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GLOBAL EXISTENCE AND BLOWUP FOR A SEMILINEAR INTEGRAL EQUATION

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I. Introduction. Several authors have investigated the "parabolic" properties of the integrodifferential equation

$$\text{(L)} \qquad \begin{gathered} u'(t) = \int_0^t a(t-\tau)Au(\tau)\,d\tau, \quad t > 0, \\ u(0) = u_0, \end{gathered}$$

where $A : D(A) \subset X \to X$ is a linear closed operator with domain $D(A)$, a dense subset of the Banach space X, and the scalar function $a : (0, \infty) \to \mathcal{R}$ is singular at $t = 0$. The abstract setting is discussed in DaPrato and Iannelli [**1**] and discussions of specific examples of related equations are found in Grimmer and Pritchard [**2**], Hannsgen and Wheeler [**3**], Hrusa and Renardy [**5**], Renardy [**8**], and references cited therein. The results establish that rough initial data is smoothed as the solution evolves.

In this paper we consider the semilinear problem

$$\text{(SL)} \qquad \begin{gathered} u'(t) = \int_0^t a(t-\tau)Au(\tau)\,d\tau + F(u(t)), \quad t > 0, \\ u(0) = u_0, \end{gathered}$$

where $F : D(F) \subset X \to X$ is nonlinear. The singularity of F is expressed in terms of a fractional power of A. In Section II, we give preliminary definitions and establish a growth estimate of a fractional power of A acting on the solution map associated with the linear initial value problem (L). For the semilinear initial value problem (SL), a local existence result is given and global existence for small, smooth initial data is established in Section III. An example is given in Section IV to illustrate that blow-up may occur in the setting of (SL) if the kernel $a(t)$ is smooth, even though the initial data is small and smooth.

II. Linear results. Throughout this paper we make the following assumptions on A. X is a Banach space with norm $|| \cdot ||$. We also use $||\cdot||$ to denote the induced norm on $B(X)$, the Banach space of bounded linear maps from X to X.

(H1) (i) $A : D(A) \subset X \to X$ is a closed linear operator, densely defined on X.

(ii) The resolvent set of $A, \rho(A)$, satisfies $\rho(A) \supset \{\lambda \in \mathcal{C} : |\arg\lambda| < \phi\} \cup V$ where $\pi/2 < \phi < \pi$ and V is a neighborhood of zero.

(iii) There exists $M > 0$ such that, for $\lambda \in \rho(A)$, the resolvent of $A, R(\lambda; A) = (\lambda I - A)^{-1}$, satisfies $||R(\lambda; A)|| \leq M/(1 + |\lambda|)$.

If the hypotheses (H1) are satisfied, then the fractional powers of $(-A)$ are defined [**7**] for $0 < \alpha < 1$ by the formula

$$\begin{aligned}(-A)^\alpha x &= (-A)((-A)^{\alpha-1}x)\\ &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1}(-A)(tI-A)^{-1}x\,dt\\ &= \frac{-1}{2\pi i}\int_\Gamma t^{\alpha-1}(-A)(tI-A)^{-1}x\,dt,\end{aligned}$$

where $\Gamma \subset \rho(A)$ with $\Gamma \cap \{t \in \mathcal{R} : t \leq 0\} = \phi$.

Let $D((-A)^\alpha) = \{x \in X : (-A)^\alpha x \in X\}$.

We also make the following assumptions on the kernel $a : (0,\infty) \to \mathcal{R}$.

(H2) There exists $\tilde{\phi} \in (\pi/2, \pi)$ such that $\hat{a}(\lambda)$, the Laplace transform of a, is analytic and bounded in $\sum(\tilde{\phi}), \hat{a}(\lambda) \neq 0$ for $\lambda \in \sum(\tilde{\phi})$, and $\lambda(\hat{a}(\lambda))^{-1} \in \rho(A)$ for $\lambda \in \sum(\tilde{\phi})$, where

$$\sum(\tilde{\phi}) = \{\lambda \in \mathcal{C} : |\arg\lambda| < \tilde{\phi}\}.$$

As shown in [**1**],

$$T(t) = \int_{\gamma(\eta,\epsilon)} e^{\lambda t}(\lambda - \hat{a}(\lambda)A)^{-1}\,d\lambda$$

is absolutely convergent for $t \geq 0$, where $\eta \in (\pi/2, \tilde{\phi}), \varepsilon > 0$, and

$$\gamma(\eta,\varepsilon) = \{\lambda = \rho e^{\pm i\eta}; \rho \geq \varepsilon\} \cup \{\lambda = \varepsilon e^{i\tau} : \tau \in (-\eta,\eta)\}.$$

Furthermore, $T(t)$ has the following properties:

(a) $T(0) = I$, the identity on X;

(b) $T(\cdot)x : [0,\infty) \to X$ is continuous for each $x \in X$. If $x \in D(A)$, then

$$T(\cdot)x \in \mathcal{C}^1([0,\infty);X) \cap \mathcal{C}([0,\infty);D(A))$$

and $u(t) = T(t)x$ is a solution to (1). It is further established in [**1**] that if $|\hat{a}(\lambda)|^{-1} \le C_1 + C_2|\lambda|^r$ for $r \ge 0$ then, for every $x \in X, T(t)x \in D(A)$ and $||AT(t)|| \le C(t^{-1} + t^{-1-r})$.

