ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 1, Spring 2000

AREA INTEGRAL CHARACTERIZATION OF $\mathcal{M}$-HARMONIC HARDY SPACES ON THE UNIT BALL

MILOŠ ARSENOVIĆ AND MIROLJUB JEVTIĆ

ABSTRACT. Characterizations of $\mathcal{M}$-harmonic Hardy spaces $\mathcal{H}^p$ on the unit ball in C^n, $n \geq 1$, in terms of area functions involving gradient and invariant gradient are proved.

1. Introduction. Let B denote the unit ball in C^n, $n \geq 1$, and m the $2n$-dimensional Lebesgue measure on B normalized so that $m(B) = 1$, while σ is the normalized surface measure on its boundary S. For the most part we will follow the notation and terminology of Rudin [**7**]. If $\alpha > 1$ and $\xi \in S$ the corresponding Koranyi approach region is defined by

$$D_\alpha(\xi) = \{z \in B : |1 - \langle z, \xi\rangle| < (\alpha/2)(1 - |z|^2)\}.$$

For any function f on B we define a scale of maximal functions by

$$M_\alpha f(\xi) = \sup\{|f(z)| : z \in D_\alpha(\xi)\}.$$

Let $\tilde{\Delta}$ be the invariant Laplacian on B. That is,

$$(\tilde{\Delta} f)(z) = \frac{1}{n+1}\Delta(f \circ \phi_z)(0), \quad f \in C^2(B),$$

where Δ is the ordinary Laplacian and ϕ_z the standard automorphism of B taking 0 to z, see [**7**]. A function f defined on B is $\mathcal{M}$-harmonic, $f \in \mathcal{M}$, if $\tilde{\Delta} f = 0$.

For $0 < p < \infty$, $\mathcal{M}$-harmonic Hardy space $\mathcal{H}^p$ is defined to be the space of all functions $f \in \mathcal{M}$ such that $M_\alpha f \in L^p(\sigma)$ for some $\alpha > 1$. We note that the definition is independent of α.

For $f \in C^1(B)$, $Df = (\partial f/\partial z_1, \dots, \partial f/\partial z_n)$ denotes the complex gradient of f, $\nabla f = (\partial f/\partial x_1, \dots, \partial f/\partial x_{2n})$, $z_k = x_{2k-1} + ix_{2k}$, $k = 1, \dots, n$, denotes the real gradient of f.

Received by the editors on May 13, 1996.

We denote the area integrals by

$$S_\alpha f(\xi) = \int_{D_\alpha(\xi)} |\nabla f(z)|^2 (1-|z|^2)^{1-n}\, dm(z), \quad \xi \in S$$

and

$$T_\alpha f(\xi) = \int_{D_\alpha(\xi)} |\tilde{\nabla} f(z)|^2\, d\tau(z), \quad \xi \in S,$$

where $\tilde{\nabla} f(z) = \nabla(f \circ \phi_z)(0)$ is the invariant gradient and

$$d\tau(z) = (1-|z|^2)^{-1-n}\, dm(z).$$

The main purpose of this paper is to prove the following theorem.

Theorem 1. *Let $0 < p < \infty$, and let $f \in \mathcal{M}$. Then the following are equivalent, with an aperture $\alpha > 1$ fixed*:

(a) $f \in \mathcal{H}^p$.

(b) $S_\alpha f \in L^p(\sigma)$.

(c) $T_\alpha f \in L^p(\sigma)$.

This paper is organized as follows. In Section 2 some preliminaries and auxiliary results are collected. In the third section we prove our theorem for the case of $\mathcal{M}$-harmonic functions. If f is holomorphic, $f \in H(B)$, the equivalence (a) $\Leftrightarrow$ (b) is known though a detailed proof seems to be lacking in the literature. The space $\mathcal{H}^p \cap H(B)$ is the usual Hardy space, and it will be denoted by H^p. In Section 4, for the reader's convenience, we give an independent proof of the theorem for the case of H^p spaces.

2. Preliminaries. In terms of ordinary differential operators, the invariant Laplacian $\tilde{\Delta}$ is as follows:

$$\tilde{\Delta} = \frac{1}{n+1}(1-|z|^2) \sum_{j,k=1}^{n} (\delta_{j,k} - z_j \bar{z}_k) \frac{\partial^2}{\partial z_j \partial \bar{z}_k}, \tag{1}$$

where $\delta_{j,k}$ denotes the Kronecker delta, see [**7**].

