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# ALMOST SURE CONVERGENCE OF SOLUTIONS OF LINEAR STOCHASTIC VOLTERRA EQUATIONS TO NONEQUILIBRIUM LIMITS

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ABSTRACT. We consider a linear stochastic Volterra equation and obtain the stochastic analogue to work by Krisztin and Terjéki for convergence and integrability in the almost sure case. We determine sufficient conditions on the resolvent, kernel and noise for the solutions to converge to an explicit nonequilibrium limit, and for the difference between the solution and the limit to be square integrable. It is proved that the conditions on the resolvent and the kernel are necessary. Necessary and sufficient conditions for almost sure convergence are provided in the scalar case. The results are applied to a biological model, and the effect that a weakly singular kernel has on the convergence of the solution is examined.

**1. Introduction.** We study the asymptotic convergence of the solution of

$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)\,ds\right)dt + \Sigma(t)\,dB(t), \quad t > 0, \tag{1.1a}$$

$$X(0) = X_0, \tag{1.1b}$$

to a nonequilibrium limit. Here, the solution $X$ is an $n \times 1$ vector-valued function on $[0,\infty)$, $A$ is a real $n \times n$ matrix, $K$ is a continuous and integrable $n \times n$ matrix-valued function on $[0,\infty)$, $\Sigma$ is a

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continuous and integrable $n \times d$ matrix-valued function on $[0, \infty)$ and $B(t) = (B_1(t), B_2(t), \dots, B_d(t))$, where each component of the Brownian motion is independent. The initial condition $X_0$ is a deterministic constant vector.

The solution of (1.1) can be written in terms of the solution of the resolvent equation

$$R'(t) = AR(t) + \int_0^t K(t-s)R(s)\,ds, \quad t > 0, \tag{1.2a}$$

$$R(0) = I, \tag{1.2b}$$

where the $n \times n$ matrix-valued function $R$ is known as the resolvent or fundamental solution of (1.2). The representation of solutions of (1.1) in terms of $R$ is given by the variation of constants formula

$$X(t) = R(t)X_0 + \int_0^t R(t-s)\Sigma(s)\,dB(s), \quad t \geq 0.$$

The case where the solutions of (1.2) are neither integrable, nor unstable, has been considered by Krisztin and Terjéki [**10**]. They considered the convergence of solutions of (1.2) to a nonequilibrium limit. In addition to determining necessary and sufficient conditions under which $R(t)$ converges to a limit $R_\infty$ as $t \to \infty$ they determined an explicit formula for $R_\infty$.

In the stochastic case the asymptotic convergence of solutions of (1.1) to the trivial solution has been studied by Appleby and Riedle [**2**], Mao [**11**] and Mao and Riedle [**12**]. Appleby, Devin and Reynolds [**1**] is the first paper that we know of to consider the convergence of solutions of (1.1) to a nonequilibrium limit. The paper [**1**] considers the mean square case and details necessary and sufficient conditions on the resolvent, kernel, noise and tail of the noise for the convergence of solutions to an explicit limiting random variable, and for the difference between the solution and the limit to be square integrable.

In this paper analogous results are proved in the almost sure case. Establishing the necessary and sufficient conditions on the resolvent, kernel and noise is complicated by the fact that $X_\infty$ is not adapted. Nonetheless, it is shown that the sufficient conditions for convergence and integrability in the mean square case also suffice in the almost sure case. However, showing that these conditions are necessary is not as straightforward as in the mean square case. This is due to the fact that we can no longer avail of the simplifying effect that taking expectations has on a random variable. Consequently, we cannot show that the condition on the tail of the noise is necessary. However, in the scalar case, for a class of noise perturbations which violate this condition, we can show that although the solution still converges to a nontrivial limit the difference between the solution and the limit is not square integrable.

An epidemiological model is studied in Section 4. The results mentioned above are exploited to highlight conditions under which the disease will become endemic, which is the interpretation when solutions settle down to a nontrivial and indeed nonequilibrium limiting value.

The behavior of Volterra equations with weakly singular kernels has been studied by several authors including Miller and Feldstein [**13**] and Brunner et al. [**4, 5**]. We briefly examine in Section 8 the effect of a weakly singular kernel of algebraic or logarithmic type on the convergence and integrability of the solution. It is found that singularities of this type have no effect on the convergence of the solution.

**2. Mathematical preliminaries.** We introduce some standard notation. We denote by $\mathbf{R}$ the set of real numbers. Let $M_{n\times d}(\mathbf{R})$ be the space of $n \times d$ matrices with real entries. The transpose of any matrix $A$ is denoted by $A^T$ and the trace of a square matrix $A$ is denoted by $\mathrm{tr}\,(A)$. Further, denote by $I$ the identity matrix in $M_{n\times n}(\mathbf{R})$ and denote by $\mathrm{diag}\,(a_1, a_2, \dots, a_n)$ the $n \times n$ matrix with the scalar entries $a_1, a_2, \dots, a_n$ on the diagonal and 0 elsewhere. We denote by $\mathbf{e}_i$ the $i$th standard basis vector in $\mathbf{R}^n$. We denote by $\langle x, y\rangle$ the standard inner product of $x$ and $y \in \mathbf{R}^n$. Let $\|\cdot\|$ denote the Euclidian norm for any vector $x \in \mathbf{R}^n$. For $A = (a_{ij}) \in M_{n\times d}(\mathbf{R})$ we denote by $\|\cdot\|$ the norm defined by

$$\|A\|^2 = \sum_{i=1}^{n} \left( \sum_{j=1}^{d} |a_{ij}| \right)^2,$$

and we denote by $\|\cdot\|_F$ the Frobenius norm defined by

$$\|A\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{d} |a_{ij}|^2.$$

Since $M_{n\times d}(\mathbf{R})$ is a finite-dimensional Banach space the two norms $\|\cdot\|$ and $\|\cdot\|_F$ are equivalent, thus we can find universal constants $0 \le d_1(n,d) \le d_2(n,d)$ such that

$$d_1\|A\| \le \|A\|_F \le d_2\|A\|, \quad A \in M_{n\times d}(\mathbf{R}).$$

If $J$ is an interval in $\mathbf{R}$ and $V$ is a finite-dimensional normed space, we denote by $C(J,V)$ the family of continuous functions $\phi : J \to V$. The space of Lebesgue integrable functions $\phi : (0,\infty) \to V$ will be denoted by $L^1((0,\infty),V)$ and the space of Lebesgue square integrable functions $\eta : (0,\infty) \to V$ will be denoted by $L^2((0,\infty),V)$. Where $V$ is clear from the context we omit it from the notation. We denote by $\mathbf{C}$ the set of complex numbers, the real part of $z$ in $\mathbf{C}$ being denoted by $\operatorname{Re} z$ and the imaginary part by $\operatorname{Im} z$. If $A : [0,\infty) \to M_{n\times n}(\mathbf{R})$, the Laplace transform of $A$ is formally defined to be

$$\widehat{A}(z) = \int_0^\infty A(t)e^{-zt}\,dt.$$

The convolution of $F$ and $G$ is denoted by $F * G$ and defined to be the function given by

$$(F * G)(t) = \int_0^t F(t-s)G(s)\,ds, \quad t \ge 0.$$

We now make our problem precise. The $n$-dimensional equation given by (1.1) is considered. We assume that the function $K : [0,\infty) \to M_{n\times n}(\mathbf{R})$ satisfies

$$K \in C([0,\infty), M_{n\times n}(\mathbf{R})) \cap L^1((0,\infty), M_{n\times n}(\mathbf{R})), \tag{2.1}$$

and the function $\Sigma : [0,\infty) \to M_{n\times d}(\mathbf{R})$ satisfies

$$\Sigma \in C([0,\infty), M_{n\times d}(\mathbf{R})). \tag{2.2}$$

Due to (2.1) we may define $K_1$ in $C([0,\infty), M_{n\times n}(\mathbf{R}))$ by

$$K_1(t) = \int_t^\infty K(s)\, ds, \quad t \geq 0, \tag{2.3}$$

so that this function defines the tail of the kernel $K$.