THEOREM 1. *Let* (H1) *and* (H2) *hold and define* $T(\cdot)$ *as above. Then, for each* $0 < \alpha < 1$, *there exists a constant* C *such that*

$$||(-A)^\alpha(\lambda - \hat{a}(\lambda)A)^{-1}|| \le C|\hat{a}(\lambda)|^{-1}|\lambda(\hat{a}(\lambda))^{-1}|^{\alpha-1}.$$

Furthermore, if $|\hat{a}(\lambda)|^{-1} \le L|\lambda|^r, r > 0$, *for* $\lambda \in \sum(\tilde{\phi})$, *then there exist positive constants* M *and* δ *such that*

$$||(-A)^\alpha T(t)|| \le Mt^{-\alpha(1+r)}e^{-\delta t}, \quad \text{for } t > 0.$$

PROOF. Let $\lambda \in \sum(\tilde{\phi})$. Then, using the resolvent identity,

$$R(\lambda;A) - R(\mu;A) = (\mu - \lambda)R(\lambda;A)R(\mu;A),$$

we obtain

$$\begin{aligned}(-A)^\alpha(\lambda - \hat{a}(\lambda)A)^{-1}x &= -\frac{1}{2\pi i}\int_\Gamma t^{\alpha-1}(-A)(tI - A)^{-1}(\lambda - \hat{a}(\lambda)A)^{-1}x\,dt \\ &= -\frac{1}{2\pi i}(\hat{a}(\lambda))^{-1}\int_\Gamma t^{\alpha-1}\left(\frac{\lambda}{\hat{a}(\lambda)} - t\right)^{-1}(-A) \\ &\qquad \left[(tI - A)^{-1} - \left(\frac{\lambda}{\hat{a}(\lambda)} - A\right)^{-1}\right]x\,dt\end{aligned}$$

for any Γ satisfying $\Gamma \subset \rho(A)$ with $\Gamma \cap \{t \in \mathcal{R} : t \le 0\} = \phi$. In particular, let $l = 3\csc\phi$ where ϕ is given in (H1) and $\Gamma : l + re^{\pm i\phi}, 0 \le r < \infty$. Then, for all $r \ge 0$,

$$\left|\frac{\lambda}{\hat{a}(\lambda)} - l\left|\frac{\lambda}{\hat{a}(\lambda)}\right| - re^{\pm i\phi}\right| \ge \left|\frac{\lambda}{\hat{a}(\lambda)}\right|.$$

Also, for $r \geq 4l|\lambda/\hat{a}(\lambda)|$,

$$|re^{\pm i\phi} + l\Big|\frac{\lambda}{\hat{a}(\lambda)}\Big| - |\frac{\lambda}{\hat{a}(\lambda)}\Big| \geq \frac{r}{2} + \frac{r}{2} - l\Big|\frac{\lambda}{\hat{a}(\lambda)}\Big| - \Big|\frac{\lambda}{\hat{a}(\lambda)}\Big| \geq \frac{r}{2}.$$

Thus,

$$\begin{aligned}
&||(-A)^{\alpha}(\lambda - \hat{a}(\lambda)A)^{-1}|| \\
&\leq C|\hat{a}(\lambda)|^{-1}\int_0^{\infty}\Big|re^{\pm i\phi} + l\Big|\frac{\lambda}{\hat{a}(\lambda)}\Big|\Big|^{\alpha-1}\cdot\Big|\frac{\lambda}{\hat{a}(\lambda)} - l\Big|\frac{\lambda}{\hat{a}(\lambda)}\Big| - re^{\pm i\phi}\Big|^{-1}\,dr \\
&\leq C|\hat{a}(\lambda)|^{-1}\int_0^{4l|\frac{\lambda}{\hat{a}(\lambda)}|}|r\sin\phi|^{\alpha-1}\Big|\frac{\lambda}{\hat{a}(\lambda)}\Big|^{-1}dr \\
&\quad + \int_{4l|\frac{\lambda}{\hat{a}(\lambda)}|}^{\infty}|r\sin\phi|^{\alpha-1}|\frac{r}{2}|^{-1}\,dr \\
&\leq C|\hat{a}(\lambda)|^{-1}\Big|\frac{\lambda}{\hat{a}(\lambda)}\Big|^{\alpha-1}.
\end{aligned}$$