Thus a straightforward calculation shows that, for $f \in \mathcal{M}$,

$$\tilde{\Delta}|f|^2(z) = \frac{4}{n+1}(1-|z|^2)(|Df(z)|^2 - |Rf(z)|^2 + |D\bar{f}(z)|^2 - |R\bar{f}(z)|^2),$$

where, as usual, $Rf(z) = \sum_{j=1}^n z_j \partial f/\partial z_j$ denotes the radial derivative of f. A simple calculation shows that:

$$\tilde{\Delta}(1-|z|^2)^n = -4\frac{n^2}{n+1}(1-|z|^2)^{n+1}. \tag{2}$$

We note that in [**6**] it is shown that, for $f \in C^1(B)$,

$$|\tilde{D}f(z)|^2 = |D(f \circ \phi_z)(0)|^2 = (1-|z|^2)(|Df(z)|^2 - |Rf(z)|^2). \tag{3}$$

The invariant Laplacian can be realized as a Laplace-Beltrami operator corresponding to the Bergman metric as follows. The Bergman metric on B is given by

$$ds^2 = \sum_{j,k=1}^n g_{jk} dz_j d\bar{z}_k,$$

where

$$g_{jk} = \frac{n+1}{(1-|z|^2)^2}[(1-|z|^2)\delta_{j,k} + \bar{z}_j z_k].$$

The inverse of the matrix g_{jk} is g^{jk} where

$$g^{jk}(z) = \frac{1}{n+1}(1-|z|^2)(\delta_{jk} - \bar{z}_j z_k)$$

and therefore the corresponding Laplace-Beltrami operator

$$4\sum_{j,k=1}^n g^{jk}\frac{\partial^2}{\partial z_j \partial \bar{z}_k}$$

is precisely the invariant Laplacian. Hence one has Green's formula for the invariant Laplacian:

If Ω is an open subset of B, $\overline{\Omega} \subset B$, whose boundary $\partial\Omega$ is smooth enough, and u, v are real valued functions such that $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then

$$\int_\Omega (u\tilde{\Delta}v - v\tilde{\Delta}u)\, d\tilde{\tau} = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \tilde{n}} - v\frac{\partial u}{\partial \tilde{n}} \right) d\tilde{\sigma}, \tag{4}$$

where $\tilde{\tau}$ is the volume element of B determined by the Bergman metric, $\tilde{\sigma}$ is the surface area element on $\partial\Omega$ determined by the Bergman metric, and $\partial/\partial\tilde{n}$ denotes outward normal differentiation across $\partial\Omega$ with respect to the Bergman metric.

By calculating the Jacobian of the identity map from "Euclidean" B onto the "Bergman" B one can verify that the volume element $\tilde{\tau}$ is given by $d\tilde{\tau}(z) = C d\tau(z)$, where C is a constant depending only on n.

Similarly, for $0 < r < 1$, by calculating the Jacobian of the map $\xi \mapsto r\xi$ from the "Euclidean" S onto the Bergman $S_r = \{r\xi : \xi \in S\}$ one can find that the surface area element $\tilde{\sigma}_r$ on S_r determined by the Bergman metric is given by

$$d\tilde{\sigma}_r(r\xi) = C\frac{r^{n-1}}{(1-r^2)^n}\, d\sigma(\xi).$$

In this paper constants will be denoted by C which may indicate a different constant from one occurrence to the next.

For $\xi \in S$ and $0 < \delta \le 2$, set $Q_\delta(\xi) = \{\eta \in S : |1 - \langle \eta, \xi\rangle| < \delta\}$.

The class BMO consists of functions $f \in L^2(\sigma)$ for which

$$\|f\|^2_{\mathrm{BMO}} = \sup \frac{1}{\sigma(Q)} \int_Q |f(\xi) - f_Q|^2\, d\sigma(\xi) < \infty,$$

where f_Q denotes the average of f over a "ball" Q and the supremum is taken over all $Q = Q_\delta(\xi)$.

