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STABILITY OF QUALOCATION METHODS FOR ELLIPTIC BOUNDARY INTEGRAL EQUATIONS

CLAUS SCHNEIDER

ABSTRACT. I.H. Sloan and W.L. Wendland analyzed thoroughly qualocation with spline trial and test spaces in the paper *Qualocation methods for elliptic boundary integral equations* [**9**]. They derived a criterion for stability and computed numerically weights and knots for some methods which are suitable for equations in which the even symbol part of the operator dominates and other methods for a dominant odd part. Stability of these methods was ensured by numerical computations. We simplify their stability result yielding a representation which allows to prove which J-point quadrature rules are leading to stable qualocation and which are not. Furthermore, the existence of at least one stable method follows for any J.

1. Introduction In previous papers [**2**], [**9**], [**10**], e.g., the qualocation method has been introduced and applied to a large class of boundary integral equations on smooth curves in the plane. Here we study the stability of this method and of tolerant qualocation, cf. [**11**], as well. Hence, we need the same assumptions as Sloan and Wendland, i.e., we suppose that the equations are expressible in the form

$$(b_+L_+ + b_-L_- + K)u = f,$$

where $L := b_+L_+ + b_-L_- + K$ is a classical periodic pseudodifferential operator of order β. We assume that $L : H^\tau \to H^{\tau-\beta}$ defines an isomorphic mapping for any $\tau \in \mathbf{R}$, $H^\tau = H^\tau[0,1]$ being the Sobolev-space of 1-periodic functions. L_+ and L_- are the even and the odd part of L, respectively. *Both* of them may have the order β. K is a sufficiently smoothing perturbation. b_+, b_- are smooth, 1-periodic coefficients. f is a given 1-periodic function in $H^{\tau_0-\beta}$ for some $\tau_0 > \beta + \frac{1}{2}$. Moreover, we assume either that L is uniformly strongly elliptic or that L is uniformly oddly elliptic, i.e., either the even symbol part of the operator dominates or the odd symbol part, see [**10**] again for a formal definition.

We consider spline qualocation on a family of uniform meshes

$$\{x_k := k/N,\ k = 0(1)N-1\}.$$

Hence, the trial space S_N^r is the space of smoothest 1-periodic splines of order $r \geq 1$ on these meshes. $S_N^{r'}$ is the test space with $r' \geq 1$.

The qualocation method may be thought of as an improvement of collocation, with respect to stability and order, which is achieved by a convex combination of collocation at different points. Qualocation can also be understood as a semi-discrete version of the Galerkin-Petrov method with the outer integration replaced by a composite quadrature rule. Surprisingly, as the analysis in [**2**] already showed, this rule has not to perform the outer integration very well (that would yield a good approximation of the Galerkin-Petrov solution); different qualities are needed to achieve excellent approximation properties. They will be summarized in the next chapter.

Anyway, we need weights $w_1, \ldots, w_J$ and knots $0 \leq \xi_1 < \cdots < \xi_J < 1$ for the basic quadrature rule or for the convex combination and the different collocation points, respectively.

Then the qualocation method is *Find* $u_N \in S_N^r$ *such that for all* $v \in S_N^{r'}$

$$\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=1}^{J} w_j((Lu_N)\bar{v})(x_k+\xi_j/N) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=1}^{J} w_j(f\bar{v})(x_k+\xi_j/N).$$

If $r' = 1$, then functions exhibiting jumps may be evaluated at breakpoints. Therefore we define in such cases $v(x_k) := \lim_{\xi\downarrow 0} v(x_k+\xi)$. We also assume that

$$r-\beta > \frac{1}{2} \quad \text{if} \quad \xi_1 > 0, \quad \text{and} \quad r-\beta > 1 \quad \text{if} \quad \xi_1 = 0,$$

in order to have a well-defined method for the equation $Lu = f$.

If the weights and knots are chosen carefully, then an additional (compared with collocation, i.e., $J = 1, b = 0$) order $b > 0$ is possible with $J > 1$, if stability can be ensured. Indeed, it has been shown in [**9**] that for $\tau > \beta + \frac{1}{2}$, $r' \geq b$, and any $s < r - \frac{1}{2}$, $\beta - b \leq s \leq \tau \leq r$

$$||u_N - u||_s \leq c\, N^{s-\tau}||u||_{\tau+\max(\beta-s,0)}$$

can be achieved if $u \in H^{\tau+\max(\beta-s,0)}$, in particular, with $r = \tau$, $r' = b$, and $s = \beta - r'$

$$||u_N - u||_{\beta-r'} \leq c\, N^{\beta-r'-r}||u||_{r+r'}.$$

Similar results are valid in the case of nonconstant coefficients, see [**10**]. The tolerant qualocation does not even need the additional smoothness of u because there these estimates remain true with $||u||_\tau$ on the righthand side.

