JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 8, Number 3, Summer 1996

A GENERAL THEOREM ON CONTINUITY AND COMPACTNESS OF THE URYSON OPERATOR

MARTIN VÄTH

ABSTRACT. We consider the Uryson operator in a very general class of spaces, which in particular contains ideal spaces (e.g., L_p-spaces and Orlicz spaces). We will prove a theorem which will allow us to construct growth conditions on the generating function, which assure that the operator is continuous and compact. The theorem is applied also for linear integral operators and for nonlinear Volterra-Uryson equations.

0. This paper is concerned with the continuity and compactness of the Uryson operator

$$Ax(t) = \int_S g(t, s, x(s))\, ds.$$

In the first section we will recall the concept of ideal spaces and prove some lemmas. In the second section we will prove the main theorem. In the last section we will give some sample applications of the theorem in Lebesgue and Orlicz spaces.

1. Ideal spaces and Carathéodory functions. We first define a rather big class of spaces:

Definition 1. Let Y be a Banach space and S some measure space. We will call a set X of (classes of) measurable functions $x : S \to Y$ together with some mapping $||\cdot|| : X \to [0, \infty]$ *projectable space*, if it has the property that for any measurable $D \subseteq S$ and any $x \in X$ the projected function $P_D x(s) = \chi_D(s)x(s)$ also belongs to X. We will call X regular, if $||P_D x|| \to 0$ whenever the measure of D tends to 0.

The most important examples of projectable spaces are ideal spaces:

Received by the editors on January 27, 1995, and in revised form on May 17, 1995.

Definition 2. A pre-ideal space X is a normed projectable space with the property that, for any $x \in X$ and measurable y with $|y(s)| \leq |x(s)|$ almost everywhere, we have $y \in X$ and $||y|| \leq ||x||$. If X is complete, it is called *ideal space*.

For example, the Lebesgue spaces $L_p(S,Y)$, $1 \leq p \leq \infty$, are ideal spaces, regular for $p < \infty$. Other examples are Lorentz, Marcinkiewicz or Orlicz spaces. Regular projectable spaces, which are not pre-ideal spaces, are $L_p(S,Y)$, $0 < p < 1$, or the space of all measurable functions with

$$||x|| = \int_S \frac{|x(s)|}{1 + |x(s)|}\, ds.$$

Definition 3. A normed linear space X of measurable functions has the *W-property*, if $||x_n|| \to 0$ implies $x_n \to 0$ in measure.

Any pre-ideal space over a finite measure space has the W-property [**10**].

Proposition 1. *X has the W-property if and only if*

$$\lim_{n\to\infty} \sup_{||x||\leq 1} \operatorname{mes}\{s : |x(s)| \geq n\} = 0, \tag{1}$$

i.e., if its unit ball is bounded in measure.

Proof. Assume that there is some sequence with $||x_n|| \to 0$ but $x_n \not\to 0$ in measure. By passing to a subsequence, we may assume that $0 < ||x_n|| \leq n^{-2}$. Putting $y_n = ||x_n||^{-1}x_n$, we have for $n^{-1} \leq \delta$ that $\operatorname{mes}\{s : |x_n(s)| \geq \delta\} \leq \operatorname{mes}\ \{s : |y_n(s)| \geq n\}$, whence (1) implies the contradiction $x_n \to 0$ in measure. Conversely, if (1) is false, there exists a sequence x_n, $||x_n|| \leq 1$, with $\operatorname{mes}\{s : |x_n(s)| \geq n\} \not\to 0$. Then $y_n = n^{-1}x_n$ converges to zero in norm, but not in measure. □

For the Uryson operator, the Carathéodory condition is often important:

Definition 4. Let T and S be compact subsets of Euclidean spaces and U and V Banach spaces. A function $g : T \times S \times U \to V$ is called *Carathéodory function*, if $g(\cdot, \cdot, u)$ is measurable on $T \times S$ for each u, and $g(t, s, \cdot)$ is continuous for almost all $(t, s) \in T \times S$.

