{j=1}^{n} (\tau - e^{i\theta_j})$ by the calculation used in proving (34) in Section 3.

6. Equiangular polygons and harmonic measures—Proof of Theorem 4. In this section we will show that, if F is a binary form of degree $n \geq 3$ with $D_F \neq 0$ and with a complete factorization over $\mathbf{R}$, and if $\mathcal{P}$ is the equiangular polygon corresponding to F under the family of mappings g_F^T defined by equation (21) in the statement of Theorem 3, then

$$D_F^{1/n(n-1)} = \frac{2}{R_{\mathcal{P}}} \left\{ \prod_{j \neq k} \sin(\pi \mu_{\mathcal{P}}(\widehat{v_j v_k})) \right\}^{1/n(n-1)}$$

where $\mathcal{R}_{\mathcal{P}}$ is the harmonic radius of $\mathcal{P}$ with respect to the origin and $\mu_{\mathcal{P}}(\widehat{v_j v_k})$ is the harmonic measure of the arc $\widehat{v_j v_k}$ on $\partial\mathcal{P}$ with respect to the origin.

We begin by defining the potential theoretic notions of harmonic measure and harmonic radius. Let Ω be a simply connected, bounded domain in $\mathbf{C}$ whose boundary is a Jordan curve, and let ζ be any point in Ω. (Of course, in our application, we will take $\Omega = \mathcal{P}$.) For any distinct points z, w on $\partial\Omega$, the boundary of Ω, let $\widehat{zw}$ denote the open arc along $\partial\Omega$ traversed counterclockwise from z to w. Let $\chi_{\widehat{zw}}$ be the characteristic function on $\partial\Omega$, i.e.,

$$\chi_{\widehat{zw}}(\xi) = \begin{cases} 1 & \text{if } \xi \text{ is on } \widehat{zw} \\ 0 & \text{otherwise.} \end{cases}$$

Then the *harmonic measure* at $\zeta \in \Omega$ of the arc $\widehat{zw}$, denoted by $\mu_\Omega(\zeta; \widehat{zw})$, is the value at ζ of the harmonic extension to Ω of the characteristic function $\chi_{\widehat{zw}}$. (By Dirichlet's theorem, bounded harmonic functions are determined by their values on the boundary; hence, $\mu_\Omega(\zeta; \widehat{zw})$ is uniquely defined.)

If $\Omega = \mathbf{D}$, then $\mu_{\mathbf{D}}(\zeta; \widehat{z_1 z_2})$ is given by the Poisson kernel, i.e.,

$$\mu_{\mathbf{D}}(\zeta; \widehat{z_1 z_2}) = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \frac{1 - |\zeta|^2}{|e^{i\theta} - \zeta|^2}\, d\theta$$

where $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$, with $\theta_1 < \theta_2 < \theta_1 + 2\pi$. In particular,

$$\mu_{\mathbf{D}}(0; \widehat{z_1 z_2}) = \frac{\theta_2 - \theta_1}{2\pi} = \frac{\text{length of the arc } \widehat{z_1 z_2}}{2\pi}.$$

This illustrates the general principle that harmonic measure is a generalization of arc length, normalized so that the harmonic measure of $\partial\Omega$ with respect to $\zeta \in \Omega$ is 1.

Note that harmonic measure is *conformally invariant* in the sense that

$$\mu_\Omega(\zeta; \widehat{z_1 z_2}) = \mu_{\mathbf{D}}(f(\zeta); \widehat{f(z_1) f(z_2)})$$

for all conformal maps f of Ω onto $\mathbf{D}$. Hence, we can compute $\mu_\Omega(\zeta; \widehat{z_1 z_2})$ by calculating