Assume that $|\hat{a}(\lambda)|^{-1} \leq L|\lambda|^{r}$. Then

$$\begin{aligned}
||(-A)^{\alpha}T(t)|| &\leq \int_{\gamma(\eta,\varepsilon)}||e^{\lambda t}(-A)^{\alpha}(\lambda - \hat{a}(\lambda)A)^{-1}||\,d\lambda \\
&\leq C\cdot L\int_{\gamma(\eta,\varepsilon)}|e^{\lambda t}|\,|\lambda|^{r\alpha+\alpha-1\,d\lambda} \\
&\leq C\cdot Lt^{-(\alpha+rd)}e^{-\delta t}\int_{\gamma(\eta,\varepsilon)}|e^{(\lambda+\delta)t}|\,|t\lambda|^{r\alpha+\alpha-1}d(t\lambda) \\
&\leq Mt^{-(\alpha+r\alpha)}e^{-\delta t}. \ \square
\end{aligned}$$

III. Local and global existence for semilinear problem. In this section, we establish local and global existence for the semilinear initial value problem (SL). The nonlinear operator F is assumed to satisfy the following hypothesis:

(H3) $F : D((-A)^{\alpha}) \to X$ is such that, for every open set $V \subset D((-A)^{\alpha})$, there exists a constant L such that

$$||F(u) - F(v)|| \leq L||u - v||_{\alpha}$$
for all $u, v \in V$ and $||u||_{\alpha} = ||(-A)^{\alpha}u||$.

As established in [**1**], if f is Hölder continuous and u is a solution to

$$u(t) = T(t)u_0 + \int_0^t T(t-\tau)f(\tau)\,d\tau,$$

then $u \in \mathcal{C}([0,t_1);X) \cap \mathcal{C}^1((0,t_1);X)$ and is a solution to (SL). Consequently, we consider the variation of parameters equation:

$$\text{(VP)} \qquad u(t) = T(t)u_0 + \int_0^t T(t-\tau)F(u(\tau))\,d\tau.$$

THEOREM 2. *Assume the hypotheses of Theorem* 1 *and* (H3) *hold with* $\alpha + r\alpha < 1$ *and* $u_0 \in V$. *Then there exists a unique local solution* $u : (0,t_1) \to D(A), u \in \mathcal{C}([0,t_1);X) \cap \mathcal{C}^1((0,t_1);X)$ *satisfying* (SL). *Furthermore, if* $\lim_{t\to t_1^-} ||u(t)||_\alpha < \infty$, *then* u *can be continued.*

PROOF. The proof of this theorem is routine and, therefore, only indicated below. For $\delta > 0$, let $t_1 > 0$ such that

$$||T(t)(-A)^\alpha u_0 - (-A)^\alpha u_0|| \leq \frac{1}{2}\delta \quad \text{for } t_0 < t < t_1.$$

Let Y be the Banach space $\mathcal{C}([0,t_1] : X)$ with

$$||u||_Y = \sup_{0\leq t\leq t_1} ||u(t)||.$$

Define $H : Y \to Y$ by

$$[Hy](t) = T(t)(-A)^\alpha u_0 + \int_0^t (-A)^\alpha T(t-\tau)F((-A)^{-\alpha}y(\tau))\,d\tau.$$

Let $S \subset Y$ be defined by

$$S = \{y \in Y : y(0) = (-A)^\alpha u_0, ||y(t) - (-A)^\alpha u_0|| \leq \delta\}.$$

$$
\begin{aligned}
&||H(y)(t) - (-A)^\alpha u_0|| \\
&\quad = \left\| T(t)(-A)^\alpha u_0 + \int_0^t (-A)^\alpha T(t-\tau) F((-A)^{-\alpha} y(\tau))\, d\tau \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. - (-A)^\alpha u_0 \right\| \\
&\quad \leq \frac{\delta}{2} + \int_0^t ||(-A)^\alpha T(t-\tau)[F((-A)^{-\alpha} y(\tau)) - F((-A)^{-\alpha} u_0)]|| d\tau \\
&\qquad + \int_0^t ||(-A)^\alpha T(t-\tau) F((-A)^{-\alpha} u_0)||\, d\tau \\
&\quad \leq \frac{\delta}{2} + LM \int_0^t (t-\tau)^{-(\alpha + r\alpha)} ||y(\tau) - u_0||\, d\tau \\
&\qquad + M \int_0^t (t-\tau)^{-(\alpha + r\alpha)} ||F((-A)^{-\alpha} u_0)||\, d\tau \\
&\quad \leq \frac{\delta}{2} + (1-\alpha-r\alpha)^{-1} ML\delta t^{1-(\alpha+r\alpha)} + M||F((-A)^{-\alpha} u_0||t^{1-(\alpha+r\alpha)} \\
&\quad < \delta \quad \text{for } 0 < t < t_1,\ t_1 \text{ sufficiently small.}
\end{aligned}
$$

Thus $H : S \to S$. Similarly, one can establish that H is a contraction map for t_1 sufficiently small. Consequently, H has a fixed point $y \in S$ such that

$$
((\mathrm{VP})_\alpha) \ \ y(t) = T(t)(-A)^\alpha u_0 + \int_0^t (-A)^\alpha T(t-\tau) F((-A)^{-\alpha} y(\tau))\, d\tau,
$$

and, defining $u(t) = (-A)^{-\alpha} y(t)$, we obtain a solution to (VP).