As final preliminary results we need the following three lemmas:

Lemma 1 [**2**]. *If $f \in H^2 \cap \mathrm{BMO}$, then $T_\alpha f \in \mathrm{BMO}$ for all $\alpha > 1$.*

Lemma 2 [**6**]. *Let $0 < p < \infty$, $0 < r < 1$, $f \in \mathcal{M}$ and $E_r(z) = \phi_z(rB)$, $z \in B$. Then there is a constant $C = C(p, r)$ such* *that*

$$|\tilde{\nabla} f(z)|^p \le C \int_{E_r(z)} |f(w)|^p \, d\tau(w), \quad z \in B. \tag{5}$$

Lemma 3 [**5**]. *Let $0 < r < 1$ and $1 \le i < j \le n$. There is a constant C such that if $f \in \mathcal{M}$, then*

a) $|T_{ij} R f(w)| \le C(1-|w|^2)^{-1/2} \int_{E_r(w)} |Rf(z)| \, d\tau(z)$, $w \in B$,

b) $|T_{ij} \overline{R} f(w)| \le C(1-|w|^2)^{-1/2} \int_{E_r(w)} |\overline{R} f(z)| \, d\tau(z)$, $w \in B$,

where $\overline{R} = \sum_{j=1}^n \bar{z}_j (\partial/\partial \bar{z}_j)$ and $T_{ij} = \bar{z}_i(\partial/\partial z_j) - \bar{z}_j(\partial/\partial z_i)$ are tangential derivatives.

3. Proof of Theorem.

(c) $\Rightarrow$ (b). It follows from (3) that

$$\begin{aligned} |\tilde{\nabla} f(z)|^2 &= 2(|\tilde{D} f(z)|^2 + |\tilde{D} \bar{f}(z)|^2) \\ &\ge 2(1-|z|^2)^2 (|Df(z)|^2 + |D\bar{f}(z)|^2) \\ &= (1-|z|^2)^2 |\nabla f(z)|^2 \end{aligned}$$

and hence (c) $\Rightarrow$ (b). (We note that it is not possible to bound $|\tilde{\nabla} f(z)|^2$ by $C(1-|z|^2)^2 |\nabla f(z)|^2$ pointwise, see [**4**].)

(b) $\Rightarrow$ (c). It is easy to check that

$$|z|^2 |Df(z)|^2 = |Rf(z)|^2 + \sum_{i<j} |T_{ij} f(z)|^2.$$

Using this and (3) we find that

$$\begin{aligned} |z|^2 |\tilde{\nabla} f(z)|^2 &= 2|z|^2 (|\tilde{D} f(z)|^2 + |\tilde{D} \bar{f}(z)|^2) \\ &= 2(1-|z|^2)[(1-|z|^2)(|Rf(z)|^2 + |R\bar{f}(z)|^2) \\ &\quad + \sum_{i<j} |T_{ij} f(z)|^2 + \sum_{i<j} |T_{ij} \bar{f}(z)|^2]. \end{aligned}$$

Since $|Rf(z)| \leq |\nabla f(z)|$ and $|R\bar{f}(z)| \leq |\nabla f(z)|$, to prove the implication (b) $\Rightarrow$ (c) it is sufficient to show that

$$\int_{D_\alpha(\eta)} |T_{ij} f(z)|^2 (1-|z|^2)^{-n}\, dm(z) \leq C \int_{D_\beta(\eta)} |\nabla f(z)|^2 (1-|z|^2)^{1-n}\, dm(z)$$

and

$$\int_{D_\alpha(\eta)} |T_{ij} \bar{f}(z)|^2 (1-|z|^2)^{-n}\, dm(z) \leq C \int_{D_\beta(\eta)} |\nabla f(z)|^2 (1-|z|^2)^{1-n}\, dm(z)$$

for all $1 \leq i < j \leq n$ where $1 < \alpha < \beta$ are fixed. We will prove the first inequality. Analogously we may prove the second one.