$$\frac{\text{length of the arc } \widehat{f(z_1) f(z_2)}}{2\pi}$$

where f is a conformal map of Ω onto $\mathbf{D}$ with $f(\zeta) = 0$. Moreover, $\mu_\Omega(\zeta; \widehat{z_1 z_2}) \in [0,1]$ for all Ω and all $\zeta \in \Omega$, and $\mu_\Omega(\zeta; E) = 1 - \mu_\Omega(\zeta; \partial\Omega \backslash E)$ for all arcs E in $\partial\Omega$. Further, the translation invariance of the Laplacian implies that harmonic measure is translation invariant, meaning that $\mu_\Omega(\zeta; \widehat{z_1 z_2}) = \mu_{\Omega-\zeta}(0; \widehat{\tilde{z}_1 \tilde{z}_2})$, where $\tilde{z}_j = z_j - \zeta$. (For more information on harmonic measure, see [**14**, pp. 114–118].)

To define the notion of harmonic radius for Ω, we will use the Green function $G_\Omega(z,w)$ of the Laplacian on Ω. The Green function $G_\Omega(z,w)$ can be constructed directly from the domain Ω using the methods of potential theory [**14**, pp. 26, 250], or can be specified by mapping Ω to the unit disk $\mathbf{D}$ [**14**, pp. 26, 30]. In the latter case,

$$G_\Omega(z,w) = \log \left| \frac{1 - f(z)\overline{f(w)}}{f(z) - f(w)} \right|$$

where $f : \Omega \to \mathbf{D}$ is a conformal map of Ω onto $\mathbf{D}$ with $f(\zeta) = 0$ (the existence of f being guaranteed by the Riemann mapping theorem and the assumption that Ω is simply connected). The function $G_\Omega(z,w)$ is harmonic in each of the variables z, w for $z \neq w$, is symmetric with respect to interchanging z and w, and has a logarithmic singularity at $z = w$. Also, the Green function is conformally invariant in the sense that

$$G_\Omega(z,w) = G_{f(\Omega)}(f(z), f(w))$$

for all conformal maps f defined on Ω. For a detailed description of Green functions and their properties, see [**14**].

Now, in terms of the Green function, the harmonic radius of Ω with respect to the point $\zeta \in \Omega$ is

$$R_\Omega(\zeta) = \lim_{z \to \zeta} |z - \zeta| e^{G_\Omega(z,\zeta)}.$$

In particular, the harmonic radius of the unit disk $\mathbf{D}$ with respect to $\zeta \in \mathbf{D}$ is

$$R_{\mathbf{D}}(\zeta) = \lim_{z \to \zeta} |1 - z\bar{\zeta}| = 1 - |\zeta|^2$$

since $G_{\mathbf{D}}(z,\zeta) = \log |(1 - z\bar{\zeta})/(z - \zeta)|$; this illustrates the general principle that the harmonic radius $R_\Omega(\zeta)$ measures the distance from ζ to the boundary $\partial\Omega$. Notice that the conformal invariance of the Green function implies in particular that the harmonic radius is invariant under translation, meaning that $R_\Omega(\zeta) = R_{\Omega-\zeta}(0)$. See [**4**] for a survey of work on the harmonic radius.

The harmonic radius can be calculated more simply when a conformal map $\tilde{f} : \mathbf{D} \to \Omega$ with $\tilde{f}(0) = \zeta$ is given. Indeed, using the continuity of $\tilde{f}$, the conformal invariance of G_Ω, and the fact that $G_{\mathbf{D}}(w, 0) = \log|1/(w-0)|$, we have

$$\begin{aligned} R_\Omega(\zeta) &= \lim_{z\to\zeta} |z-\zeta| e^{G_\Omega(z,\zeta)} \\ &= \lim_{w\to 0} |\tilde{f}(w) - \tilde{f}(0)| e^{G_\Omega(\tilde{f}(w),\tilde{f}(0))} \\ &= \lim_{w\to 0} |\tilde{f}(w) - \tilde{f}(0)| e^{G_{\mathbf{D}}(w,0)} \\ &= \lim_{w\to 0} \left| \frac{\tilde{f}(w) - \tilde{f}(0)}{w-0} \right| \\ &= |\tilde{f}'(0)|. \end{aligned}$$

(This explains why $R_\Omega(\zeta)$ is often called the *inner conformal radius.*) We will later apply this with $\Omega = \mathcal{P}$, $\zeta = 0$, $\tilde{f} = h_F$, which gives $R_{\mathcal{P}}(0) = |h_F'(0)|$.