Let $(B(t))_{t\geq 0}$ denote $d$-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}^B, \mathbf{P})$ where the filtration is the natural one $\mathcal{F}^B(t) = \sigma\{B(s) : 0 \leq s \leq t\}$. Here we define by $\sigma\{c\}$ the smallest $\sigma$-algebra which contains the family of subsets $c$. We define the function $t \mapsto X(t; X_0, \Sigma)$ to be the unique continuous adapted process which satisfies the initial value problem (1.1). Results concerning the existence and uniqueness of solutions may be found in [**3**, Theorem 2E] or [**14**, Chapter 5] for example. Under the hypothesis (2.1), it is well known that (1.2) has a unique continuous solution $R$, which is continuously differentiable. Moreover, if $\Sigma$ is continuous, then for any deterministic initial condition $X_0$, the unique almost surely continuous solution to (1.1) is given by

$$X(t; X_0, \Sigma) = R(t)X_0 + \int_0^t R(t-s)\Sigma(s)\, dB(s), \quad t \geq 0. \tag{2.4}$$

Where $X_0$ and $\Sigma$ are clear from the context, we omit them from the notation $X(t; X_0, \Sigma)$.

We also consider a deterministically and stochastically perturbed version of (1.2),

$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)ds + f(t)\right)dt \tag{2.5a}$$
$$+\Sigma(t)dB(t), \quad t > 0,$$

$$X(0) = X_0, \tag{2.5b}$$

with $A, K, \Sigma$ and $B$ defined as before. We assume that the function $f : [0,\infty) \to \mathbf{R}^n$ satisfies

$$f \in C([0,\infty), \mathbf{R}^n) \cap L^1((0,\infty), \mathbf{R}^n). \tag{2.6}$$

We define the function $t \mapsto X(t; X_0, \Sigma, f)$ to be the unique solution of the initial value problem (2.5). Moreover, if $\Sigma$ and $f$ are continuous, then for any deterministic initial condition $X_0$ there exists a unique almost surely continuous solution to (2.5) given by

$$X(t; X_0, \Sigma, f) = R(t)X_0 + \int_0^t R(t-s)f(s)\,ds + \int_0^t R(t-s)\Sigma(s)\,dB(s), \tag{2.7}$$

where $t \geq 0$. Where $X_0, \Sigma$ and $f$ are clear from the context we omit them from the notation.

We denote $\mathbf{E}[X^2]$ by $\mathbf{E}X^2$ except in cases where the meaning may be ambiguous. We now define the notion of convergence in mean square and almost sure convergence.

**Definition 2.1.** The $\mathbf{R}^n$-valued stochastic process $(X(t))_{t\geq 0}$ converges in mean square to $X_\infty$ if

$$\lim_{t\to\infty} \mathbf{E}\|X(t) - X_\infty\|^2 = 0,$$

and we say that the difference between the stochastic process $(X(t))_{t\geq 0}$ and $X_\infty$ is integrable in the mean square sense if

$$\int_0^\infty \mathbf{E}\|X(t) - X_\infty\|^2\,dt < \infty.$$

**Definition 2.2.** If there exists a $\mathbf{P}$-null set $\Omega_0$ such that for every $\omega \notin \Omega_0$ the following holds

$$\lim_{t\to\infty} X(t,\omega) = X_\infty(\omega),$$

then we say $X$ converges almost surely to $X_\infty$, and we say that the difference between the stochastic process $(X(t))_{t\geq 0}$ and $X_\infty$ is square integrable in the almost sure sense if

$$\int_0^\infty \|X(t,\omega) - X_\infty(\omega)\|^2\,dt < \infty.$$

In this paper we are particularly interested in the case where the random variable $X_\infty$ is nonzero almost surely.

**3. Discussion of main results.** The main results of the paper are presented in this section. We discuss necessary and sufficient conditions for asymptotic convergence of the solution of (1.1) to a nontrivial limit and the integrability of this solution in the almost sure case.

In the deterministic case Krisztin and Terjéki [**10**] considered the necessary and sufficient conditions for asymptotic convergence of solutions of (1.2) to a nontrivial limit and the integrability of these solutions. Before stating their main result, we define the following notation introduced in [**10**] and adopted in this paper. We let $M = A + \int_0^\infty K(s)\,ds$ and $T$ be an invertible matrix such that $T^{-1}MT$ has Jordan canonical form. Let $e_i = 1$ if all the elements of the $i$th row of $T^{-1}MT$ are zero, and $e_i = 0$ otherwise. Put $P = T\,\mathrm{diag}\,(e_1, e_2, \dots, e_n)T^{-1}$ and $Q = I - P$.

**Theorem 3.1.** *Let $K$ satisfy*

$$\int_0^\infty t^2 \|K(t)\|\,dt < \infty. \tag{3.1}$$

*The resolvent $R$ of* (1.2) *satisfies*

$$R(\cdot) - R_\infty \in L^1((0,\infty), M_{n\times n}(\mathbf{R})), \tag{3.2}$$

*if and only if*

$$\det\,[zI - A - \widehat{K}(z)] \neq 0 \ \text{ for } \ \mathrm{Re}\, z \geq 0 \ \text{ and } \ z \neq 0,$$

*and*

$$\det\left[P - M - \int_0^\infty \int_s^\infty PK(u)\,du\,ds\right] \neq 0.$$

Krisztin and Terjéki consider the case where $R - R_\infty$ exists in the space of $L^1$ functions. However, for stochastic equations it is more natural to consider the case where $R - R_\infty$ lies in the $L^2$ space of functions. In [**1**] the convergence of solutions of (1.1) to a nonequilibrium limit was considered and the following theorem was obtained.

**Theorem 3.2.** *Let $K$ satisfy* (2.1) *and*

$$\int_0^\infty t\|K(t)\|\,dt < \infty, \tag{3.3}$$

*and let $\Sigma$ satisfy* (2.2) *and*

$$\int_0^\infty \|\Sigma(t)\|^2\,dt < \infty. \tag{3.4}$$

*If the resolvent $R$ of* (1.2) *satisfies*

$$R(\cdot) - R_\infty \in L^2((0,\infty), M_{n\times n}(\mathbf{R})), \tag{3.5}$$

*then the solution $X$ of* (1.1) *satisfies* $\lim_{t\to\infty} X(t) = X_\infty$ *almost surely, where*

$$X_\infty = R_\infty\left(X_0 + \int_0^\infty \Sigma(t)\,dB(t)\right) \quad a.s.$$

*and $X_\infty$ is almost surely finite.*

In this theorem the existence of the first moment of $K$ is required rather than the existence of the second moment of $K$ as in Theorem 3.1.

The following theorem was proved in [**1**]; it details necessary and sufficient conditions for convergence in mean square.

**Theorem 3.3.** *Let $K$ satisfy* (2.1) *and* (3.3), *and let $\Sigma$ satisfy* (2.2). *The following are equivalent.*

(i) *The function $\Sigma$ satisfies* (3.4), *and there exists a constant matrix $R_\infty$ such that the solution $R$ of* (1.2) *satisfies* (3.5) *and*

$$\int_0^\infty t\|R_\infty\Sigma(t)\|^2\,dt < \infty. \tag{3.6}$$

(ii) *For all initial conditions $X_0$ there is an almost surely finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(X_0,\Sigma)$ with* $\mathbf{E}\|X_\infty(X_0,\Sigma)\|^2 <$

$\infty$ *such that the unique continuous adapted process* $X(\cdot; X_0, \Sigma)$ *which obeys* (1.1) *satisfies*

$$\lim_{t\to\infty} \mathbf{E}\|X(t; X_0, \Sigma) - X_\infty(X_0, \Sigma)\|^2 = 0 \tag{3.7}$$

*and*

$$\mathbf{E}\|X(\cdot; X_0, \Sigma) - X_\infty(X_0, \Sigma)\|^2 \in L^1((0,\infty), \mathbf{R}). \tag{3.8}$$

The sufficient conditions for the asymptotic convergence of the solution $X$ of (1.1) to a nontrivial limit $X_\infty$, and for the integrability of $X - X_\infty$ in the almost sure sense are considered in Theorem 3.4. As in the mean square case, we find that conditions (3.4) and (3.5) are required for convergence; in addition, (3.6) is required for integrability.

**Theorem 3.4.** *Let* $K$ *satisfy* (2.1) *and* (3.3), *and let* $\Sigma$ *satisfy* (2.2). *If* $\Sigma$ *satisfies* (3.4) *and if there exists a constant matrix* $R_\infty$ *such that the solution* $R$ *of* (1.2) *satisfies* (3.5), *then for all initial conditions* $X_0$ *there is an almost surely finite* $\mathcal{F}^B(\infty)$*-measurable random variable* $X_\infty(X_0, \Sigma)$ *such that the unique continuous adapted process* $X(\cdot; X_0, \Sigma)$ *which obeys* (1.1) *satisfies*

$$\lim_{t\to\infty} X(t; X_0, \Sigma) = X_\infty(X_0, \Sigma) \quad a.s.. \tag{3.9}$$

*Moreover, if the function* $\Sigma$ *also satisfies* (3.6), *then*

$$X(\cdot; X_0, \Sigma) - X_\infty(X_0, \Sigma) \in L^2((0,\infty), \mathbf{R}^n) \quad a.s.. \tag{3.10}$$

We now state the necessary conditions for the asymptotic convergence of the solution $X$ of (1.1) to a nontrivial limit $X_\infty$, and for the square integrability of $X - X_\infty$ in the almost sure sense.