Suppose u is a solution to (VP) on $[0, t_1)$ and $\lim_{t \to t_1^-} ||(-A)^\alpha u(t)|| < \infty$. By the above argument there exists a unique solution $v(t)$ on $[0, t')$ for some $t' > 0$ to the equations

$$
\begin{aligned}
v(t) &= T(t+t_1)u_0 + \int_0^t T(t-\tau) F(v(\tau))\, d\tau + \int_0^{t_1} T(t+t_1-\tau) F(u(\tau)) d\tau \\
v(0) &= T(t_1)u_0 + \int_0^{t_1} T(t_1 - \tau) F(u(\tau))\, d\tau \\
&= u(t_1).
\end{aligned}
$$

Then $\tilde{u}$ defined on $[0, t_1 + t')$ is the solution to (VP) which extends u, where

$$
\tilde{u}(t) = \begin{cases} u(t), & 0 \leq t \leq t_1, \\ v(t - t_1), & t_1 \leq t < t_1 + t'. \end{cases}
$$

We now establish the necessary regularity of a solution u to (VP) to obtain that $u \in \mathcal{C}([0,t_1);X) \cap \mathcal{C}^1((0,t_1);X)$ and u satisfies (SL). We first establish that a solution y to $((\mathrm{VP})_\alpha)$ is locally Hölder continuous. Let $0 < t < t_1, h > 0$, and $\beta = \alpha + r\alpha < 1$. It follows from the proof of Theorem 1 that there exists $M > 0$ such that $||T'(t)|| \le Mt^{-1}$ and $||(-A)^\alpha T'(t)|| \le Mt^{-1-\beta}$ for $t > 0$. Since $F((-A)^{-\alpha}y(t))$ is continuous, there exists $N > 0$ such that $||F(-A)^{-\alpha}y(t)|| \le N$ for all $t \in [0,t_1]$.

$$\begin{aligned}
||y(t+h)-y(t)|| = \Big\| & T(t+h)(-A)^\alpha u_0 \\
& + \int_0^{t+h} (-A)^\alpha T(t+h-\tau)F((-A)^{-\alpha}y(\tau))\,d\tau \\
& - T(t)(-A)^\alpha u_0 - \int_0^t (-A)^\alpha T(t-\tau)F((-A)^{-\alpha}y(\tau))d\tau \Big\| \\
\le\ & ||(-A)^\alpha u_0|| \int_t^{t+h} ||T'(\tau)||\,d\tau \\
& + N\int_t^{t+h} ||((-A)^\alpha T(t+h-\tau))||\,d\tau \\
& + N\int_0^t \int_{t-\tau}^{t+h-\tau} ||(-A)^\alpha T'(s)||\,ds\,d\tau \\
\le\ & M||(-A)^\alpha u_0||t^{\beta-1}\int_t^{t+h} \tau^{-\beta-1}\int_t^{t+h} \tau^{-\beta}\,d\tau \\
& + MN\int_t^{t+h} (t+h-\tau)^{-\beta}\,d\tau \\
& + MN\int_0^t \int_{t-\tau}^{t+h-\tau} s^{-1-\beta}\,ds\,d\tau \\
\le\ & M||(-A)^\alpha u_0|| * 1-\beta)^{-1}t^{\beta-1}h^{1-\beta} \\
& + MN(1-\beta)^{-1}h^{1-\beta} \\
& + MN[\beta(1-\beta)]^{-1}h^{1-\beta}
\end{aligned}$$

It follows that $y(t)$ is locally Hölder continuous and, by (H3), $F((-A)^{-\alpha}$ $y(t))$ is locally Hölder continuous. Defining $u(t) = (-A)^{-\alpha}y(t)$ and applying results obtained in [**1**], we have that $u(t) \in D(A)$ for $t > 0$, $u \in \mathcal{C}([0,t_1);X) \cap \mathcal{C}^1((0,t_1);X)$, and u is the unique solution to the initial value problem (SL).

THEOREM 3. *Suppose hypotheses of Theorem* 2 *hold with* $\beta = \alpha + r\alpha$ *satisfying* $0 < \beta < 1$. *If there exist positive constants* γ, C *and* $p > 1$ *such that* $||F(u)|| \le C||(-A)^\alpha u||^p$ *for* $||(-A)^\alpha u|| \le \gamma$, *then there exists* $\rho > 0$ *such that if* $||(-A)^\alpha u_0|| \le \rho$ *then there is a global solution to* (SL).

PROOF. By the previous theorem, it suffices to show that $||(-A)^\alpha u(t)||$ remains bounded. Suppose u is a solution to (VP). Then

$$\begin{aligned} ||(-A)^\alpha u(t)|| &\le ||T(t)(-A)^\alpha u_0|| + \int_0^t ||(-A)^\alpha T(t-\tau)F(u(\tau))||\,d\tau \\ &\le M||(-A)^\alpha u_0|| + M\int_0^t (t-\tau)^{-\beta} e^{-\delta(t-\tau)} ||(-A)^\alpha u(\tau)||^p d\tau. \end{aligned}$$

Let $K(t) = \max_{0 \le \tau \le t}\{||(-A)^\alpha u(\tau)||\}$. Then

$$K(t) \le M||(-A)^\alpha u_0|| + M\left(\int_0^\infty r^{-\beta} e^{-\delta r}\,dr\right)[K(t)]^p.$$

It follows that if $||(-A)^\alpha u_0||$ is sufficiently small, then there exists $\eta > 0$ such that $||K(t)|| \le \eta$ for $t > 0$.