Now let $F(X,Y) = \prod_{j=1}^n (\alpha_j X - \beta_j Y)$ be a binary form of degree $n \geq 3$ with $D_F \neq 0$ and with $\alpha_j, \beta_j \in \mathbf{R}$, and let $\mathcal{P}$ be the equiangular polygon corresponding to F under the Schwarz-Christoffel map

$$h_F(w) = \frac{-i}{2} \kappa^{-2/n} \int_0^w \prod_{j=1}^n (\tau - e^{i\theta_j})^{-2/n} \, d\tau$$

discussed in Corollary 3.1, where $\kappa = \prod_{j=1}^n (\alpha_j + i\beta_j)/2$ and $\theta_j = 2\arctan(\alpha_j/\beta_j)$. Suppose that the roots $s_j = \alpha_j/\beta_j$ of $F(1,v)$ are ordered such that $s_1 < \cdots < s_n$. Then $-\pi < \theta_1 < \cdots < \theta_n \leq \pi$ and the vertices $v_j = h_F(e^{i\theta_j})$ of $\mathcal{P}$ are in counterclockwise order. Let $\mu_{\mathcal{P}}(\widehat{v_j v_k})$ denote the harmonic measure of the arc $\widehat{v_j v_k}$ on $\partial\mathcal{P}$, and let $R_{\mathcal{P}}$ denote the harmonic radius of $\mathcal{P}$, both taken with respect to the origin.

Recall, from formula (29) of Section 2, that

$$D_F^{1/n(n-1)} = \frac{1}{r_F^2} \prod_{j\neq k} d(L_j, L_k)^{1/n(n-1)}$$

where $d(L_j, L_k) = |\sin((\theta_k - \theta_j)/2)|$ and $r_F = (\prod_{j=1}^n \sqrt{\alpha_j^2 + \beta_j^2})^{-1/n}$. As we explained in Section 2, $d(L_j, L_k)$ measures the separation of the asymptotic lines L_j, L_k, while r_F measures the average distance of the curve $|F(x,y)| = 1$ from the origin. We will show $d(L_j, L_k) = \sin(\pi\mu_{\mathcal{P}}(v_j \frown v_k))$ and $r_F = (R_{\mathcal{P}}/2)^{1/2}$, so that

$$D_F^{1/n(n-1)} = \frac{2}{R_{\mathcal{P}}}\Big\{ \prod_{j\neq k} \sin(\pi\mu_{\mathcal{P}}(\widehat{v_j v_k}))\Big\}^{1/n(n-1)}.$$

The proof of Theorem 4 will then be complete since $A_F = |\partial\mathcal{P}|$ by Theorem 3.

Hence, let $\omega_j = e^{i\theta_j}$. Then, using the properties of harmonic measure developed above, we have for $j < k$ that

$$\begin{aligned} d(L_j, L_k) &= \sin\left(\frac{\theta_k - \theta_j}{2}\right) \\ &= \sin\left(\pi \cdot \frac{\theta_k - \theta_j}{2\pi}\right) \\ &= \sin(\pi\mu_{\mathbf{D}}(0; \widehat{\omega_j\omega_k})) \\ &= \sin(\pi\mu_{\mathcal{P}}(h_F(0); \widehat{h_F(\omega_j) h_F(\omega_k)})) \\ &\qquad\qquad \text{by conformal invariance} \\ &= \sin(\pi\mu_{\mathcal{P}}(\widehat{v_j v_k})) \qquad \text{since } h_F(0) = 0 \text{ and } h_F(\omega_j) = v_j. \end{aligned}$$