A well-known consequence is that measurability is no problem: If g is a Carathéodory function, then for any measurable x the function $(t, s) \mapsto g(t, s, x(s))$ is measurable. In fact, this is obvious for simple functions. Otherwise approximate x by simple functions x_n and observe that $g(t, s, x_n(s)) \to g(t, s, x(s))$.

Another consequence is an extension of Luzin's theorem, the so-called Scorza-Dragoni lemma. We recall a generalized form of the lemma:

Lemma 1. *Let $g : T \times S \times U \to V$ be a Carathéodory function where U is separable. Then, for any $\gamma > 0$, there exists a compact set $M \subseteq T \times S$ with $\operatorname{mes}(T \times S \backslash M) \leq \gamma$ such that the restriction $g : M \times U \to V$ is continuous.*

The proof may be found (even for more general measure spaces than $T \times S$) in [**9**]. We remark that the condition of separability may not be dropped.

2. The main theorem. Let S and T be compact subsets of Euclidean spaces. Let X be a normed linear space of measurable functions over S, which take values in some separable Banach space, and let Y be a Banach space of measurable functions over T which take values in a finite-dimensional space. Let $r > 0$ and $B_r = \{x \in X : ||x||_X \leq r\}$. The Uryson operator

$$Ax(t) = \int_S g(t, s, x(s))\, ds \tag{2}$$

is called k-bounded (with respect to some given functions a and b), if g satisfies the Carathéodory condition and the growth condition

$$|g(t, s, u)| \leq k(t, s)[a(t, s) + b(|u|)]. \tag{3}$$

A projectable space Z of functions $k : T \times S \to \mathbf{R}$ is called forcing (with respect to a, b, c, X, Y, r), if for any $k \in Z$ we have that any k-bounded

Uryson operator A maps B_r into Y, and

$$||Ax||_Y \leq c(||k||_Z), \qquad x \in B_r. \tag{4}$$

Theorem 1. *Let S, T, X and Y be as above, $r > 0$. Assume that X has the W-property and that $C(T)$ is continuously embedded in Y. Let a be measurable and nonnegative, b monotone increasing and positive, and c continuous at $c(0) = 0$. Let Z be forcing and regular, containing the constant functions. Then each Uryson operator A, which is k-bounded by some $k \in Z$, defines a compact and continuous mapping $A : B_r \to Y$.*

Observe that the assumptions on X are satisfied if X is a subspace of a pre-ideal space. For Y, it suffices that Y is an ideal space which contains a nontrivial constant function.

We emphasize that the theorem does not imply that all linear integral operators acting between $X = L_p(S, \mathbf{R})$ and $Y = L_q(T, \mathbf{R})$ are compact (which is not true), although such operators are always bounded (see, e.g., [**6**]), and thus for a proper choice of a, b, c, r and Z, each linear integral operator with kernel function k is k-bounded and satisfies (4).

But in order to apply the theorem for some given k-bounded Uryson operator A, it is not enough to check (4) for this fixed k or for this fixed A. Besides, it must also be verified that (4) is satisfied for all other choices of pairs (k, A), where $k \in Z$ and A is k-bounded. The crucial point here is that by the regularity of Z the size $c(||P_D k||_Z)$ becomes arbitrarily small for $\operatorname{mes} D \to 0$ and thus all $P_D k$-bounded operators must become 'uniformly small' with D.

Proof. We use a reduction technique similar to [**7**]. Let such an Uryson operator (2) be given.

a) We may assume that there exists some $B > 0$ with

$$|g(t, s, u)| \leq Bb(|u|).$$

To see this, observe that, by Lemma 1, there exist compact subsets $M_n \subseteq T \times S$ such that the measure of $Q_n = T \times S \backslash M_n$ tends to zero, and such that all of the functions $g(t, s, u)$, $k(t, s)$, $k(t, s)a(t, s)$ are continuous, if restricted to $M_n \times U$. Then each $g_n(t, s, u) = \chi_{M_n}(t, s)g(t, s, u)$ satisfies the Carathéodory condition, (3), and the additional assumption. Furthermore, the Uryson operator A_n generated by g_n converges on B_r uniformly to A. Indeed, (3) holds if we replace g by $\tilde{g}_n = g - g_n$ and k by $\tilde{k}_n = \chi_{Q_n}k$, and thus, by (4),