EXAMPLE. Examples for which the preceding results apply are readily found in the literature. For example, let

$$X = \mathcal{L}^2(0,1), \quad A = \tfrac{-d^2}{dx^2}, \quad D(A) = H^2(0,1) \cap H_0^1(0,1), \quad F(u) = \tfrac{d}{dx}u^2,$$

$a(t) = t^{-\eta}, \gamma > 0$ and $\eta \in (0,1)$. The verification that the required assumptions are satisfied in this example may be found in [**1**] and [**7**]. In this example, $\alpha > 1/2$ and $0 < \alpha + r\alpha < 1$ implies $(2-\eta)\alpha < 1$. Increased regularity of solutions is obtained by an increase in α and, consequently, an increase in η.

IV. Breakdown of smooth solutions. In this section we discuss the breakdown of smooth solutions to the differential equation

$$u_t + f(u)_x = \int_{-\infty}^t a(t-\tau)u_{xx}\,d\tau, \tag{4.1}$$

with initial datum

$$u(x,0) = u_0(x). \tag{4.2}$$

Here, we assume that the kernel $a(t)$ is regular and $a(0) = 1$, and that $f' > 0$ and $f'' > 0$ so that f is monotone and genuinely nonlinear. We also normalize f so that $f(0) = 0$.

In order to discuss the breakdown of smooth solutions, it turns out that it is convenient to rewrite the equation in the following form. Setting

$$v(x,t) = \int_{-\infty}^{t} a(t-\tau)u_x \, d\tau, \tag{4.3}$$

we obtain

$$\begin{aligned} u_t + f(u)_x - v_x &= 0, \\ v_t - u_x &= \int_{-\infty}^{t} a'(t-\tau)u_x \, d\tau. \end{aligned} \tag{4.4}$$

Notice that

$$v(x,0) = \int_{-\infty}^{0} a(-\tau)u_x \, d\tau.$$

Therefore, this gives the initial datum for v and we assume that this past history can be controlled so that we can prescribe arbitrary initial datum for v.

Remark. Actually it is not natural to think this way. It is more natural to think that we have the breakdown of smooth solutions if the past history satisfies the conditions which come from the initial data (4.14).

We now apply the transformation due to MacCamy [**6**]. Let $r(t)$ be the resolvent kernel associated with the solution of linear Volterra equation

$$r(t) + \int_0^t a'(t-\tau)r(\tau) \, d\tau = a'(t), \quad t \geq 0.$$

Then, convoluting $(4.4)_2$ with $r(t)$, we obtain

$$
\begin{aligned}
&u_t + f(u)_x - v_x = 0,\\
&v_t - u_x = \mathcal{F}[v],
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
\mathcal{F}[v] &= -\alpha v + \phi(x,t) + \int_0^t r'(t-\tau) v(x,\tau)\, d\tau,\\
\phi(x,t) &= -r(t) v_0(x) + \int_0^t r(t-\tau) \int_{-\infty}^0 a'(\tau - s) u_x(x,s)\, ds\, d\tau,
\end{aligned}
$$

and $-\alpha = r(0) = a'(0) < 0$. In what follows, for simplicity we assume that $r'(t)$ is bounded and set

$$K_0 = \sup_{0 \le t \le \infty} |r'(t)|.$$

System (4.5) has a structure similar to the viscoelasticity model considered in [**4**]. It turns out that the proof is essentially the same. Therefore, we indicate the modification necessary to prove the a priori estimate and the proof of breakdown.

The characteristics of system (4.5) are

$$\left.\begin{matrix}\lambda\\ \mu\end{matrix}\right) = (f' \pm \sqrt{(f')^2 + 4})/2,$$

and in what follows we assume that

$$\lambda' = \lambda f'' / \sqrt{(f')^2 + 4} > \varepsilon > 0. \tag{4.6}$$

The Riemann invariants for (4.5) can be taken as

$$\left.\begin{matrix}r\\ s\end{matrix}\right) = \int \big(f' \pm \sqrt{(f')^2 + 4}\big)/2\, du - v. \tag{4.7}$$