For $j > k$,

$$\begin{aligned} d(L_j, L_k) = d(L_k, L_j) &= \sin(\pi\mu_{\mathcal{P}}(\widehat{v_k v_j})) \qquad \text{by above} \\ &= \sin(\pi[1 - \mu_{\mathcal{P}}(\widehat{v_j v_k})]) = \sin(\pi\mu_{\mathcal{P}}(\widehat{v_j v_k})) \end{aligned}$$

once more. On the other hand, using the fact that

$$h_F'(w) = \frac{-i}{2}\kappa^{-2/n}\prod_{j=1}^n (w - e^{i\theta_j})^{-2/n}$$

(which follows immediately from the formula for $h_F(w)$ given above), we have

$$\begin{aligned} r_F &= \left(\prod_{j=1}^n \sqrt{\alpha_j^2 + \beta_j^2}\right)^{-1/n} = 2^{-1}|\kappa|^{-1/n} \\ &= 2^{-1}|2h_F'(0)|^{1/2} = \left(\frac{R_{\mathcal{P}}}{2}\right)^{1/2} \qquad \text{since } R_{\mathcal{P}} = |h_F'(0)|. \end{aligned}$$

Consequently, $d(L_j, L_k) = \sin(\pi\mu_{\mathcal{P}}(\widehat{v_j v_k}))$ and $r_F = (R_{\mathcal{P}}/2)^{1/2}$ as claimed. Theorem 4 now follows.

Finally we show that $R_{\mathcal{P}}$ and $\mu_{\mathcal{P}}$ can be replaced in Theorem 4 by the harmonic radius and the harmonic measure with respect to any point $\zeta \in \mathcal{P}$. To this end, fix $\zeta \in \mathcal{P}$ and let $z_0 = g_F^{-1}(\zeta) \in \mathbf{H}$. Choose $T \in SL_2(\mathbf{R})$ to be such that $V_T^{-1}(i) = z_0$, where $V_T : \mathbf{H} \to \mathbf{H}$ was defined in Theorem 3. Let $G = F_{T^{-1}}$ (so that $G_T = F$) and observe that the form G has a complete factorization over $\mathbf{R}$ and has $D_G \neq 0$ since F is assumed in Theorem 4 to have the same properties. The Schwarz-Christoffel maps g_F, g_G^T are defined on $\mathbf{H}$, and

$$\begin{aligned} g_F(z) &= \int_i^z F(1,\tau)^{-2/n}\, d\tau \\ &= \int_i^{z_0} F(1,\tau)^{-2/n}\, d\tau + \int_{z_0}^z F(1,\tau)^{-2/n}\, d\tau \\ &= g_F(z_0) + (\det T) \int_{z_0}^z G_T(1,\tau)^{-2/n}\, d\tau \\ &= \zeta + g_G^T(z), \end{aligned}$$

so that

$$\mathcal{P}(F) = \zeta + \mathcal{P}(G).$$

Next, $SL_2(\mathbf{R})$ invariance [**6**, p. 119] shows that $A_F = A_G$, $D_F = D_G$, $Q(F) = Q(G)$, while the translation invariance of harmonic measure and harmonic radius gives that

$$R_{\mathcal{P}(F)}(\zeta) = R_{\mathcal{P}(G)}(0) = R_{\mathcal{P}(G)}$$

and

$$\mu_{\mathcal{P}(F)}(\zeta; \widehat{v_j v_k}) = \mu_{\mathcal{P}(G)}(0; \widehat{\tilde{v}_j \tilde{v}_k}) = \mu_{\mathcal{P}(G)}(\widehat{\tilde{v}_j \tilde{v}_k}),$$