**Theorem 3.5.** *Let* $K$ *satisfy* (2.1) *and* (3.3), *and let* $\Sigma$ *satisfy* (2.2). *Suppose for all initial conditions* $X_0$ *there is an almost surely finite* $\mathcal{F}^B(\infty)$*-measurable random variable* $X_\infty(X_0, \Sigma)$ *such that the unique continuous adapted process* $X(\cdot; X_0, \Sigma)$ *which obeys* (1.1) *satisfies* (3.9) *and* (3.10). *Then there exists a constant matrix* $R_\infty$ *such that the solution* $R$ *of* (1.2) *satisfies* (3.5) *and the function* $\Sigma$ *satisfies* (3.4) *and*

$$\int_0^\infty \left\| \int_t^\infty R_\infty \Sigma(s)\, dB(s) \right\|^2 dt < \infty \quad a.s.. \tag{3.11}$$

We have stated that conditions (3.4) and (3.5) are both necessary and sufficient for convergence and square integrability. However, we have not succeeded in showing that (3.6) is a necessary condition. By taking expectations it is clear that (3.6) implies (3.11) but it is not immediate that (3.11) implies (3.6). We conjecture that these two conditions are equivalent and that (3.6) is in fact a necessary condition for almost sure convergence and integrability. In order to support this conjecture we consider (1.1) in the scalar case and state the following theorem.

**Theorem 3.6.** *Let* $n = d = 1$, $K$ *satisfy* (2.1) *and* (3.3), $\Sigma$ *satisfy* (2.2) *and* (3.4), *and suppose there exists a nontrivial constant* $R_\infty$ *such that the scalar solution* $R$ *of* (1.2) *satisfies* (3.5). *Suppose the function* $\Sigma$ *satisfies*

$$\Sigma(t)^2 > 0, \quad t \geq 0, \tag{3.12}$$

*and*

$$\limsup_{t\to\infty} \frac{\Sigma(t)^2}{(\int_t^\infty \Sigma(s)^2\, ds)^2} < \infty. \tag{3.13}$$

*Then for all initial conditions* $X_0$ *there is an almost surely finite* $\mathcal{F}^B(\infty)$*-measurable random variable* $X_\infty(X_0, \Sigma)$ *such that the unique continuous adapted process* $X(\cdot; X_0, \Sigma)$ *which obeys* (1.1) *satisfies* (3.9) *but*

$$X(\cdot; X_0, \Sigma) - X_\infty(X_0, \Sigma) \notin L^2((0,\infty), \mathbf{R}) \quad a.s. \tag{3.14}$$

In this theorem we have assumed that $\Sigma$ is square integrable. However (3.13) ensures that

$$\int_0^\infty t|\Sigma(t)|^2\, dt = \infty.$$

Of course, if $\int_0^\infty t|\Sigma(t)|^2\,dt < \infty$, then Theorem 3.4 guarantees that $X - X_\infty \in L^2(0,\infty)$ almost surely. Hence, we are able to prove that although the solution tends to a limit in the almost sure sense the difference between the solution and the limit is not square integrable.

In the scalar case, and for certain families of noise intensity, Theorems 3.4 and 3.6 complement one another. Consider, for example, the family of noise intensities which behave asymptotically polynomially in the sense that $\lim_{t\to\infty}\Sigma(t)^2 t^{2\beta} = c$, where $c$ and $\beta$ are positive constants. If $\beta > 1/2$, then $\Sigma$ is square integrable so we see that $\lim_{t\to\infty} X(t) = X_\infty$ almost surely using Theorem 3.2. Now, if $\beta > 1$ it is clear from Theorem 3.4 that $X - X_\infty \in L^2(0,\infty)$ almost surely. If $1/2 < \beta \le 1$ the noise term $\Sigma$ is not square integrable but condition (3.13) is satisfied and so Theorem 3.6 states that $X - X_\infty \notin L^2(0,\infty)$ almost surely.

Analogous results may be obtained in the case where the equation is both stochastically and deterministically perturbed. The following theorem places sufficient conditions under which solutions tend to a nonequilibrium limit.

**Theorem 3.7.** *Let $K$ satisfy* (2.1) *and* (3.3)*, let $\Sigma$ satisfy* (2.2) *and* (3.4)*, and let $f$ satisfy* (2.6)*. Suppose the resolvent $R$ of* (1.2) *satisfies* (3.5)*. Then the solution $X(t;X_0,\Sigma,f)$ of* (2.5) *satisfies $X(\cdot;X_0,\Sigma,f) \to X_\infty(X_0,\Sigma,f)$ almost surely, where*

$$X_\infty(X_0,\Sigma,f) = R_\infty\left(X_0 + \int_0^\infty f(t)\,dt + \int_0^\infty \Sigma(t)\,dB(t)\right) \quad a.s. \tag{3.15}$$

*and $X_\infty$ is almost surely finite. Moreover, if $\Sigma$ satisfies* (3.6) *and $f$ satisfies*

$$\int_0^\infty t\|R_\infty f(t)\|\,dt < \infty, \tag{3.16}$$

*then*

$$X(\cdot;X_0,\Sigma,f) - X_\infty(X_0,\Sigma,f) \in L^2((0,\infty),\mathbf{R}^n) \quad a.s. \tag{3.17}$$

This theorem has applications in the study of infinite-delay equations. In particular, it provides useful insights into the epidemiological model examined in Section 4.

Theorem 3.8 deals with the convergence of solutions to a nontrivial limit in the scalar case. This theorem illustrates the necessity of (3.4) for convergence, a fact which is not obvious in the finite-dimensional case.

**Theorem 3.8.** *Suppose that $n = d = 1$, $\Sigma$ is non-trivial, $K$ obeys* (2.1), (3.3) *and*

$$A + \int_0^\infty K(s)\,ds = 0. \tag{3.18}$$

*The following are equivalent.*

(i) *There exists a unique continuous $\mathcal{F}^B$-adapted process $X$ which obeys* (1.1) *and an $\mathcal{F}^B(\infty)$-measurable and almost surely finite random variable $X_\infty$ such that* (3.9) *holds.*

(ii) *The function $\Sigma$ obeys* (3.4)*, the function $K$ obeys*

$$1 + \int_0^\infty sK(s)\,ds \neq 0, \tag{3.19}$$

*and there exists a constant $R_\infty$ such that the solution $R$ of* (1.2) *satisfies*

$$\lim_{t\to\infty} R(t) = R_\infty.$$

The proofs of Theorems 3.4 and 3.5 may be found in Section 5, the proof of Theorem 3.6 is located in Section 6 and Theorems 3.7 and 3.8 are proved in Section 7.

**4. Application.** In this section we consider the following epidemiological model:

$$dx(t) = \left(g(x(t)) - \int_{-\infty}^t w(t-s)g(x(s))\,ds\right)dt + \Sigma(t)\,dB(t), \tag{4.1a}$$

$$x(t) = \phi(t), \quad t \le 0. \tag{4.1b}$$

Here the solution $x(\cdot;\phi,\Sigma)$ is a scalar function on $[0,\infty)$, the function $g$ is a scalar linear function satisfying $g(x)=\alpha x$ for some constant $\alpha>0$, $w$ is a positive scalar weighting function satisfying

$$\int_0^\infty w(s)\,ds = 1,$$

$\Sigma$ is a continuous and square integrable scalar function on $[0,\infty)$, $(B(t))_{t\geq 0}$ denotes one-dimensional Brownian motion on a complete filtered probability space $(\Omega,\mathcal{F},\mathcal{F}^B(t)_{t\geq 0},\mathbf{P})$ where the filtration is the natural one $\mathcal{F}^B(t)=\sigma\{B(s):0\leq s\leq t\}$ and the initial function $\phi$ satisfies

$$\sup_{t\leq 0}|\phi(t)|\leq\bar{\phi}. \tag{4.2}$$

Various authors have considered similar models in the deterministic case where $x(t)$ represents the population at time $t$. Cooke and Yorke [**7**] proposed the nonlinear delay-differential equation $x'(t)=g(x(t))-g(x(t-L))$ as a model for the growth of an epidemic where $g(x(t))$ represents the birth rate when the current population is $x(t)$, while death is certain at an age of $L$ time units. A generalization of this model was considered by Haddock and Terjéki [**9**], in which a convolution term was incorporated to allow for deaths at a distribution of ages. Indeed, Burton [**6**] extended their model and considered

$$x'(t)=\int_{t-L}^{t}p(s-t)g(x(s))\,ds-\int_{-\infty}^{t}q(s-t)g(x(s))\,ds,$$

in which both births and deaths are distributed. Here, death can occur at any time while the number of births is related to the number of conceptions which occurred up to $L$ time units ago. A simple calculation illustrates that this equation is fundamentally the same as the deterministic version of (4.1) when appropriate conditions are imposed on the functions $p$ and $q$. Many more authors have considered biological models of this type. We direct the interested reader to [**6**] for a comprehensive list of references.