$$||Ax - A_n x||_Y \le c(||\tilde{k}_n||_Z) \longrightarrow 0 \quad \text{uniformly in } x \in B_r.$$

b) We may assume that g is bounded, and for some $N > 0$ we have $g(t, s, u) = 0$ for $|u| \ge N$. We argue as in a) and define

$$g_n(t, s, u) = \begin{cases} g(t, s, u) & \text{if } |u| \le n \\ (n + 1 - |u|)g(t, s, (n/|u|)u) & \text{if } n \le |u| \le n + 1 \\ 0 & \text{if } |u| \ge n + 1. \end{cases}$$

By the monotonicity of b and by a), each g_n satisfies (3) with $k \equiv B$ and the additional assumption (and the Carathéodory condition of course). The Uryson operator A_n generated by g_n converges on B_r uniformly to A. To see this, define $M_n^x = \{s \in S : |x(s)| > n\}$ for $x \in B_r$. Then (3) holds for g replaced by $g_n^x(t, s, u) = \chi_{M_n^x}(s)[g(t, s, u) - g_n(t, s, u)]$, and k by $k_n^x(t, s) = \chi_{M_n^x}(s)2B$; hence, by (4),

$$||Ax - A_n x||_Y \le c(||k_n^x||_Z), \qquad x \in B_r.$$

By (1), the righthand side tends to zero uniformly in $x \in B_r$.

c) We may assume additionally that g is continuous. Let g be as in b), bounded by some $D > 0$. By Lemma 1 there exist compact sets $M_n \subseteq T \times S$ such that the measure of $Q_n = T \times S \backslash M_n$ tends to zero, and such that the restriction of g to $M_n \times U$ is continuous. Define $g_n(t, s, u) = g(t, s, u)$ for $(t, s) \in M_n$, and $g(t, s, u) = 0$ for $|u| \ge N$. By the Tietze-Uryson lemma (see, e.g., [**3**, Theorem 7.2]) we may extend each g_n to $T \times S \times U$ such that g_n is continuous and still bounded by D. Each g_n satisfies the Carathéodory condition and (3) with k replaced by $\tilde{k} \equiv D/b(0)$. Again, the Uryson operator A_n generated by g_n converges on B_r uniformly to A. If we replace g in (3) by $\tilde{g}_n = g - g_n$ and k by $\chi_{Q_n}2\tilde{k}$, we have

$$||Ax - A_n x||_Y \le c(||\tilde{k}_n||_Z) \longrightarrow 0 \quad \text{uniformly in } x \in B_r.$$

d) Assume that g satisfies b) and c). Then g is uniformly continuous and bounded. This implies that the family of functions AB_r is equicontinuous and uniformly bounded, hence precompact in $C(T)$ by the theorem of Arzelà-Ascoli (see, e.g., [**4**]). Thus, A is compact as a mapping of B_r into the space $C(T)$. This mapping is also continuous. It suffices to prove that any sequence $x_n \to x$ in B_r contains a subsequence with $Ax_{n_k} \to Ax$ in $C(T)$. Choose a subsequence such that $Ax_{n_k} \to y$ converges in $C(T)$ and $x_{n_k} \to x$ almost everywhere. Since, for almost all (t,s), we have $g(t,s,x_{n_k}(s)) \to g(t,s,x(s))$, we have $Ax_{n_k}(t) \to Ax(t)$ for almost all t by Lebesgue's dominated convergence theorem; hence, $y = Ax$ as stated.