where $\tilde{v}_j = v_j - \zeta$ is the jth vertex of $\mathcal{P}(G)$. Hence by applying Theorem 4 to G we deduce that

$$\begin{aligned} D_F^{1/n(n-1)} = D_G^{1/n(n-1)} &= \frac{2}{R_{\mathcal{P}(G)}} \left\{ \prod_{j \neq k} \sin(\pi\mu_{\mathcal{P}(G)}(\widehat{\tilde{v}_j \tilde{v}_k})) \right\}^{1/n(n-1)} \\ &= \frac{2}{R_{\mathcal{P}(F)}(\zeta)} \left\{ \prod_{j \neq k} \sin(\pi\mu_{\mathcal{P}(F)}(\zeta; \widehat{v_j v_k})) \right\}^{1/n(n-1)} \end{aligned}$$

and

$$\begin{aligned} Q(F) = Q(G) &= \frac{2|\partial \mathcal{P}(G)|}{R_{\mathcal{P}(G)}} \left\{ \prod_{j \neq k} \sin(\pi \mu_{\mathcal{P}(G)}(\widehat{\tilde{v}_j \tilde{v}_k})) \right\}^{1/n(n-1)} \\ &= \frac{2|\partial \mathcal{P}(F)|}{R_{\mathcal{P}(F)}(\zeta)} \left\{ \prod_{j \neq k} \sin(\pi \mu_{\mathcal{P}(F)}(\zeta; \widehat{v_j v_k})) \right\}^{1/n(n-1)}. \end{aligned}$$

Thus, Theorem 4 remains true when $R_{\mathcal{P}}$ and $\mu_{\mathcal{P}}$ are replaced by the harmonic radius and harmonic measure with respect to ζ, instead of the origin.

TABLE 1.

	affine symmetric	affine asymmetric
$F(X,Y)$	$\prod_{j=1}^n (\alpha_j X - \beta_j Y)$	$a \prod_{j=1}^n (Y - \sigma_j X)$
$d(L_j, L_k)$	$\frac{\lvert\alpha_j \beta_k - \alpha_k \beta_j\rvert}{\sqrt{\alpha_j^2 + \beta_j^2}\sqrt{\alpha_k^2 + \beta_k^2}}$	$\frac{\lvert s_j - s_k\rvert}{\sqrt{1+s_j^2}\sqrt{1+s_k^2}}$
r_F	$(\prod_{j=1}^n \sqrt{\alpha_j^2! + \beta_j^2})^{-1/n}$	$(\lvert a\rvert \prod_{j=1}^n \sqrt{1+s_j^2})^{-1/n}$
$D_F^{1/n(n-1)}$	$\prod_{j \neq k} \lvert\alpha_j \beta_k - \alpha_k \beta_j\rvert^{1/n(n-1)}$	$\lvert a\rvert^{2/n} \prod_{j \neq k} \lvert s_j - s_k\rvert^{1/n(n-1)}$
A_F	$\iint_{\lvert F(x,y)\rvert \leq 1} dx\, dy$	$\lvert a\rvert^{-2/n} \int_{-\infty}^{\infty} \prod_{j=1}^n \lvert s - s_j\rvert^{-2/n} ds$

TABLE 2.