The following theorem, the proof of which may be found in Section 7, considers the conditions under which the solution of our model converges to a nontrivial limit.

**Theorem 4.1.** *Let $w$ satisfy*

$$w \in C([0,\infty),\mathbf{R}) \cap L^1((0,\infty),\mathbf{R})$$

*and*

$$\int_0^\infty t|w(t)|\,dt < \infty. \tag{4.3}$$

*Let $\Sigma$ satisfy* (2.2) *and* (3.4) *where $n = d = 1$, and let $\phi$ satisfy* (4.2). *Suppose the resolvent $R$ of* (1.2) *satisfies* (3.5). *Then the solution $x(\cdot;\phi,\Sigma)$ of* (4.1) *satisfies $x(\cdot;\phi,\Sigma) \to x_\infty(\phi,\Sigma)$ almost surely, where*

$$x_\infty(\phi,\Sigma) = R_\infty\left(\phi(0) + \int_0^\infty \int_{-\infty}^0 w(t-s)\phi(s)\,ds\,dt + \int_0^\infty \Sigma(t)\,dB(t)\right) \quad a.s.$$

*and $x_\infty$ is almost surely finite. Moreover, if $\Sigma$ satisfies* (3.6) *and $w$ satisfies*

$$\int_0^\infty t^2|w(t)|\,dt < \infty, \tag{4.5}$$

*then*

$$x(\cdot;\phi,\Sigma) - x_\infty(\phi,\Sigma) \in L^2((0,\infty),\mathbf{R}^n) \quad a.s. \tag{4.6}$$

The function $w$ represents the distribution of deaths within a population. It is evident from Theorem 4.1 that the growth of a population is influenced by the decay rate of $w$, that is, if the first moment of $w$ exists, then the population will converge to a finite limit.

**5. Conditions for asymptotic convergence and integrability in the almost sure sense.** In this section we begin by considering sufficient conditions for asymptotic convergence of solutions of (1.1) to a nontrivial random variable in the almost sure sense. The necessity of these conditions is also considered. Two technical lemmas used in the proof of Theorem 3.5 are presented. Lemma 5.1 concerns the structure of $X_\infty$. This enables us to prove Lemma 5.2 which concerns the necessity of (3.4) for stability of the system. Consequently, we need only assume the continuity of the noise intensity $\Sigma$ to ensure the existence of solutions at the outset. Lemma 5.2 in turn allows us to show the necessity of (3.11); the proof of this inference may be found in the proof of Theorem 3.5 below.

**Lemma 5.1.** *Let $K$ satisfy* (2.1) *and* (3.3). *Suppose that for all initial conditions $X_0$ there is an almost surely finite random variable $X_\infty(X_0,\Sigma)$ such that the solution $t \mapsto X(t;X_0,\Sigma)$ of* (1.1) *satisfies* (3.9) *and* (3.10). *Then*

$$\left(A+\int_0^\infty K(s)\,ds\right)X_\infty = 0 \quad a.s.. \tag{5.1}$$

**Lemma 5.2.** *Let $K$ satisfy* (2.1) *and* (3.3). *Suppose for all initial conditions $X_0$ there is an almost surely finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(X_0,\Sigma)$ such that the solution $t \mapsto X(t;X_0,\Sigma)$ of* (1.1) *satisfies* (3.9) *and* (3.10). *Then $\Sigma$ satisfies* (3.4).

We defer the proof of Lemmas 5.1 and 5.2 to Section 7.

*Proof of Theorem* 3.4. From Theorem 3.2 we know that $X_\infty$ is almost surely finite and (3.9) holds if (3.3), (3.4) and (3.5) hold.

We know from Theorem 3.3 that $\int_0^\infty \mathbf{E}\|X(t)-X_\infty\|^2\,dt<\infty$ since (3.4), (3.5) and (3.6) hold. Fubini's theorem allows us to interchange the order of integration of this term; thus, $\mathbf{E}[\int_0^\infty \|X(t)-X_\infty\|^2\,dt]<\infty$. If the expectation of a non-negative random variable is finite, then the random variable itself is almost surely finite; applying this here means that (3.10) holds. $\square$

*Proof of Theorem* 3.5. We begin by proving (3.5). Consider the $n+1$ solutions $X_j(t)$ of (1.1) with initial conditions $X_j(0)=\mathbf{e}_j$ for $j=1,\dots,n$ and $X_{n+1}(0)=0$ where $\mathbf{e}_1,\dots,\mathbf{e}_n$ is the standard basis. Note that $X_j(t)=R(t)\mathbf{e}_j+\mu(t)$ and $X_{n+1}(t)=\mu(t)$ where $\mu(t)=\int_0^t R(t-s)\Sigma(s)\,dB(s)$. Since $X_{n+1}(t)\to X_{n+1}(\infty)$ as $t\to\infty$, this implies that $\mu(t) \to \mu(\infty)$. Now, since $R(t)\mathbf{e}_j = X_j(t) - X_{n+1}(t)$ for $j = 1, \dots, n$, we see that $R(t) \to R_\infty$. Thus, for $j = 1, \dots, n$, we can write

$$\begin{aligned}(R(t) - R_\infty)\mathbf{e}_j &= (X_j(t) - X_{n+1}(t)) - (X_j(\infty) - X_{n+1}(\infty)) \\ &= (X_j(t) - X_j(\infty)) - (X_{n+1}(t) - X_{n+1}(\infty)).\end{aligned}$$

Since (3.10) holds, we see that $(R(\cdot) - R_\infty)\mathbf{e}_j \in L^2(0, \infty)$ for $j = 1, \dots, n$, hence (3.5) holds.

In order to show (3.4) holds, we apply Lemma 5.2.

Finally, we turn to (3.11). Expressing the solution of (1.1) using variation of parameters, subtracting $X_\infty$ from both sides and rearranging the equation, we obtain

$$\begin{aligned}\int_t^\infty R_\infty \Sigma(s)\, dB(s) \; &= (R(t) - R_\infty)X_0 \\ &\quad + \int_0^t (R(t-s) - R_\infty)\Sigma(s)\, dB(s) - (X(t) - X_\infty).\end{aligned} \tag{5.2}$$

The first term on the righthand side of (5.2) is in $L^2(0, \infty)$ due to the above argument. Using the fact that (3.4) and (3.5) hold, we see that

$$\begin{aligned}\mathbf{E}\left[\int_0^\infty \left\| \int_0^t (R(t-s) - R_\infty)\Sigma(s)\, dB(s) \right\|^2 dt\right] \\ = \int_0^\infty \int_0^t \|(R(t-s) - R_\infty)\Sigma(s)\|^2\, ds\, dt < \infty.\end{aligned} \tag{5.3}$$

If the expectation of a random variable is finite, then the random variable itself is finite almost surely which means that the second term is in $L^2(0, \infty)$. The third term on the righthand side of (5.2) is in $L^2(0, \infty)$ using (3.10), thus (3.11) holds. This completes our proof. $\square$

**6. On the necessity of condition (3.6) for convergence and integrability of solutions.** We make use of Lemmas 6.1 and 6.2 in the proof of Theorem 3.6. The proof of these lemmata is deferred to Section 7.