From Lemma 3 we see that if $r\zeta \in D_\alpha(\eta)$, then

$$\begin{aligned} |T_{ij} Rf(r\zeta)| &\leq \left\{ \frac{C}{1-r} \int_{S_\beta(r,\eta)} |Rf(w)|^2\, d\tau(w) \right\}^{1/2} \\ &\leq \left\{ \frac{C}{1-r} \int_{S_\beta(r,\eta)} |\nabla f(w)|^2\, d\tau(w) \right\}^{1/2} \\ &= J_r \end{aligned}$$

and

$$|T_{ij} \overline{R} f(r\zeta)| \leq \left\{ \frac{C}{1-r} \int_{S_\beta(r,\eta)} |R\bar{f}(w)|^2\, d\tau(w) \right\}^{1/2} \leq J_r$$

where $S_\beta(r,\eta)$ denotes the region

$$S_\beta(r,\eta) = \{z \in D_\beta(\eta) : ((1-r^2)/2) < 1-|z|^2 < 2(1-r^2)\}.$$

An integration by parts shows that

$$f(r\zeta) = \int_0^1 [Rf(tr\,\zeta) + \overline{R} f(tr\,\zeta) + f(tr\,\zeta)]\, dt.$$

Hence

$$|T_{ij}f(r\zeta)| \le \frac{C}{r}\int_0^r J_t\,dt.$$

Having obtained a bound for $|T_{ij}f(r\zeta)|$ which depends only on r, we integrate in polar coordinates in $D_\alpha(\eta)$ using the fact that, for fixed r,

$$\sigma(\{\zeta \in S : r\zeta \in D_\alpha(\eta)\}) \le C(1-r)^n.$$

This gives that

$$\begin{aligned}\int_{D_\alpha(\eta)} |T_{ij}f(z)|^2(1-|z|^2)^{-n}\,dm(z) &\le C\int_0^1 \left(\int_0^r J_t\,dt\right)^2 dr\\ &\le C\int_0^1 (1-r)^2 J_r^2\,dr,\end{aligned}$$

by Hardy's inequality. Inserting the definition of J_r we obtain the bound

$$\int_0^1 (1-r)\int_{S_\beta(r,\eta)} |\nabla f(z)|^2\,d\tau(z)\,dr.$$

If $z \in S_\beta(r,\eta)$, then $1-|z|$ is comparable to $1-r$, hence the above integral is dominated by $\int_{D_\beta(\eta)} |\nabla f(z)|^2(1-|z|^2)^{1-n}\,dm(z)$.

Implication (c) $\Rightarrow$ (a). Let us fix $1<\alpha<\beta$ and $0<r<1$. Put

$$N_{\alpha,r}f(\xi) = \sup|f(r_1\eta) - f(r_2\eta)|, \quad \xi \in S,$$

where sup is taken over all $r_1\eta, r_2\eta \in D_\alpha(\xi)\cap rB$. We shall show that

$$\|N_{\alpha,r}f\|_{L^p(\sigma)} \le C\|T_\beta f\|_{L^p(\sigma)}, \tag{6}$$

where C is independent of r. From this it easily follows that $M_\alpha f \in L^p(\sigma)$ if $T_\beta f \in L^p(\sigma)$.

For $\lambda > 0$, let χ_λ be the characteristic function of $\{\xi : T_\beta f > \lambda\}$, $R_\lambda(f) = M(\chi_\lambda)$ and $\Omega_{r,\lambda} = \{\xi \in S : N_{\alpha,r}f > \lambda\}$. Here, as usual, $M(\chi_\lambda)$ denotes the maximal function of χ_λ defined by

$$M(\chi_\lambda)(\xi) = \sup_{t>0}\frac{1}{\sigma(Q_t(\xi))}\int_{Q_t(\xi)} \chi_\lambda(\eta)\,d\sigma(\eta).$$

Let $\mathcal{B}$ denote the collection of all balls contained in $\Omega_{r,\lambda}$ which touch the boundary. There is a disjoint subcollection $\{Q_k\}$ of $\mathcal{B}$ and a $C > 0$ such that if $\{\tilde{Q}_k\}$ is the ball with the same center as $\{Q_k\}$ and C times the radius, then $\{\tilde{Q}_k\}$ covers $\Omega_{r,\lambda}$. An adaptation of the argument given in [**3**] shows that the following is true.