	complex symmetric	complex asymmetric
$F(X,Y)$	$\prod_{j=1}^n (\bar{\gamma}_j \bar{z} + \gamma_j z)$	$\kappa \prod_{j=1}^n (z - e^{i\theta_j} \bar{z})$
$d(L_j, L_k)$	$\frac{1}{2} \frac{\lvert\gamma_j \bar{\gamma}_k - \bar{\gamma}_j \gamma_k\rvert}{\lvert\gamma_j\rvert \cdot \lvert\gamma_k\rvert}$	$\frac{1}{2} \lvert e^{i\theta_j} - e^{i\theta_k}\rvert$
r_F	$\frac{1}{2}(\lvert\gamma_1\rvert \cdots \lvert\gamma_n\rvert)^{-1/n}$	$\frac{1}{2}\lvert\kappa\rvert^{-1/n}$
$D_F^{1/n(n-1)}$	$2 \prod_{j \neq k} \lvert\gamma_j \bar{\gamma}_k - \bar{\gamma}_j \gamma_k\rvert^{1/n(n-1)}$	$2\lvert\kappa\rvert^{2/n} \prod_{j \neq k} \lvert e^{i\theta_j} - e^{i\theta_k}\rvert^{1/n(n-1)}$
A_F	$\frac{1}{2} \int_{-\pi}^{\pi} \prod_{j=1}^n \lvert\gamma_j e^{i\theta/2} + \bar{\gamma}_j e^{-i\theta/2}\rvert^{-2/n} d\theta$	$\frac{1}{2}\lvert\kappa\rvert^{-2/n} \int_{-\pi}^{\pi} \prod_{j=1}^n \lvert e^{i\theta} - e^{i\theta_j}\rvert^{-2/n} d\theta$

TABLE 3.

	polar	equiangular polygon
$F(X,Y)$	$\pm\prod_{j=1}^{n}\rho_j(X\sin\psi_j - Y\cos\psi_j)$	$\mathcal{P}(v_1,\ldots,v_n)$
$d(L_j,L_k)$	$\lvert\sin(\psi_k-\psi_j)\rvert$	$\sin(\pi\cdot\mu_{\mathcal{P}}(\widehat{v_jv_k}))$
r_F	$(\rho_1\cdots\rho_n)^{-1/n}$	$(\frac{R_{\mathcal{P}}}{2})^{1/2}$
$D_F^{1/n(n-1)}$	$\frac{1}{r_F^2}\prod_{j\neq k}\lvert\sin(\psi_k-\psi_j)\rvert^{1/n(n-1)}$	$\frac{2}{R_{\mathcal{P}}}\{\prod_{j\neq k}\sin(\pi\cdot\mu_{\mathcal{P}}(\widehat{v_jv_k}))\}^{1/n(n-1)}$
A_F	$r_F^2\int_{-\pi/2}^{\pi/2}\lvert\sin(\psi-\psi_j)\rvert^{-2/n}\,d\psi$	$\lvert\partial\mathcal{P}\rvert$

7. Summary of the representations of A_F, D_F, $Q(F)$ and related quantities. In this final section we summarize the more important formulas derived in the paper, for binary forms with a complete factorization over $\mathbf{R}$.

Tables 1, 2 and 3 present six equivalent formulas for each of the five quantities $F(X,Y)$, $d(L_j,L_k)$, r_F, $D_F^{1/n(n-1)}$, A_F. (Note that the area A_F is calculated in the real affine plane $\mathbf{R}^2$.) In particular, the entries in corresponding rows of the three tables will all be equal, when the assumptions listed below are satisfied. For example,

$$
\begin{aligned}
A_F &= \iint_{|F(x,y)|\le 1} dx\,dy \\
&= |a|^{-2/n}\int_{-\infty}^{\infty}\prod_{j=1}^{n}|s-s_j|^{-2/n}\,ds \\
&= \frac{1}{2}\int_{\pi}^{\pi}\prod_{j=1}^{n}|\gamma_j e^{i\theta/2}+\bar{\gamma}_j e^{-\theta/2}|^{-2/n}\,d\theta \\
&= \frac{1}{2}|\kappa|^{-2/n}\int_{-\pi}^{\pi}\prod_{j=1}^{n}|e^{i\theta}-e^{i\theta_j}|^{-2/n}\,d\theta \\
&= r_F^2\int_{-\pi/2}^{\pi/2}\prod_{j=1}^{n}|\sin(\psi-\psi_j)|^{-2/n}\,d\psi \\
&= |\partial\mathcal{P}|.
\end{aligned}
$$

Some formulas in the tables have not been written down explicitly elsewhere in the paper, but these all follow easily from other formulas that have been derived in this paper.