Now we have proved that $A : B_r \to C(T)$ is continuous and compact. Since $C(T)$ is continuously embedded in Y, we are done. $\square$

An obvious modification of part a) of the proof shows that we may replace (3) by the apparently more general formula

$$|g(t,s,u)| \leq k(t,s)[a(t,s) + d(t,s)b(|u|)],$$

where $d : T \times S \to [1,\infty)$ is a given measurable function. But this yields the same result since we just have to apply the theorem, where we replace a by a/d, Z by $\tilde{Z} = \{kd : k \in Z\} \cup \{\chi_D \text{const} : D \text{ measurable}\}$, $||h||_{\tilde{Z}} = \inf\{c(||k||_Z) : k \in Z, kd \geq h\}$, and c by the identity.

3. Applications. Observe that especially part b) of the proof essentially makes use of nonlinear operators, even if g is linear in u. However, we will first use the theorem to regain some well-known results for linear integral operators

$$Kx(t) = \int_S k(t,s)x(s)\,ds. \tag{5}$$

For those, it is usually enough to choose $a \equiv 0$, $b(|u|) = 1 + |u|$, $c(t) = (||1||_X + r)t$.

First consider $X = L_p(S)$, $Y = L_q(T)$, $1 < p \leq \infty$, $1 \leq q < \infty$, where the functions in X and Y take values in finite-dimensional spaces. Let Z_1 and Z_2 consist of $T \times S$-measurable matrix-valued functions with finite norms

$$||k||_{Z_1} = \left(\int_T \left(\int_S |k(t,s)|^{p'} \, ds \right)^{q/p'} dt \right)^{1/q},$$
$$||k||_{Z_2} = \left(\int_S \left(\int_T |k(t,s)|^{q} \, dt \right)^{p'/q} ds \right)^{1/p'},$$

where $1/p + 1/p' = 1$. A straightforward application of Hölder's and Minkowski's inequality shows that, for $k \in Z_i$, (5) maps X into Y with

$$(6) \qquad ||Kx||_Y \le ||k||_{Z_i} ||x||_X, \qquad x \in X, \ i = 1, 2.$$

Thus, it is easy to see that Z_1 and Z_2 are forcing for each $r > 0$ (a, b, c as above). In particular, K is a compact mapping $L_p(S) \to L_q(T)$ if $k \in Z_1$ or $k \in Z_2$.

On the other hand, the linear estimate (6) can be used in connection with the following simple observation to get more nonlinear results:

Theorem 2. *Let S, T, X and Y be spaces as in Theorem* 1*, Y being an ideal space, Z a regular projectable space over $T \times S$ and W a linear space of real measurable functions over S with the property that for each $k \in Z$ the linear integral operator* (5) *maps W into Y with norm*

$$||K|| \le d(||k||_Z),$$

where d is continuous at $d(0) = 0$. Assume that $b : [0, \infty) \to (0, \infty)$ is monotone increasing, such that the stationary superposition operator $Bx(s) = b(|x(s)|)$ maps $B_r = \{x \in X : ||x||_X \le r\}$ into a bounded set of W (let this set be bounded by $M \ge 0$).

Then, for each $a(t,s) = a(s)$ with $a \in W$, $a \ge 0$, the space Z is forcing with $c(t) = (M + ||a||_W)\, d(t)$.

In particular, if a Carathéodory function g satisfies the growth condition

$$|g(t,s,u)| \le k(t,s)(a(s) + b(|u|))$$

for some $a \in W$, $a \ge 0$ and $k \in Z$, the Uryson operator (2) *maps B_r into Y and is continuous and compact. The image is bounded by $(M + ||a||_W) d(||k||_Z)$.*

Proof. Consider for $a \in W$, $k \in Z$, the operator

$$Cx(t) = \int_S k(t,s)(a(s) + b(|x(s)|))\, ds, \qquad x \in B_r.$$

Then $Cx = K(a + Bx)$, hence $Cx \in Y$, and

$$||Cx||_Y \le ||K||\, ||a + Bx||_W \le d(||k||_Z)(||a||_W + M), \qquad x \in B_r,$$

which implies $Ax \in Y$ and $||Ax||_Y \le c(||k||_Z)$. $\square$

The same idea for estimation is used for Orlicz spaces in [**7**, Lemma 19.1], but the lemma there is inadvertently stated with $a(t,s) = a(t)$ instead of $a(t,s) = a(s)$ (the lemma as stated has easy counterexamples).