Suppose $\delta > 0$, then there exists an $\varepsilon > 0$ such that for all $\lambda > 0$ and all k we have

$$\sigma(\{\xi \in \tilde{Q}_k : N_{\alpha,r} f > 2\lambda, R_{\varepsilon\lambda} f(\xi) \leq 1/2\}) \leq \delta\sigma(\tilde{Q}_k). \tag{7}$$

This implies (6). Put

$$G_k = \{\xi \in \tilde{Q}_k : N_{\alpha,r} f(\xi) > 2\lambda, R_{\varepsilon\lambda} f(\xi) \leq 1/2\}.$$

Now

$$\{\xi \in S : N_{\alpha,r} > 2\lambda\} \subset \{\xi \in S : R_{\varepsilon\lambda} f(\xi) > 1/2\} \cup (\cup G_k)$$

so

$$\begin{aligned}\sigma(\{\xi \in S : N_{\alpha,r}(\xi) > 2\lambda\}) &\leq \sigma(\{\xi \in S : R_{\varepsilon\lambda} f(\xi) > 1/2\}) \\ &\quad + \delta \sum_k \sigma(\tilde{Q}_k) \\ &\leq C(\sigma(\{\xi \in S : T_\beta f(\xi) > \varepsilon\lambda\}) \\ &\quad + \delta\sigma(\{\xi \in S : N_{\alpha,r} f(\xi) > \lambda\}).\end{aligned}$$

(Here we have used that

$$\sigma(\{\xi \in S : R_{\varepsilon\lambda} f(\xi) > 1/2\}) \leq C\sigma(\{\xi \in S : T_\alpha f(\xi) > \varepsilon\lambda\}),$$

by the maximal theorem, and (7).)

Multiply by $p\lambda^{p-1}$ and integrate in λ from 0 to ∞ to find

$$\|N_{\alpha,r} f\|^p_{L^p(\sigma)} \leq C(\|T_\beta f\|^p_{L^p(\sigma)} + \delta\|N_{\alpha,r} f\|^p_{L^p(\sigma)}),$$

where C is independent of r, which gives (6) if δ is chosen sufficiently small.

Implication (a) $\Rightarrow$ (c). Assume $1 < \alpha < \beta$ and $0 < r < 1$. Put

$$T_{\alpha,r} f(\xi) = \int_{D_\alpha(\xi)\cap rB} |\tilde{\nabla} f(z)|^2 \, d\tau(z).$$

We shall show

$$\|T_{\alpha,r} f\|_{L^p(\sigma)} \le C \|M_\beta f\|_{L^p(\sigma)}, \tag{8}$$

with C independent of r, and then let rB expand to B.

Suppose $M_\beta f \in L^p(\sigma)$. For $\lambda > 0$, consider an open set $\Omega_{r,\lambda} = \{\xi \in S : T_{\alpha,r} f(\xi) > \lambda\}$. Let $\mathcal{B}$ denote the collection of balls contained in $\Omega_{r,\lambda}$ which touch the boundary. There is a subcollection $\{\tilde{Q}_k\}$ of $\mathcal{B}$ and a $C > 0$ such that if $\{\tilde{Q}_k\}$ is the ball with the same center as Q_k and C times the radius, then $\{\tilde{Q}_k\}$ covers $\Omega_{r,\lambda}$. Arguing as above we find that the following is true.

Suppose that $\delta > 0$; then there exists an $\varepsilon > 0$ such that

$$\sigma(\{\xi \in \tilde{Q}_k : T_{\alpha,r} f(\xi) > 2\lambda, M_\beta f(\xi) \le \varepsilon\lambda\}) \le \delta\sigma(\tilde{Q}_k). \tag{9}$$

This implies (8).

Put $G_k = \{\xi \in \tilde{Q}_k : T_{\alpha,r} f(\xi) > 2\lambda, M_\beta f(\xi) \le \varepsilon\lambda\}$. Now

$$\{\xi \in S : T_{\alpha,r} f(\xi) > 2\lambda\} \subset \{\xi \in S : M_\beta f(\xi) > \varepsilon\lambda\} \cup (\cup G_k),$$

so

$$\begin{aligned}
\sigma(\{\xi \in S : T_{\alpha,r} f(\xi) > 2\lambda\}) &\le \sigma(\{\xi \in S : M_\beta f(\xi) > \varepsilon\lambda\}) + \delta \sum_k \sigma(\tilde{Q}_k) \\
&\le C(\sigma(\{\xi \in S : M_\beta f(\xi) > \varepsilon\lambda\}) \\
&\quad + \delta\sigma(\{\xi \in S : T_{\alpha,r} f(\xi) > \lambda\})).
\end{aligned}$$