The transformation given by (4.7) is one-to-one. These Riemann invariants satisfy the diagonal system

$$
\begin{aligned}
&r_t + \lambda r_x = G(r,s) + \phi(x,t),\\
&s_t + \mu s_x = G(r,s) + \phi(x,t),
\end{aligned}
\tag{4.8}
$$

where $G(r, s)$ is given by

$$G(r, s) = -\alpha g(r, s) + \int_0^t r'(t - \tau) g(r, s)\, d\tau$$

and $g(r, s)$ is given by

$$g(r, s) = -(r + s)/2 - h(r - s)/2.$$

In the above relation, $h(r - s)$ is given by $h(r - s) = f(k(r - s))$, where $u = k(r - s)$ is the solution of $r - s = \int_0^u \sqrt{(f')^2 + 4}\, du$. It is not difficult to show that $h(0) = 0$ and $h'(0)$ is finite. If we write

$${}' = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}, \quad {}^{\backprime} = \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x},$$

(4.8) can be written as

$$r' = G(r, s) + \phi, \quad s^{\backprime} = G(r, s) + \phi. \tag{4.9}$$

Modifying Nishida's argument, we have the following a priori estimate of sup norm for the smooth solutions.

Lemma 4.1. *Suppose*

$$|r_0| = \sup_{-\infty<x<\infty} |r_0(x)|, \quad |s_0| = \sup_{-\infty<x<\infty} |s_0(x)|,$$
$$|\Phi(t)| = \sup_{0\le\tau\le t, -\infty<x<\infty} |\phi(x, \tau)|,$$

and that

$$|h(r - s)| \le K|r - s|$$

on the interval $-A \le r - s \le B$, *where* K, A, *and* B *are positive constants. Then, if* $|r_0| + |s_0| < \min(A, B)$, *there is a* T_0 *such that the estimate*

$$|r| + |s| \le \left(|r_0| + |s_0| + 2\int_0^t |\Phi|\, d\tau\right) \exp\{Kt + K_0(\alpha/2 + K)t^2/2\}$$

holds on $0 \le t \le T_0$.

PROOF. We introduce the characteristic curves

$$x_1 = x_1(p,m) = m + \int_0^p \lambda dq, \quad -\infty < m < \infty, \tag{4.10}$$

$$x_2 = x_2(p,n) = n + \int_0^p \mu dq, \quad -\infty < n < \infty. \tag{4.11}$$

Then, along the characteristic curve defined by (4.10), from (4.9) we see

$$\begin{aligned} e^{\alpha t/2} r' + \alpha e^{\alpha t/2} r/2 = &- e^{\alpha t/2}[\{\alpha s + h(r-s)\}/2 + \phi] \\ &+ e^{\alpha t/2} \int_0^t r'(t-\tau) g(x_1(\tau,m),\tau)\, d\tau. \end{aligned}$$

Therefore, after the integration,

$$\begin{aligned} e^{\alpha p/2} r = r_0(m) &- \int_0^p e^{\alpha\tau/2}[\{\alpha s + h(r-s)\}/2 + \phi](x_1(\tau,m),\tau)\, d\tau \\ &+ \int_0^p e^{\alpha t/2} \int_0^t r'(t-\tau) g(x_1(\tau,m),\tau)\, d\tau\, dt. \end{aligned}$$

Since we assume the existence of smooth solutions, using the mean value theorem, we obtain

$$\begin{aligned} e^{\alpha p/2} r(x_1(p,m),p) = r_0(m) & \\ &- \int_0^p e^{\alpha\tau/2}[\{\alpha s + h(r-s)\}/2 + \phi](x_1(\tau,m),\tau) d\tau \\ &+ \int_0^p e^{\alpha\tau/2} \tau r'(\tau - q) g(x_1(\tau,m), q(\tau))\, d\tau, \end{aligned}$$

where $q(\tau)$ satisfies $0 \le q(\tau) \le \tau$. Define

$$\begin{aligned} \overline{r}(x_1(t,m),t) &= \max_{0\le p\le t} e^{\alpha p/2} |r(x_1(p,m),p)|, \\ \overline{s}(x_2(t,n),t) &= \max_{0\le p\le t} e^{\alpha p/2} |s(x_2(p,n),p)|. \end{aligned}$$

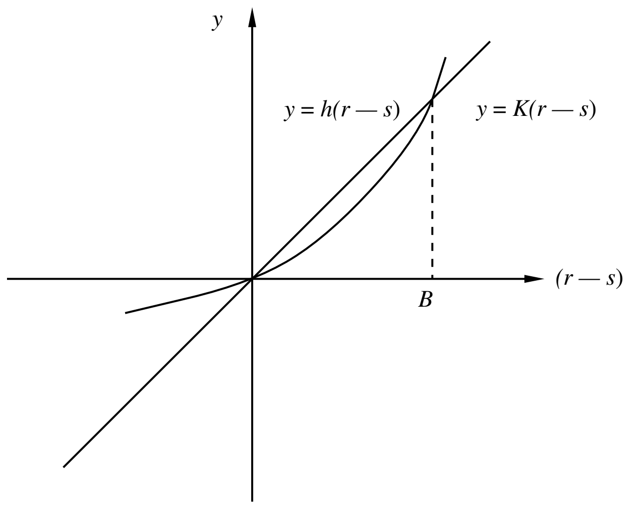

FIGURE 4.1. A typical example of h.