Recall that L_j denotes the jth asymptote of the curve $|F(x,y)| = 1$ (as ordered from the negative y-axis); that $d(L_j, L_k)$ denotes the "distance" between the asymptotes L_j, L_k, as defined by equation (27) of Section 2; and that r_F is the "average distance" of the curve $|F(x,y)| = 1$ from the origin, as measured by equation (28) of Section 2. Further, $\mathcal{P}$ is the equiangular polygon corresponding to the form F under the Schwarz-Christoffel map of Theorem 3; $R_{\mathcal{P}}$ denotes the harmonic radius of $\mathcal{P}$ with respect to the origin and $\mu_{\mathcal{P}}(\widehat{v_j v_k})$ is the harmonic measure of the arc $\widehat{v_j v_k}$ on $\partial\mathcal{P}$ with respect to the origin (see Section 6 for definitions).

The formulas in Tables 1, 2 and 3 hold under the following assumptions: (1) $n \geq 2$, (2) $\alpha_j, \beta_j \in \mathbf{R}$, (3) $\rho_j = \sqrt{\alpha_j^2 + \beta_j^2}$, (4) $\rho_j \neq 0$, (5) for the "affine asymmetric" column of Table 1, assume $\beta_j \neq 0$ for all j, (6) $\psi_j = \arctan(\alpha_j/\beta_j) \in (-\pi/2, \pi/2]$, (7) $\psi_1 \leq \cdots \leq \psi_n$, (8) $s_j = \alpha_j/\beta_j = \tan\psi_j$, (9) $a = (-1)^n \beta_1 \cdots \beta_n$, (10) $z = X + iY$, $\bar{z} = X - iY$, (11) $\gamma_j = (\alpha_j + i\beta_j)/2$, $|\gamma_j| = \rho_j/2$, (12) $\kappa = \gamma_1 \cdots \gamma_n$, (13) $\theta_j = 2\psi_j \in (-\pi, \pi]$; consequently, $s_j = \tan(\theta_j/2)$ and $\bar{\gamma}_j/\gamma_j = -e^{i\theta_j}$, (14) for the "equiangular polygon" column of Table 3, assume $\psi_1 < \cdots < \psi_n$ and $n \geq 3$, (15) $v_1, \ldots, v_n$ are the consecutive vertices of the equiangular polygon $\mathcal{P}$ and are given by

$$v_j = g_F(s_j) = h_F(e^{i\theta_j})$$

where $g_F : \mathbf{H} \to \mathcal{P}$, $h_F : \mathbf{D} \to \mathcal{P}$ are the conformal maps defined by

$$g_F(z) = a^{-2/n} \int_i^z \prod_{j=1}^n (v - s_j)^{-2/n}\, dv,$$

$$h_F(w) = \frac{-i}{2} \kappa^{-2/n} \int_0^w \prod_{j=1}^n (\tau - e^{i\theta_j})^{-2/n}\, d\tau,$$

not forgetting the notational conventions for g_F described after Corollary 3.1.

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ENDNOTES

1. For any ring K, $K[X,Y]$ is the ring of polynomials in X and Y with coefficients in K. Note that the collection of binary forms over K, i.e., bivariate polynomials of homogeneous degree, is a *proper* subset of $K[X,Y]$. See [**15**].

2. Though if we consider A_F over the class of forms F with *integer* coefficients and nonzero discriminant, then $A_F \leq 3B[(1/3),(1/3)]$ since the discriminant of a form with integer coefficients must be an integer, see [**6**].

3. On the other hand, property (P2) does hold.

4. Notationally, it may seem more natural here to interchange the roles of a, s_j and b, t_j. However, we are going to actually only use the representation involving a, s_j, and so to maintain consistency with the papers [**5, 6**], the stated notation is preferable.

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