Multiply by $p\lambda^{p-1}$ and integrate in λ from 0 to ∞ to find

$$\|T_{\alpha,r} f\|^p_{L^p(\sigma)} \le C(\|M_\beta f\|^p_{L^p(\sigma)} + \delta\|T_{\alpha,r} f\|^p_{L^p(\sigma)}),$$

by (9). This gives (8), if δ is chosen sufficiently small.

4. Analytic case. In this section we give an independent proof, based on an interpolation theorem, for the case of H^p spaces.

The implication (a) $\Rightarrow$ (c). We consider first case $0 < p < 2$. First we show the implication is true in case $f \in H^\infty(B)$, the space of all holomorphic bounded functions on B. Assume that $f \in H^\infty(B)$ and let $1 < \alpha < \beta$ be fixed. Let $E = \{\xi \in S : M_\beta f(\xi) \le h\}$, $h > 0$, and let F be the complement of the set E. If $\lambda_{M_\beta f}(t) = \sigma(\{\xi \in S : M_\beta f(\xi) > t\})$, $t > 0$ is the distribution function of $M_\beta f$, then $\lambda_{M_\beta f}(h) = \sigma(F)$.

Let $R = \cup_{\xi\in E} D_\alpha(\xi)$. From (1) and (3) we see that $\tilde{\Delta}|f|^2 = (4/(n+1))|\tilde{D}f|^2$. Hence,

$$\begin{aligned}\frac{2}{n+1}\int_E [T_\alpha f(\xi)]^2\, d\sigma(\xi) &= \int_E \int_{D_\alpha(\xi)} \tilde{\Delta}|f|^2(z)\, d\tau(z)\, d\sigma(\xi)\\ &= \int_R \tilde{\Delta}|f|^2(z)\sigma(\{\xi \in E : z \in D_\alpha(\xi)\})\, d\tau(z),\end{aligned}$$

by Fubini's theorem. Since $\sigma(\{\xi \in S : z \in D_\alpha(\xi)\}) \cong (1-|z|^2)^n$, we see that

$$\int_E [T_\alpha f(\xi)]^2\, d\sigma(\xi) \le C \int_R \tilde{\Delta}|f|^2(z)(1-|z|^2)^n\, d\tau(z). \tag{10}$$

To calculate the right integral, we apply (4). But we have to replace the region R by smooth regions $R_\varepsilon \subset R$ approximating R. See [**9**] for this argument. We put $u = (1-|z|^2)^n$, $v = |f|^2$. Then we have

$$\begin{aligned}\int_{R_\varepsilon} \tilde{\Delta}|f|^2(z)(1-|z|^2)^n \tau(z) &= \int_{R_\varepsilon} |f(z)|^2 \tilde{\Delta}(1-|z|^2)^n\, d\tau(z)\\ &\quad + \int_{\partial R_\varepsilon} (1-|z|^2)^n \frac{\partial}{\partial \tilde{n}} |f(z)|^2\, d\tilde{\sigma}(z)\\ &\quad - \int_{\partial R_\varepsilon} |f(z)|^2 \frac{\partial}{\partial \tilde{n}}(1-|z|^2)^n\, d\tilde{\sigma}(z)\\ &= I_1 + I_2 - I_3.\end{aligned}$$

It follows from (2) that $I_1 < 0$. To evaluate I_2 and I_3 we divide the boundary ∂R_ε into two parts ∂R_ε^E and ∂R_ε^F, where ∂R_ε^E (respectively

∂R_ε^F) is the part lying above the set E (respectively F). Set

$$I_1^E = \int_{\partial R_\varepsilon^E} (1-|z|^2)^n \frac{\partial}{\partial \tilde{n}} |f(z)|^2 \, d\tilde{\sigma}(z),$$
$$I_1^F = \int_{\partial R_\varepsilon^F} (1-|z|^2)^n \frac{\partial}{\partial \tilde{n}} |f(z)|^2 \, d\tilde{\sigma}(z),$$
$$I_2^E = \int_{\partial R_\varepsilon^E} |f(z)|^2 \frac{\partial}{\partial \tilde{n}} (1-|z|^2)^n \, d\tilde{\sigma}(z)$$

and

$$I_2^F = \int_{\partial R_\varepsilon^F} |f(z)|^2 \frac{\partial}{\partial \tilde{n}} (1-|z|^2)^n \, d\tilde{\sigma}(z).$$