Then,

$$\begin{aligned}\overline{r}(x_1(t,m),t) \le |r_0| &+ \int_0^t e^{\alpha\tau/2}[\{\alpha|s| + |h(r-s)|\}/2 + |\phi|]\,d\tau \\ &+ \int_0^t e^{\alpha\tau/2}\tau|r'(\tau-q)g(x_1(\tau,m),q(\tau))|\,d\tau.\end{aligned}$$

Since $h(0)=0$ and $h'(0)$ is finite, we estimate h by

$$|h(r-s)| \le K|r-s|.$$

A typical example of h is given in Figure 4.1. Since h'' is positive, B is finite and A is infinite. If the initial data is small in the supremum norm, there is a time interval in which this majorization holds. Then in this time interval, we have

$$\begin{aligned}&\overline{r}(x_1(t,m),t) \\ &\quad \le |r_0| + \int_0^t e^{\alpha\tau/2}\{\alpha|s|/2 + |\phi|\}(x_1(\tau,m),\tau)\,d\tau \\ &\quad\quad + 1/2\int_0^t e^{\alpha\tau/2}(K + K_0(\alpha/2 + K\tau)\ [(|r|+|s|)(x_1(\tau,m),\tau]\,d\tau,\end{aligned}$$

Define

$$R(t) = \sup_{-\infty<x<\infty, 0\le p\le t} e^{\alpha p/2}|r(x,p)|,$$
$$S(t) = \sup_{-\infty<x<\infty, 0\le p\le t} e^{\alpha p/2}|s(s,p)|.$$

Then the above inequality implies

$$\begin{aligned}\overline{r}(x_1(t,m),t) \le |r_0| + \int_0^t \{\alpha S(\tau)/2 + |\Phi(\tau)|\}\,d\tau \\ + 1/2\int_0^t (K + K_0(\alpha/2+K)\tau)(R(\tau)+S(\tau))\,d\tau.\end{aligned}$$

We can go through the same argument for s along the characteristic curve defined by (4.11) and obtain

$$\begin{aligned}\overline{s}(x_2(t,n),t) \le |s_0| + \int_0^t \{\alpha R(\tau)/2 + |\Phi(\tau)|\}\,d\tau \\ + 1/2\int_0^t (K + K_0(\alpha/2+K)\tau)(R(\tau)+S(\tau))\,d\tau.\end{aligned}$$

Since we assume that the solution is smooth, the slopes of characteristics are finite. So, for each t, there are characteristic curves so that

$$\overline{r}(x_1(t,\overline{m}),t) = R(t), \quad \overline{s}(x_2(t,\overline{n}),t) = S(t).$$

If we choose these $\overline{m}, \overline{n}$ for each t, we have

$$\begin{aligned}R(t)+S(t) \le |r_0| + |s_0| + 2\int_0^t |\Phi(\tau)|\,d\tau \\ + \int_0^t (\alpha/2 + K + K_0(\alpha/2+K)\tau)(R(\tau)+S(\tau))\,d\tau.\end{aligned}$$

Then, setting

$$\beta(t) = |r_0| + |s_0| + 2\int_0^t |\phi(\tau)|\,d\tau,$$
$$\gamma(t) = (\alpha/2 + K + K_0(\alpha/2+K)t)$$

and using the generalized Gronwall inequality, we obtain

$$R(t)+S(t) \le \beta(t)+\int_0^t \gamma(\tau)\beta(\tau)\Big(\exp\int_\tau^t \gamma(\xi)\,d\xi\Big)\,d\tau$$
$$\le \beta(t)\exp\int_0^t \gamma(\xi)\,d\xi.$$

Therefore,

$$|r(t)|+|s(t)| \le \beta(t)\exp\{Kt+K_0(\alpha/2+K)t^2/2\}. \tag{4.12}$$

As $\beta(0)=|r_0|+|s_0|$, if $\beta(0)<\min(A,B)$, there is a $T_0>0$ such that the majorization and, hence, the a priori estimate is valid. If $\min(A,B)$ exists, we can find an estimate for T_0 by solving

$$\min(A,B)=\beta(T_0)\exp\left\{KT_0+\frac{K_0}{2}\left(\frac{\alpha}{2}+K\right)T_0^2\right\}.$$

Now we state the theorem for the breakdown of smooth solutions.

THEOREM 4.2. *Suppose the condition in* (4.6) *is satisfied. Then, for an appropriate smooth initial datum and the past history, the breakdown of smooth solutions to* (4.1) *will occur in finite time.*

PROOF. The proof is basically the same as in [**4**]. The idea of proof is to show that two characteristic curves of the same family will cross each other and have the different values of the solutions, which indicates that the solution is not smooth. Suppose $x_1(t)$ and $x_2(t)$ are r-characteristic curves with $x_1(0)=x_1^0$ and $x_2(0)=x_2^0$ and that $x_1^0<x_2^0$. Then, this is equivalent to finding a positive $t_1(<T_0)$ such that

$$r_1(t)>r_2(t), \quad 0\le t\le t_1, \tag{4.13}$$

$$x_1(t_1)\ge x_2(t_1). \tag{4.14}$$

To show the above we choose points $(x_1^0,0)$ and $(x_2^0,0)$ on the initial line such that $x_2^0-x_1^0=\delta$, where δ is a small positive constant which will be determined later in this proof. On the initial line we give the smooth and bounded data $r_0(x)$ and $s_0(x)$ which take the values

$$r_0(x_1^0) = \overline{r}_0 + \Delta r/2, \quad r_0(x_2^0) = \overline{r}_0 - \Delta r/2, \tag{4.15}$$

$$s_0(x) = \overline{s}_0,$$

where $\overline{r}_0, \overline{s}_0$, and $\Delta r(> 0)$ are constants.