To exploit (6), we have to know conditions which ensure that $Bx(s) = b(|x(s)|)$ maps L_r into L_p. Sadly, by [**5**], a restrictive growth condition is necessary: $b(|u|) \le \beta + \gamma |u|^{r/p}$. For $b(|u|) = 1 + |u|^{r/p}$, we arrive at the

Corollary 1. *Let* $X = L_r(S)$, $Y = L_q(s)$, $W = L_p(S)$ *with* $1 \le r < \infty$, $1 < p \le \infty$, $1 \le q < \infty$. *Let the Carathéodory function* g *satisfy the growth condition*

$$|g(t,s,u)| \le |k(t,s)|(|a(s)| + |u|^{r/p})$$

for some $a \in W$, $k \in Z_1$ *or* $k \in Z_2$. *Then the Uryson operator* (2) *maps* X *into* Y *and is compact and continuous.*

One sees that the worse the nonlinearity in u, the more restrictive are the growth conditions on k. If g grows exponentially in u, Lebesgue spaces are too small. Here the appropriate spaces are Orlicz spaces:

Definition 5. A *Young function* $\Phi : \mathbf{R} \to [0,\infty]$ is a convex even function $\Phi : \mathbf{R} \to [0,\infty]$ with $\Phi(0) = 0$, $\Phi(t) \to \infty$ for $t \to \infty$, $\Phi(t) < \infty$ for some $t > 0$. Its complementary function is defined by

$$\Psi(s) = \sup\{t|s| - \Phi(t) : t \ge 0\}.$$

The *Orlicz* space $L_\Phi(S)$ consists of all measurable functions, for which the (Luxemburg) norm

$$||x||_\Phi = \inf\left\{\alpha > 0 : \int_S \Phi\left(\frac{|x(s)|}{\alpha}\right) ds \le 1\right\}$$

is finite.

Similarly as in [**8**, Proposition 6.1.1], one can prove the linear result:

Theorem 3. *Let Φ, Φ_1, Φ_2 be Young functions, Ψ and Ψ_2 complementary to Φ and Φ_2 such that, for some $\alpha > 0$, $u_0 \ge 0$*

$$\Phi(\alpha uv) \le \Phi_1(u)\Psi_2(v), \quad u, v \ge u_0,$$
$$C = \Phi(\alpha u_0^2)\text{mes}\, S\text{mes}\, T + \Phi_1(u_0) mes\, S + \Psi_2(u_0) mes\, T + 1 < \infty.$$

Then, for any $k \in L_\Psi(T \times S)$ the linear integral operator (5) *maps $L_{\Phi_1}(S)$ into $L_{\Phi_2}(T)$ with $||K|| \le 2\alpha^{-1}C||k||_{L_\Psi(T\times S)}$.*

If $Z \subseteq L_\Psi(T \times S)$ is a regular projectable space which contains the constant functions, Theorem 1 yields the linear result that the mapping in the previous theorem is even compact for $k \in Z$. Recall that you may (and should) choose (see [**8**], respectively [**7**])

$$\begin{aligned} Z &= M_\Psi(T \times S) \\ &= \left\{k \in L_\Psi : \int_{T\times S} \Psi(\alpha|k(t,s)|)\, d(t,s) < \infty \text{ for all } \alpha > 0\right\}, \end{aligned}$$

which contains the constant function, if Ψ is finite everywhere. Observe that we don't have to know that Z is a linear space!

If we use the well-known conditions for the superposition operator to act in Orlicz spaces (see [**2**]) we get similarly as before the

Corollary 2. *Let the conditions of the previous theorem be satisfied, Ψ being finite. Let Φ_3 be a Young function, $X = L_{\Phi_3}(S)$, $W = L_{\Phi_1}(S)$, $Y = L_{\Phi_2}(T)$ and $Z = M_\Psi(T \times S)$.*

Assume that, for some monotone increasing continuous $b : [0, \infty) \to (0, \infty)$ and for some $r, \gamma, \beta > 0$, we have

$$\Phi_1(b(u)) \le \gamma + \beta\Phi_3(u/r), \qquad u > 0.$$

Then, if a Carathéodory function g satisfies the growth condition

$$|g(t,s,u)| \leq |k(t,s)|(|a(s)| + b(|u|))$$

for some $k \in Z$ and $a \in W$, the Uryson operator (2) maps $B_r = \{x \in X : ||x||_X \leq r\}$ into Y, and this mapping is continuous and compact.