It is easily verified that if v is real valued and C^1 in B then the outward normal derivative $\partial v / \partial \tilde{n}$ at $t\xi$ along S_t is given by

$$\frac{\partial v}{\partial \tilde{n}}(t\xi) = \frac{2}{\sqrt{n+1}} (1-t^2) \operatorname{Re} \sum_{j=1}^n \xi_j \frac{\partial v}{\partial z_j}(t\xi).$$

Using this the Schwarz inequality and (5), Lemma 2, we find that

$$\begin{aligned} |I_1^E| &\le C \int_{\partial R_\varepsilon^E} (1-|z|^2) |f(z)| \, |\nabla f(z)| \, ds(z) \\ &\le C \left(\int_{\partial R_\varepsilon^E} |f(z)|^2 \, ds(z) \right)^{1/2} \left(\int_{\partial R_\varepsilon^E} |\tilde{\nabla} f(z)|^2 \, ds(z) \right)^{1/2} \\ &\le C \int_E [M_\beta f(\xi)]^2 \, d\sigma(\xi). \end{aligned} \tag{11}$$

(Here we denoted the area measure on ∂R_ε^E by ds.)

Next we estimate I_1^F. By the definition of E, we know that $|f(z)| \le h$ for all $z \in \cup \{ D_\beta(\xi) : \xi \in E \}$ and so by (5) we have $|\tilde{\nabla} f(z)| \le Ch$ for all $z \in R$. (We may choose $0 < r < 1$ so that if $w \in D_\alpha(\xi)$ then $E_r(w) \subset D_\beta(\xi)$.) This gives that

$$\begin{aligned} I_1^F &\le C \int_{\partial R_\varepsilon^F} (1-|z|^2)^n |f(z)| \, |\tilde{\nabla} f(z)| \, d\tilde{\sigma}(z) \le C h^2 \sigma(F) \\ &= C h^2 \lambda_{M_\beta f}(h). \end{aligned} \tag{12}$$

Using the same argument as in the previous step we find that

$$I_2^E \le C \int_E [M_\beta f(z)]^2 \, d\sigma(\xi) \le C \int_0^h t\lambda_{M_\beta f}(t)\, dt. \tag{13}$$

Finally, since $|f(z)| \le h$ on R_ε, we get

$$I_2^F \le Ch^2\sigma(F) = Ch^2\lambda_{M_\beta f}(h). \tag{14}$$

Combining (11), (12), (13) and (14) we can replace (10) by the following inequality:

$$\int_E [T_\alpha f(\xi)]^2 \, d\sigma(\xi) \le C\left[h^2\lambda_{M_\beta f}(h) + \int_0^h t\lambda_{M_\beta f}(t)\, dt\right].$$

From this and the fact that $\sigma(F) = \lambda_{M_\beta f}(h)$, it follows that

$$\lambda_{T_\alpha f}(h) \le C\left[\lambda_{M_\beta f}(h) + \frac{1}{h^2}\int_0^h t\lambda_{M_\beta f}(t)\, dt\right].$$

Therefore we get that

$$\begin{aligned} &\int_S [T_\alpha f(\xi)]^p \, d\sigma(\xi) \\ &\le C\left[\int_0^\infty h^{p-1}\left(\lambda_{M_\beta f}(h) + \frac{1}{h^2}\int_0^h t\lambda_{M_\beta f}(t)\, dt\right) dh\right] \\ &\le C\left[\int_0^\infty h^{p-1}\lambda_{M_\beta f}(h)\, dh + \int_0^\infty t\lambda_{M_\beta f}(t)\left(\int_t^\infty h^{p-3}\, dh\right) dt\right] \\ &\le C\int_S [M_\beta f(\xi)]^p \, d\sigma(\xi). \end{aligned} \tag{15}$$

This proves the theorem for the case $f \in H^\infty(B)$.