**Lemma 6.1.** *Let $n = d = 1$, and let $B$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}^B(t))_{t\geq 0}, \mathbf{P})$. Then, for any constant $c > 0$,*

$$\int_c^\infty B(t)^2 t^{-2}\,dt = \infty \quad a.s.. \tag{6.1}$$

**Lemma 6.2.** *Let $n = d = 1$, let the function $\Sigma$ satisfy* (2.2), (3.4), (3.12) *and* (3.13), *and let $B$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}^B(t))_{t\geq 0}, \mathbf{P})$. Then*

$$\int_0^\infty \left(\int_t^\infty \Sigma(s)\,dB(s)\right)^2 dt = \infty \quad a.s.. \tag{6.2}$$

*Proof of Theorem* 3.6. Using the fact that $K$ satisfies (2.1), (3.3), the fact that $R$ satisfies (3.5) and the fact that $\Sigma$ satisfies (2.2) and (3.4), we can apply the scalar version of Theorem 3.2 to obtain (3.9). We now show that (3.14) holds. Subtract $X_\infty$ from both sides of (1.1) to obtain

$$\begin{aligned} X(t)-X_\infty = {} & (R(t)-R_\infty)X_0 + \int_0^t (R(t-s)-R_\infty)\Sigma(s)\,dB(s) \\ & - \int_t^\infty R_\infty \Sigma(s)\,dB(s). \end{aligned} \tag{6.3}$$

Although the first term on the righthand side of (6.3) is square integrable as (3.5) holds, and the second term is in $L^2(0,\infty)$ almost surely as (3.4) and (3.5) hold, it is clear from Lemma 6.2 that $X(\cdot) - X_\infty \notin L^2(0,\infty)$, as (6.2) holds. This completes our proof. $\square$

**7. Proofs.** In this section we give the proofs of results which were postponed earlier in the paper.

*Proof of Theorem* 3.7. The solution $X(t; X_0, \Sigma)$ of (1.1) satisfies (2.4), and the solution $X(t; X_0, \Sigma, f)$ satisfies (2.7); thus,

$$X(t; X_0, \Sigma, f) = X(t; X_0, \Sigma) + \int_0^t R(t-s) f(s)\,ds, \quad t \geq 0.$$

As $t \to \infty$, we know from Theorem 3.2 that $X(t; X_0, \Sigma) \to X_\infty(X_0, \Sigma)$. Also, from our assumptions,

$$\lim_{t\to\infty} \int_0^t R(t-s) f(s)\, ds = R_\infty \int_0^\infty f(s)\, ds,$$

and so $X(t; X_0, \Sigma, f) \to X_\infty(X_0, \Sigma, f)$ where $X_\infty(X_0, \Sigma, f)$ is given by (3.15).

We now prove (3.17). Consider

$$\begin{aligned} X(t; X_0, \Sigma, f) - X_\infty(X_0, \Sigma, f) &= (X(t; X_0, \Sigma) - X_\infty(X_0, \Sigma)) \\ &\quad + \int_0^t (R(t-s) - R_\infty) f(s)\, ds - \int_t^\infty R_\infty f(s)\, ds. \end{aligned} \tag{7.1}$$

Consider the righthand side of (7.1). We know that $X(t; X_0, \Sigma) - X_\infty(X_0, \Sigma) \in L^2(0, \infty)$ using Theorem 3.4. An $L^2(0, \infty)$ term convolved with an $L^1(0, \infty)$ term lies in the space of $L^2(0, \infty)$ functions and so the second term on the righthand side of (7.1) must lie in $L^2(0, \infty)$. Finally, (3.16) guarantees that the last term on the righthand side of (7.1) is in $L^2(0, \infty)$. Combining the arguments given in this paragraph, we see that (3.17) must hold. This completes our proof. □

*Proof of Theorem* 3.8. We begin by proving that (i) implies (ii). Let $Y$ be the process defined by

$$Y(t) = X_0 + \int_0^t \Sigma(s)\, dB(s), \quad t \geq 0. \tag{7.2}$$

Then $Z = X - Y$ obeys $Z(0) = 0$ and

$$Z'(t) = AZ(t) + \int_0^t K(t-s) Z(s)\, ds + f(t), \quad t > 0,$$

where $f(t) = AY(t) + (K * Y)(t)$. Now, with $K_1$ as defined in (2.3), if we define $p$ by $p(t) = (K_1 * Z)(t)$ for $t \geq 0$, we have

$$-p'(t) = -\int_0^t K_1'(t-s) Z(s)\, ds - K_1(0) Z(t) = (K * Z)(t) + AZ(t), \quad t > 0.$$

Hence, $Z'(t) = -p'(t) + f(t)$, for $t > 0$ and so, by integration, we get $Z(t) + p(t) = Z(0) + p(0) + \int_0^t f(s)\,ds$, $t \geq 0$. Therefore,

$$X(t) + (K_1 * X)(t) = Y(t) + (K_1 * Y)(t) + \int_0^t f(s)\,ds, \quad t \geq 0.$$

Finally, by reversing the order of integration, we get

$$\begin{aligned}\int_0^t f(s)\,ds &= \int_0^t AY(s)\,ds + \int_0^t \int_u^t K(s-u)\,ds Y(u)\,du \\ &= \int_0^t \left(A + \int_0^{t-s} K(v)\,dv\right) Y(s)\,ds \\ &= -(K_1 * Y)(t),\end{aligned}$$

so

$$X(t) + (K_1 * X)(t) = Y(t), \quad t \geq 0. \tag{7.3}$$

By (2.3) and (3.3), $K_1$ is integrable, and so, as $X(t) \to X_\infty$ as $t \to \infty$, it follows that

$$\lim_{t\to\infty} Y(t) = X_\infty \left(1 + \int_0^\infty sK(s)\,ds\right), \quad \text{a.s..} \tag{7.4}$$

Therefore, it follows by the definition of $Y$ that

$$\lim_{t\to\infty} \int_0^t \Sigma(s)\,dB(s) \text{ exists a.s. and is a.s. finite,}$$

from which $\Sigma \in L^2(0,\infty)$ automatically follows. We now prove that (3.19) holds by providing a proof by contradiction. Using (7.4), we see that

$$X_0 + \int_0^\infty \Sigma(s)\,dB(s) = X_\infty \left(1 + \int_0^\infty sK(s)\,ds\right) \text{ a.s..}$$

We suppose that

$$1 + \int_0^\infty sK(s)\,ds = 0,$$

then
$$\int_0^\infty \Sigma(s)\,dB(s) = -X_0 \ \text{ a.s..}$$
But
$$\int_0^\infty \Sigma(s)\,dB(s) \sim \mathcal{N}\Big(0, \int_0^\infty \Sigma(s)^2\,ds\Big),$$
and $X_0$ is purely deterministic; this is only possible if $X_0$ and $\Sigma$ are both zero. As we excluded this trivial case by assumption, it is clear that (3.19) holds. Finally, in the proof of Theorem 3.5, we provide an argument to show that $R(t) \to R_\infty < \infty$ as $t \to \infty$ when (3.3) and (3.9) hold.

We now show that (ii) implies (i). Consider (7.3). From [**8**, Theorem 2.3.5], we know that $X$ can be expressed as

$$X(t) = Y(t) - \int_0^t r(t-s)Y(s)\,ds, \tag{7.5}$$

where the function $r$ satisfies $r + K_1 * r = K_1$. Letting $t \to \infty$, the first term on the righthand side of (7.5) becomes

$$Y(\infty) = X_0 + \int_0^\infty \Sigma(s)\,dB(s). \tag{7.6}$$

From [8, Theorem 2.4.1], we know that $r \in L^1(0,\infty)$ if $1 + \widehat{K}_1(z) \neq 0$ for $\operatorname{Re} z \geq 0$. We show in the sequel that $1 + \widehat{K}_1(z) \neq 0$ for $\operatorname{Re} z \geq 0$, thus we can integrate $r + K_1 * r = K_1$ over $[0,\infty)$ and rearrange the equation to obtain

$$\int_0^\infty r(s)\,ds = \Big(1 + \int_0^\infty K_1(s)\,ds\Big)^{-1} \int_0^\infty K_1(s)\,ds.$$

Using this we see that the second term on the righthand side of (7.5) becomes

$$\begin{aligned} &\lim_{t\to\infty} \int_0^t r(t-s)Y(s)\,ds \\ &\qquad = \Big(1 + \int_0^\infty K_1(s)\,ds\Big)^{-1} \int_0^\infty K_1(s)\,ds Y(\infty). \end{aligned} \tag{7.7}$$

Again, letting $t \to \infty$ we see that (7.5) becomes

$$\begin{aligned} X_\infty &= Y(\infty) - \int_0^\infty r(s)\,ds Y(\infty) \\ &= \left(1 + \int_0^\infty K_1(s)\,ds\right)^{-1} Y(\infty) \\ &= \left(1 + \int_0^\infty K_1(s)\,ds\right)^{-1} \left(X_0 + \int_0^\infty \Sigma(t)\,dB(t)\right) \end{aligned}$$

by combining (7.6) and (7.7).