By making use of the a priori estimate, we define

$$M_0 = \beta(T_0)\exp(KT_0 + K_0(\alpha/2 + K)T_0^2/2),$$

where T_0 is the constant obtained in Lemma 4.1. We also define the values

$$\begin{aligned} M_1 &= \max_{|r|+|s|\le M_0} \alpha|g(r,s)| + |\Phi(T_0)|, \\ M_2 &= \max_{|r|+|s|\le M_0} |\lambda'(r,s)|, \\ M_3 &= \max_{|r|+|s|\le M_0} |\lambda(r,s)|, \\ M_4 &= \max_{|r|+|s|\le M_0} K_0|g(r,s)|. \end{aligned} \tag{4.16}$$

Now we can obtain the following inequalities along r-characteristic curves $x_1(t)$ and $x_2(t)$. Along $x_1(t)$ we have

$$\begin{aligned} r_1(t) = \overline{r}_0 + \Delta r/2 + \int_0^t \{-\alpha g(r,s) + \phi\}\, dt \\ - \int_0^t \int_0^t r'(t-\tau) g(r,s)(x,\tau)\, d\tau\, dt, \end{aligned}$$

$$\overline{r}_0 + \Delta r/2 - M_1 t - M_2 t^2 \le r_1(t) \le \overline{r}_0 + \Delta r/2 + M_1 t + M_2 t^2, \tag{4.17}$$

$$\overline{s}_0 - M_1 t - M_2 t^2 \le s_1(t) \le \overline{s}_0 + M_1 t + M_2 t^2, \tag{4.18}$$

$$x_1(t) = x_1 + \int_0^t \lambda(r_1 - s_1)\, dt. \tag{4.19}$$

And along $x_2(t)$,

$$\overline{r}_0 - \Delta r/2 - M_1 t - M_2 t^2 \le r_2(t) \le \overline{r}_0 - \Delta r/2 + M_1 t + M_2 t^2, \tag{4.20}$$

$$\overline{s}_0 - M_1 t - M_2 t^2 \le s_2(t) \le \overline{s}_0 + M_1 t + M_2 t^2, \tag{4.21}$$

$$x_2(t) = x_2 + \int_0^t \lambda(r_2 - s_2)\,dt. \tag{4.22}$$

From (4.17) and (4.20), if

$$r_1 - r_2 > \Delta r - 2M_1 t - 2M_4 t^2 > 0, \tag{4.23}$$

then (4.12) is satisfied. And, from (4.18) and (4.21), we obtain

$$\begin{aligned} x_2(t) - x_1(t) &= x_2^0 - x_1^0 + \int_0^t \{\lambda(r_2 - s_2) - \lambda(r_1 - s_1)\}\,dt \\ &= x_2^0 - x_1^0 + \int_0^t \lambda'(\xi)(r_2 - r_1 + s_1 - s_2)\,dt, \end{aligned}$$

where ξ is between $(r_2 - s_2)$ and $(r_1 - s_1)$. The condition (4.14) is equivalent to

$$\int_0^t \lambda'(\xi)(r_2 - r_1 + s_1 - s_2)\,dt \le x_1^0 - x_2^0 = -\,\delta.$$

We estimate the above integral. From (4.6), (4.16), and (4.20),

$$\lambda'(\xi)(r_2 - r_1) \le -\,\varepsilon(\Delta r - 2M_1 t - M_4 t^2),$$

and, from (4.16), (4.18), and (4.21),

$$\lambda'(\xi)(s_1 - s_2) \le 2M_2(M_1 t + M_2 t^2).$$

Thus, if

$$\begin{aligned} &\int_0^t \lambda'(\xi)(r_2 - r_1 + s_1 - s_2)\,dt \\ &\quad \ge -\,\varepsilon\Delta r t + M_1(\varepsilon + M_2)t^2 + 2M_4(\varepsilon + M_2)t^3/3 \le x_1^0 - x_2^0 = -\,\delta \end{aligned} \tag{4.24}$$

is satisfied, (4.14) is satisfied. If we choose small enough δ, it is easy to find $t_1 \leq T_0$ which satisfies (4.23) and (4.24). Since the a priori estimate in Lemma 3.1 depends only on the maximum absolute values $|r_0|$ and $|s_0|$ of initial data, we can change δ without changing $|r_0|$ and $|s_0|$.

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