To show the general case, suppose that $f \in H^p(B)$, $0 < p < 2$. Define f_ε, for $0 < \varepsilon < 1$ and $z \in B$ by $f_\varepsilon(z) = f(\varepsilon z)$. Then we have $f_\varepsilon \in H^\infty(B)$. Replace f by f_ε in (15) to get that

$$\|T_\alpha f_\varepsilon\|_{L^p(\sigma)} \le C\|M_\beta f\|_p. \tag{16}$$

To complete the proof we have to eliminate ε in the above inequality.

Since $|\tilde{\nabla} f_\varepsilon(z)| \to |\tilde{\nabla} f(z)|$ as $\varepsilon \to 1$, by Fatou's lemma we have

$$\begin{aligned}
\|T_\alpha f\|^p_{L^p(\sigma)} &\le \int_S \left(\liminf_{\varepsilon\to 1} \int_{D_\alpha(\xi)} |\tilde{\nabla} f_\varepsilon(z)|^2 \, d\tau(z) \right)^{p/2} d\sigma(\xi) \\
&= \int_S \liminf_{\varepsilon\to 1} \left(\int_{D_\alpha(\xi)} |\tilde{\nabla} f_\varepsilon(z)|^2 \, d\tau(z) \right)^{p/2} d\sigma(\xi) \\
&\le \liminf_{\varepsilon\to 1} \int_S \left(\int_{D_\alpha(\xi)} |\tilde{\nabla} f_\varepsilon(z)|^2 \, d\tau(z) \right)^{p/2} d\sigma(\xi) \\
&\le C \liminf_{\varepsilon\to 1} \|M_\beta f_\varepsilon\|^p_{L^p(\sigma)} \\
&\le C \|M_\beta f\|^p_{L^p(\sigma)},
\end{aligned}$$

by (16).

Now we apply Lemma 1 and interpolation theorem [**8**] to conclude that the implication (a) $\Rightarrow$ (c) is true for all $0 < p < \infty$.

The proof of the implication (b) $\Rightarrow$ (a) can be easily reduced to the harmonic case already proved in [**1**]. More precisely, if $S_\alpha f \in L^p(\sigma)$, then by Lemma 3 on page 61 of [**9**] the standard area integral of f taken over cones lies in $L^p(\sigma)$. Now by the result of [**1**], Lemma 2.2 the nontangential maximal function of f lies in $L^p(\sigma)$. This certainly implies that $f \in H^p$.

REFERENCES

1. P. Ahern and J. Bruna, *Maximal and area characterization of Hardy-Sobolev spaces in the unit ball of* C^n, Revista Math. Iber. **4** (1988), 123–153.

2. M. Arsenović and M. Jevtić, *BMO-boundedness of Lusin's area function*, Proc. Amer. Math. Soc., to appear.

3. Daryl Geller, *Some results in* H^p *theory for the Heisenberg group*, Duke Math. J. **47** (1980), 365–390.

4. Miroljub Jevtić, *On the Carleson measure characterization of* BMOA *functions on the unit ball*, Proc. Amer. Math. Soc. **114** (1992), 379–386.

5. Miroljub Jevtić and Miroslav Pavlović, *On* m*-harmonic Bloch space*, Proc. Amer. Math. Soc. **123** (1995), 1385–1392.

6. Miroslav Pavlović, *Inequalities for the gradient of eigenfunctions of the invariant Laplacian in the unit ball*, Indag. Math. **2** (1991), 89–98.

7. Walter Rudin, *Function theory in the unit ball in* C^n, in Grundlehren Math. Wiss. **241** Springer-Verlag, New York, 1980.

8. G. Stampacchia, $L^{p,\lambda}$ *spaces and interpolation*, Comm. Pure Appl. Math. **17** (1964), 293–306.

9. Elias M. Stein, *Boundary behavior of holomorphic functions of several complex variables*, Math. Notes, Princeton University Press, 1972.

Faculty of Mathematics, University Belgrade, Studentski trg 16, 11000 Beograd, Serbia
E-mail address: arsenovic@matf.bg.ac.yu

Faculty of Mathematics, University Belgrade, Studentski trg 16, 11000 Beograd, Serbia
E-mail address: jevtic@matf.bg.ac.yu