We now show that $1 + \widehat{K}_1(z) \neq 0$ for $\operatorname{Re} z \geq 0$. If $z = 0$, then

$$1 + \widehat{K}_1(0) = 1 + \int_0^\infty sK(s)\,ds \neq 0$$

from our assumptions. For $\operatorname{Re} z \geq 0$ and $z \neq 0$, we have

$$1 + \widehat{K}_1(z) = \frac{1}{z}(z - A - \widehat{K}(z)).$$

A proof by contradiction is provided to show that this is nonzero. Suppose that there exists $z_0 \neq 0$, $\operatorname{Re} z_0 \geq 0$ such that $z_0 - A - \widehat{K}(z_0) = 0$. Thus, $e^{z_0 t}$ is a solution of

$$y'(t) = Ay(t) + \int_0^\infty K(s)y(t-s)\,ds.$$

Using variation of parameters, we see that

$$e^{z_0 t} = R(t) + \int_0^t R(t-s) \int_s^\infty K(u)e^{z_0(s-u)}\,du\,ds. \tag{7.8}$$

We consider the cases where $\operatorname{Re} z_0 > 0$ and $\operatorname{Re} z_0 = 0$ separately. When $\operatorname{Re} z_0 > 0$, the real part of the lefthand side of (7.8) tends to $\infty$ as $t \to \infty$. Now consider the righthand side. The first term on the righthand side of (7.8) converges to a finite limit as $R \to R_\infty$ as $t \to \infty$. Now we consider the second term. Since $t \mapsto \int_t^\infty K(u)e^{z_0(t-u)}\,du$ is integrable and $R(t) \to R_\infty$ as $t \to \infty$ their convolution tends to a finite constant. Thus, the real part of the righthand side approaches a finite constant while the real part of the lefthand side tends to $\infty$. This yields a contradiction and so $1+\widehat{K}_1(z) \neq 0$ for $\operatorname{Re} z > 0$. We now look at the case when $\operatorname{Re} z_0 = 0$. By considering the real part of both sides of (7.8), we see that the lefthand side is identically equal to zero while the righthand side is not. This yields a contradiction and so

$$1+\widehat{K}_1(z) \neq 0 \quad \text{for} \quad \operatorname{Re} z \geq 0.$$

*Proof of Theorem* 4.1. We begin by splitting the convolution term as follows

$$dx(t) = \alpha\left(x(t) - \int_{-\infty}^{0} w(t-s)x(s)\,ds - \int_{0}^{t} w(t-s)x(s)\,ds\right)dt + \Sigma(t)\,dB(t).$$

Clearly $-\alpha\int_{-\infty}^{0} w(t-s)x(s)\,ds$ corresponds to $f(t)$ of (2.5) for $t \geq 0$. We see that this term is in $L^1(0,\infty)$ using (4.3). Thus, we can apply Theorem 3.7 to show that the solution $x(t;\phi,\Sigma)$ of (4.1) satisfies $x(\cdot;\phi,\Sigma) \to x_\infty(\phi,\Sigma)$ almost surely, where $x_\infty(\phi,\Sigma)$ is given by (4.4).

Furthermore, as (4.5) holds, a simple calculation shows that condition (3.16) of Theorem 3.7 is satisfied and so (4.6) must hold.

*Proof of Lemma* 5.1. Define the random vector

$$\Lambda := \left(A + \int_{0}^{\infty} K(s)\,ds\right)X_\infty. \tag{7.9}$$

Writing (1.1) in integral form, adding and subtracting $X_\infty$ from both sides, and then dividing both sides of the equation by $t$, we obtain

$$\begin{aligned}\frac{X(t)-X_\infty}{t} &= \frac{X_0 - X_\infty}{t} + \frac{\int_0^t A(X(s)-X_\infty)\,ds}{t} \\ &\quad + \frac{\int_0^t\int_0^s K(s-u)(X(u)-X_\infty)\,du\,ds}{t} \\ &\quad - \frac{\int_0^t\int_s^\infty K(u)\,du\,ds X_\infty}{t} + \frac{\int_0^t \Sigma(s)\,dB(s)}{t} + \Lambda.\end{aligned} \tag{7.10}$$

As $t\to\infty$ we see that the term on the lefthand side of (7.10) tends to zero since (3.9) holds. Now consider the righthand side of (7.10).

The first term tends to zero as $t \to \infty$ since $X_0$ is a finite deterministic vector and $X_\infty$ is almost surely finite by hypothesis. The second term tends to zero since (3.10) holds. Consider the third term. Using the Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{1}{t}&\left\| \int_0^t \int_0^s K(s-u)(X(u)-X_\infty)\,du\,ds \right\| \\ &\le \left( \frac{1}{t^2} \left\| \int_0^t \int_0^s K(s-u)(X(u)-X_\infty)\,du\,ds \right\|^2 \right)^{1/2} \\ &\le \left( \frac{\overline{K}}{t} \int_0^t \int_0^s \|K(s-u)\| \|X(u)-X_\infty\|^2\,du\,ds \right)^{1/2},\end{aligned}$$

where $\overline{K} = \int_0^\infty \|K(t)\|\,dt$. Using (2.1) and (3.10) we see that the righthand side of this inequality tends to zero as $t \to \infty$. Thus the third term on the righthand side of (7.10) tends to zero. Since (3.3) holds, we see that the fourth term tends to zero as $t \to \infty$. Therefore, if we take limits on both sides of (7.10), we obtain

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \Sigma(s)\,dB(s) = -\Lambda. \tag{7.11}$$

We now show that $\Lambda = 0$ almost surely. Each individual entry of the vector $(1/t)\int_0^t \Sigma(s)\,dB(s)$ is given by

$$\left[ \frac{1}{t} \int_0^t \Sigma(s)\,dB(s) \right]_i = \frac{1}{t} \sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s).$$

Since $\Lambda$ is almost surely finite by hypothesis, we know that $\mathbf{P}[C_i] = 1$ where $C_i \subset \Omega$ is defined by

$$C_i = \left\{ \omega : \left[ \lim_{t\to\infty} \frac{1}{t} \int_0^t \Sigma(s)\,dB(s) \right]_i \quad \text{exists} \right\}, \quad i = 1, \dots, d.$$

For each $i = 1, \dots, d$, define $\sigma_i \in C([0,\infty),[0,\infty))$ by

$$\sigma_i^2(t) = \sum_{j=1}^d \Sigma_{ij}^2(t), \quad t \ge 0, \tag{7.12}$$

and consider the cases when $\sigma_i^2 \in L^1(0,\infty)$ and $\sigma_i^2 \notin L^1(0,\infty)$ individually. If $\sigma_i^2 \in L^1(0,\infty)$, then $\lim_{t\to\infty}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s)$ exists and is almost surely finite, and so

$$\lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) = 0, \quad \text{a.s..}$$

Thus, if $\sigma_i^2 \in L^1(0,\infty)$, then $\Lambda_i = 0$, a.s.

In the case when $\sigma_i^2 \notin L^1(0,\infty)$, we have that

$$\liminf_{t\to\infty}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dBj(s) = -\infty,$$
$$\limsup_{t\to\infty}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) = \infty, \quad \text{a.s.}$$

Therefore

$$\liminf_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) \le 0,$$
$$\limsup_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) \ge 0, \quad \text{a.s.}$$

Since $\lim_{t\to\infty}(1/t)\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) = \Lambda_i$ a.s., and $\Lambda_i$ is almost surely finite, we have

$$\Lambda_i = \lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) = \liminf_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) \le 0,$$

so $\Lambda_i \le 0$, a.s. Similarly,

$$\Lambda_i = \lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) = \limsup_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) \ge 0,$$

so $\Lambda_i \ge 0$, almost surely. Therefore, in the case when $\sigma_i^2 \notin L^2(0,\infty)$, we have that $\Lambda_i = 0$, almost surely. Hence, $\Lambda_i = 0$ for all $i = 1,\dots,d$, almost surely, and so $\Lambda = 0$, almost surely. Thus, (5.1) must hold.