Finally we consider the Volterra-Uryson operator

$$Vx(t) = \int_\tau^t g(t,s,x(s))\,ds \tag{7}$$

and prove similarly to [**1**] that the conditions of Theorem 1 ensure that the nonlinear Volterra equation

$$x(t) = \int_\tau^t g(t,s,x(s))\,ds + f(t) \tag{8}$$

has a local solution in B_r for $||f|| < r$. Observe that in all previous applications the conditions of the following theorem are satisfied:

Theorem 4. *Let $T = S = [\tau, \tau + \delta_0]$, and assume that all conditions of Theorem 1 are satisfied with $X = Y$. Then to each Carathéodory function g satisfying the growth condition (3) for some $k \in Z$, and to each $0 < \varepsilon \leq r$, there exists some $0 < \delta \leq \delta_0$, depending only on ε and k such that (8) has, for any $f \in X$, $||f||_X \leq r - \varepsilon$ a solution $x \in B_r$ on $I = [\tau, \tau + \delta]$ satisfying $||x - f||_X \leq \varepsilon$.*

Proof. Let $\Delta = \{(t,s) : \tau \leq s \leq t \leq \tau + \delta\}$ and $P_\delta k(t,s) = \chi_\Delta(t,s)k(t,s)$, $P_\delta g(t,s,u) = \chi_\Delta(t,s)g(t,s,u)$. For δ small enough, we have $c(||P_\delta k||) \leq \varepsilon$. Theorem 1 implies that

$$A_\delta x(t) = \int_S P_\delta g(t,s,x(s))\,ds$$

maps $B = \{x \in X : ||x - f||_X \leq \varepsilon\} \subseteq B_r$ continuously and compact into X with $||A_\delta x||_X \leq \varepsilon$. Hence, by Schauder's fixed point theorem the mapping $x \mapsto A_\delta x + f$ has some fixed point $x \in B$. Since on I the operators A_δ and (7) coincide, we are done. $\square$

Acknowledgments. This paper was written in the framework of a DFG project (Ap 40/6-1). Financial support by the DFG is gratefully acknowledged.

REFERENCES

1. J. Appell and M. Väth, *Weakly singular Hammerstein-Volterra operators in Orlicz and Hölder spaces*, Z. Anal. Anwendungen **12** (1993), 663–676.

2. J. Appell and P.P. Zabrejko, *Nonlinear superposition operators*, Cambridge University Press, 1990.

3. K. Deimling, *Nonlinear functional analysis*, Springer, Berlin, 1985.

4. F. Hirzebruch and W. Scharlau, *Einführung in die Funktionalanalysis*, Mannheim, 1988 (in German).

5. M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Pergamon Press, Oxford, 1964.

6. M.A. Krasnoselskii, P.P. Zabrejko, E.I. Pustylnik and P.E. Sobolevskii, *Integral operators in spaces of summable functions*, Noordhoff, Leyden, 1976.

7. M.A. Krasnoselskii and Ya.B. Rutickii, *Convex functions and Orlicz spaces*, Noordhoff, Groningen, 1961.

8. M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces*, Marcel Dekker, New York, 1991.

9. B. Ricceri and A. Villani, *Separability and Scorza-Dragoni's property*, Matematiche **37** (1982), 156–161.

10. P.P. Zabrejko, *Ideal spaces of functions* I, Vestnik Jaroslavskogo Universiteta, Jaroslavl, 1974, 12–52.

Department of Mathematics, University of Würzburg, Am Hubland, D-97074 Würzburg, Germany