*Proof of Lemma* 5.2. By Itô's rule,

$$\|X(t)\|^2 = \|X_0\|^2 + 2\int_0^t \langle X(s), AX(s) + (K*X)(s)\rangle\, ds + \int_0^t \|\Sigma(s)\|_F^2\, ds + M(t), \tag{7.13}$$

where

$$M(t) = 2\sum_{j=1}^{d}\sum_{i=1}^{n}\int_0^t X_i(s)\Sigma_{ij}(s)\, dB_j(s). \tag{7.14}$$

Introducing the function $\Delta$ defined by $\Delta(t) = X(t) - X_\infty$, and by using the fact that

$$\int_0^t AX(s) + (K*X)(s)\, ds = X(t) - X_0 - \int_0^t \Sigma(s)\, dB(s),$$

we have

$$\begin{aligned}\int_0^t \langle X(s), AX(s) + (K*X)(s)\rangle\, ds &= \int_0^t \langle \Delta(s), A(\Delta(s) + X_\infty) + (K*[\Delta + X_\infty])(s)\rangle\, ds \\ &\quad + \langle X_\infty, X(t) - X_0 - \int_0^t \Sigma(s)\, dB(s)\rangle.\end{aligned}$$

Therefore, by Lemma 5.1, and the definition of $K_1$, we get

$$\begin{aligned}\int_0^t \langle X(s), AX(s) + (K*X)(s)\rangle\, ds &= \int_0^t \langle \Delta(s), A\Delta(s) + (K*\Delta)(s)\rangle\, ds - \int_0^t \langle \Delta(s), K_1(s)X_\infty\rangle\, ds \\ &\quad + \langle X_\infty, X(t) - X_0 - \int_0^t \Sigma(s)\, dB(s)\rangle.\end{aligned} \tag{7.15}$$

We suppose that

$$\lim_{t\to\infty}\int_0^t \|\Sigma(s)\|_F^2\, ds = \infty, \tag{7.16}$$

and prove that this is false by contradiction. The quadratic variation of $M$ is given by

$$\langle M\rangle(t) = 4\sum_{i=1}^{n}\sum_{j=1}^{d}\int_0^t (X_i(s)\Sigma_{ij}(s))^2\,ds.$$

Therefore,

$$\begin{aligned}\langle M\rangle(t) &\le 4\sum_{j=1}^{d}\int_0^t \sum_{i=1}^{n} X_i(s)^2 \sum_{i=1}^{n}\Sigma_{ij}(s)^2\,ds\\ &\le 4\int_0^t \|X(s)\|^2\|\Sigma(s)\|_F^2\,ds.\end{aligned}$$

If we define

$$C_1 = \{\omega : \lim_{t\to\infty}\langle M\rangle(t,\omega) = \infty\},$$

then by L'Hôpital's rule, (7.16) and (3.9), we get

$$\limsup_{t\to\infty}\frac{\langle M\rangle(t)}{\int_0^t \|\Sigma(s)\|_F^2\,ds} \le 4\|X_\infty\|^2, \text{ a.s. on } C_1.$$

Therefore, by the law of large numbers for martingales, we get

$$\lim_{t\to\infty}\frac{|M(t)|}{\int_0^t \|\Sigma(s)\|_F^2\,ds} = \lim_{t\to\infty}\frac{|M(t)|}{\langle M\rangle(t)}\cdot\frac{\langle M\rangle(t)}{\int_0^t \|\Sigma(s)\|_F^2\,ds} = 0, \text{ a.s. on } C_1.$$

On $\overline{C}_1$, we have that $\lim_{t\to\infty} M(t)$ exists a.s. and is almost surely finite. Therefore, on account of (7.16), we have

$$\lim_{t\to\infty}\frac{|M(t)|}{\int_0^t \|\Sigma(s)\|_F^2\,ds} = 0, \text{ a.s. on } \overline{C}_1.$$

Hence,

$$\lim_{t\to\infty}\frac{M(t)}{\int_0^t \|\Sigma(s)\|_F^2\,ds} = 0, \text{ a.s.}.$$

By applying this result and using (3.9) in (7.13), we now may conclude

$$\lim_{t\to\infty}\frac{\int_0^t \langle X(s), AX(s) + (K*X)(s)\rangle\,ds}{\int_0^t \|\Sigma(s)\|_F^2\,ds} = -\frac{1}{2}, \text{ a.s.} \tag{7.17}$$

We now analyze the limit on the lefthand side above by using the representation (7.15) and show that its limit must be zero, thereby inducing a contradiction to the hypothesis that (7.16) holds. Dividing (7.15) by $\int_0^t \|\Sigma(s)\|_F^2 \, ds$, we get

$$\begin{aligned}(7.18)\quad &\frac{1}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} \int_0^t \langle X(s), AX(s) + (K * X)(s) \rangle \, ds \\ &= \frac{1}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} \int_0^t \langle \Delta(s), A\Delta(s) + (K * \Delta)(s) \rangle \, ds \\ &\quad - \frac{\int_0^t \langle \Delta(s), K_1(s) X_\infty \rangle \, ds}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} + \frac{\langle X_\infty, X(t) - X_0 \rangle}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} \\ &\quad - \frac{1}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} \Big\langle X_\infty, \int_0^t \Sigma(s) \, dB(s) \Big\rangle.\end{aligned}$$

Assumption (3.10) states that $\|X - X_\infty\|^2 \in L^1(0, \infty)$ almost surely. Therefore, as $K$ obeys (2.1), it follows that the numerator in the first term on the righthand side of (7.18) tends to a finite limit as $t \to \infty$. Consequently, the first term has zero limit as $t \to \infty$, almost surely. By (3.9) and (3.10), it follows that for each $\omega$ in an almost sure event $t \mapsto |\Delta(t, \omega)|$ is uniformly bounded. As (3.3) holds $K_1$ is integrable, and so the numerator in the second term on the righthand side of (7.18) tends to a limit for each outcome in an almost sure set. Hence, the second term has zero limit as $t \to \infty$, almost surely. Equations (3.9) and (7.16) guarantee that the third term has zero limit as $t \to \infty$, almost surely. Thus, by considering the final term on the righthand side of (7.18), it is evident that

$$(7.19)\quad \lim_{t \to \infty} \frac{1}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} \int_0^t \langle X(s), AX(s) + (K * X)(s) \rangle \, ds = 0, \quad \text{a.s.}$$

if it can be shown that (7.16) implies

$$(7.20)\qquad \lim_{t \to \infty} \frac{\int_0^t \Sigma(s) \, dB(s)}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} = 0, \quad \text{a.s..}$$

Hence, proving (7.20) provides the desired contradiction to (7.17) in the shape of (7.19).

The proof of (7.20) is quite straightforward. Define $N(t)=\int_0^t \Sigma(s)\,dB(s)$, for $t\geq 0$ and

$$N_i(t)=\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s),\quad t\geq 0,$$

so that $N_i(t)=\langle N(t),\mathbf{e}_i\rangle$. Then each $N_i$ is a local martingale with square variation

$$\langle N_i\rangle(t)=\int_0^t \sigma_i^2(s)\,ds,\quad t\geq 0,$$

where $\sigma_i$ is defined by (7.12). It is easily seen that

$$\langle N_i\rangle(t)\leq \int_0^t \|\Sigma(s)\|_F^2\,ds. \tag{7.21}$$

In the case when $\lim_{t\to\infty}\langle N_i\rangle(t)=\infty$, the law of large numbers for martingales and (7.21) give

$$\lim_{t\to\infty}\frac{|N_i(t)|}{\int_0^t \|\Sigma(s)\|_F^2\,ds}=\lim_{t\to\infty}\frac{|N_i(t)|}{\langle N_i\rangle(t)}\cdot\frac{\langle N_i\rangle(t)}{\int_0^t \|\Sigma(s)\|_F^2\,ds}=0,\quad \text{a.s.}$$

On the other hand, if $\lim_{t\to\infty}\langle N_i\rangle(t)<\infty$, then $\lim_{t\to\infty}N_i(t)$ exists almost surely and is almost surely finite. Since $\Sigma$ obeys (7.16), it is immediate that once more

$$\lim_{t\to\infty}\frac{|N_i(t)|}{\int_0^t \|\Sigma(s)\|_F^2\,ds}=0,\quad \text{a.s.}.$$

Therefore,

$$\lim_{t\to\infty}\frac{|N_i(t)|}{\int_0^t \|\Sigma(s)\|_F^2\,ds}=0,\quad \text{for all }\ i=1,\dots,d\ \text{ a.s.},$$

from which (7.20) follows immediately.

*Proof of Lemma* 6.1. Using integration by parts over $[c,t]$ and Itô's lemma, we obtain

$$B^2(t)/t-\log t=B^2(c)/c-\log c+2M(t)-\langle M\rangle(t), \tag{7.22}$$

where $M(t) = \int_c^t s^{-1} B(s)\, dB(s)$ and the square variation of $M$ is given by $\langle M\rangle(t) = \int_c^t s^{-2} B^2(s)\, ds$. Define the event $D = \{\omega : \lim_{t\to\infty} \langle M\rangle(t,\omega) = L < \infty\}$ and suppose that $\mathbf{P}[D] > 0$. On the event $D$ we know that $\lim_{t\to\infty} M(t,\omega) < \infty$, and so each term on the righthand side of (7.22) is finite. This implies that the lefthand side of (7.22) is finite, which in turn implies that $\lim_{t\to\infty} B^2(t) = \infty$ on an event $D$ of nonzero probability. This contradicts the Law of the Iterated Logarithm for standard Brownian motion and so (6.1) holds.

*Proof of Lemma* 6.2. Define the event $A$ by

$$A = \left\{\omega : \int_0^\infty \left(\int_t^\infty \Sigma(s)\, dB(s)\right)^2 dt < \infty\right\}. \tag{7.23}$$

In the sequel we show that

$$\int_0^\infty \left(\int_t^\infty \Sigma(s)\, dB(s)\right)^2 dt > \varepsilon \int_c^\infty B_3(t)^2 t^{-2}\, dt, \tag{7.24}$$

where $B_3$ is standard Brownian motion on the probability triple $(\Omega, \mathcal{F}, \mathbf{P})$, and $\varepsilon$ and $c$ are positive constants. From Lemma 6.1 we see that under our hypotheses the righthand side of (7.24) is infinite, and hence that $\mathbf{P}[A] = 0$.

We now show that (7.24) holds. Define $M(t) = \int_0^t \Sigma(s)\, dB(s)$. Then $M$ is a martingale with square variation $\langle M\rangle(t) = \int_0^t \Sigma^2(s)\, ds$. Define $T := \int_0^\infty \Sigma^2(s)\, ds = \langle M\rangle(\infty)$. By the martingale time change theorem, there is a standard Brownian motion $B_1$ such that $M(t) = B_1(\langle M\rangle(t))$. Using (3.12) and the continuity of $\Sigma$, we may define $\theta : [0,T) \to [0,\infty) : t \mapsto \theta(t)$ by $\langle M\rangle(\theta(t)) = t$, $t \in [0,T)$. Thus, because $M(\infty) - M(t) = \int_t^\infty \Sigma(s)\, dB(s)$, we obtain

$$\begin{aligned}
&\int_0^\infty \left(\int_t^\infty \Sigma(s)\, dB(s)\right)^2 dt \\
&\qquad = \int_0^\infty \left\{B_1(\langle M\rangle(\infty)) - B_1(\langle M\rangle(t))\right\}^2 dt \\
&\qquad = \int_0^T \left(B_1(T) - B_1(s)\right)^2 \frac{1}{\langle M\rangle'(\theta(s))}\, ds \\
&\qquad = \int_0^T \left(B_1(T) - B_1(T-u)\right)^2 \Sigma(\theta(T-u))^{-2}\, du.
\end{aligned}$$

Now, the process $B_2 = \{B_2(t); 0 \le t \le T; \mathcal{F}^{B_2}(t)\}$ defined by $B_2(t) = B_1(T) - B_1(T-t)$, $t \in [0,T]$ is a standard Brownian motion. Hence

$$\begin{aligned}(7.25)\quad \int_0^\infty \left(\int_t^\infty \Sigma(s)\,dB(s)\right)^2 dt &= \int_0^T B_2^2(u)\Sigma(\theta(T-u))^{-2}\,du \\ &= \int_{1/T}^\infty B_2^2(1/v)\Sigma(\theta(T-1/v))^{-2}v^{-2}\,dv \\ &= \int_{1/T}^\infty (vB_2(1/v))^2\,\Sigma(\theta(T-1/v))^{-2}v^{-4}\,dv \\ &= \int_{1/T}^\infty B_3^2(v)\Sigma(\theta(T-1/v))^{-2}v^{-4}\,dv,\end{aligned}$$

where $B_3$ defined by $B_3(t) = tB_2(1/t)$ for $t > 0$ and $B_3(0) = 0$ is a standard Brownian motion.

Since $\theta = \langle M\rangle^{-1}$, we have that $\int_{\theta(T-1/v)}^\infty \Sigma^2(u)\,du = v^{-1}$, so using (3.13) we see that for $v > 1/T$

$$\begin{aligned}(7.26)\quad v^{-2}\Sigma(\theta(T-1/v))^{-2} &= \left(\int_{\theta(T-1/v)}^\infty \Sigma^2(u)\,du\right)^2 \Sigma(\theta(T-1/v))^{-2} > \varepsilon.\end{aligned}$$

Using (7.25) and (7.26), we obtain the inequality in (7.24), where $c = 1/T$.

**8. Equations with weakly singular kernels.** In this section we consider the behavior of the solution of equation (1.1) when the kernel is weakly singular. While Miller and Feldstein considered a general definition for weak singularities in the kernel, Brunner et al. [**4, 5**] considered Volterra equations with weakly singular kernels of algebraic or logarithmic type. In these papers singularities not only in the kernel itself but also in its derivatives were considered. In keeping with earlier assumptions made in this paper no new assumptions concerning the existence of the derivatives of the kernel are made in this section. Instead, we restrict our investigation to the study of singularities in the kernel alone. Consequently, we can adopt an abridged version of the definition of a weakly singular function used in [**5**]. We say that the function $K : (0,T] \to M_{n\times n}(\mathbf{R})$ satisfies

$$K \in \mathcal{W}^{\nu}((0,T], M_{n\times n}(\mathbf{R})), \tag{8.1}$$

if $K$ is continuous on $(0,T]$ and

$$\|K(t)\| \le c(K)\begin{cases} 1 + |\log(t)| & \nu = 0,\ 0 < t \le T, \\ t^{-\nu} & 0 < \nu < 1,\, 0 < t \le T. \end{cases}$$

We do not know of any work concerned with the behavior of stochastic Volterra equations with weakly singular kernels. Consequently, before considering the effect of a weakly singular kernel on the results in this paper, it is necessary to prove that a solution exists under assumption (8.1).

**Theorem 8.1.** *Let $K$ satisfy* (8.1)*, and let $\Sigma$ satisfy* (2.2)*. Then for every $T > 0$ there is a unique adapted process $X(\cdot, X_0, \Sigma) \in C([0,T), \mathbf{R}^n)$ obeying* (1.1).

We now provide a sketch the proof of Theorem 8.1. Due to the presence of the weakly singular kernel, our analysis is simplified if we consider the following equation

$$X(t) = X_0 + \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X(s)\,ds + \mu(t), \quad 0 < t \le T, \tag{8.2a}$$

$$X(0) = X_0, \tag{8.2b}$$

where $\mu(t) = \int_0^t \Sigma(s)\,dB(s)$. Using standard arguments, the existence of an adapted process $X \in C[0,T]$ which satisfies (8.2) can be shown. Moreover, by applying Fubini's theorem to (8.2), we can show that $X$ is in fact a solution of (1.1). A Gronwall-type argument can be implemented to show that this is in fact a unique process. Again, standard arguments can be applied to show that $X$ is in fact a unique, continuous, adapted process on $[0,\infty)$.

A consequence of Theorem 8.1 is that assumption (2.1) may be replaced by

$$K \in \mathcal{W}^{\nu}((0,\infty), M_{n\times n}(\mathbf{R})) \cap L^1((0,\infty), M_{n\times n}(\mathbf{R})), \quad 0 \le \nu < 1,$$

in Theorem 3.2. The conclusion of this theorem and its proof is essentially unaltered. The primary reason for this is that the reformulation of equation (1.1) found in the proof of [**1**, Theorem 3.2] still holds. In fact, the structure of the reformulated equation ensures that the type of singularity considered in the kernel has no influence on the convergence of the solutions.

The question of integrability of solutions is more delicate and requires careful analysis. The proof of this result requires the use of the variation of parameters representation of the solution. It will be necessary to prove the validity of this formula, which will involve a close examination of stochastic Fubini theorems, before we can tackle the integrability of the solution. The authors intend to examine this in future work.

In [**5, 13**], the extent to which the regularity in the kernel influences the regularity of the solution of the deterministic equation was investigated. However, the presence of the nondifferentiable Brownian motion in the stochastic equation prohibits the existence of a derivative in the solution; indeed it is known that the solution to the stochastic equation will be Hölder continuous with exponent $1/2$. Consequently, we cannot expect to obtain the same amount of regularity in the solution of the stochastic equation as obtained in the deterministic case regardless of the regularity of the kernel. An interesting question is what effect a stronger singularity in the kernel, for example a singularity in its tail, has on the behavior of the solution of the stochastic equation.

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