Anyway, the crucial assumption for these estimates to hold is stability of the qualocation method. Up to now, stability was controlled by examining numerically a stability criterion (see Chapter 3) for a given quadrature rule. The results of such tests are given in [**9**] for some J-point rules ($J \le 4$) and for different parameters r, r', β. Here we will prove which J-point rules ($J \in \mathbf{N}$) ensure stability. And there will be one for any case.

2. The quadrature rules $l_{J,b,r-\beta}$ and $g_{J,b,r-\beta}$. For an additional order of convergence $b > 0$ the quadrature rule

$$Q_J(f) := \sum_{j=1}^{J} w_j \, f(\xi_j)$$

has to satisfy

(i) $w_j > 0$, $j = 1(1)J$, $\sum_{j=1}^{J} w_j = 1$, i.e., $Q_J(1) = \int_0^1 1 \, dt$, and

(ii) $\xi_1 = 0$ and the remaining knots $0 < \xi_2 < \cdots < \xi_J < 1$ are symmetrical with respect to $\frac{1}{2}$ (Lobatto-type) or the knots $0 < \xi_1 < \cdots < \xi_J < 1$ are symmetrical with respect to $\frac{1}{2}$ (Gauss-type) and

(iii) symmetrical knots have the same weight, and

(iv) $Q_J(G_\alpha) = \int_0^1 G_\alpha(t) \, dt = 0$ for $\alpha = r - \beta(1) r - \beta + b - 1$ (constant coefficient case) or for $\alpha = 2(2)2\left[\frac{r-\beta-1}{2}\right], r-\beta(1)r-\beta+b-1$, respectively, if variable coefficients appear and $\beta \in \mathbf{Z}$. Here

$$G_\alpha(t) := 2 \sum_{j=1}^{\infty} \cos(2\pi j t)/j^\alpha.$$

The highest value of b we can expect to achieve with the given number of free parameters is $b = J-1$ or $b = J-1-\left[\frac{r-\beta-1}{2}\right]$, respectively. Such rules are called $l_{J,b,r-\beta}$ or $L_{J,b,r-\beta}$ if $\xi_1 = 0$, and $g_{J,b,r-\beta}$ or $G_{J,b,r-\beta}$ otherwise (see [**9**]). In the sequel, we will not distinguish between l, g- and L, G-rules because they do not differ with respect to existence and stability.

The unique existence of these rules has been shown in [**4**] and with full details in [**3**]. The main result, which we will need later, tells us that $\{1, -G_{\alpha_1}, \cdots, -G_{\alpha_{J-1}}\}$ is a Chebyshev-system over $[0, \frac{1}{2}]$ with positive determinant for arbitrary $1 < \alpha_1 < \alpha_2 < \cdots < \alpha_{J-1}$. (If $0 < \alpha_1 \le 1$ then some technical modifications have to be performed with an appropriate weight-function, because now $G_{\alpha_1}(t)$ behaves like $t^{\alpha_1 - 1}$ if $\alpha_1 < 1$ or like $\log t$ if $\alpha_1 = 1$ for $t \to 0$).

Therefore, optimal order rules exist integrating exactly the appropriate G_α over $[0, \frac{1}{2}]$ (but $\int_0^{1/2} G_\alpha(t)\,dt = 0$ again). Their symmetric extension to $[0,1)$ then yields the required unique l- and g-rules because $G_\alpha(t) = G_\alpha(1-t)$.

In order to decide which of these rules are leading to a stable qualocation method, we introduce shortly the stability result from [**9**] and prove a simplified version.

3. The stability function. At first we have to repeat some definitions from [**9**] and earlier papers. Let $\alpha > \frac{1}{2}$ and $x \in [0,1]$.

$$G_\alpha^\pm(x;y) := \sum_{k=1}^\infty \Big[\frac{1}{(k+y)^\alpha} \pm \frac{1}{(k-y)^\alpha}\Big] \cos(2\pi kx), \quad y \in \Big[-\frac{1}{2}, \frac{1}{2}\Big],$$
$$H_\alpha^\pm(x;y) := \sum_{k=1}^\infty \Big[\frac{1}{(k+y)^\alpha} \mp \frac{1}{(k-y)^\alpha}\Big] \sin(2\pi kx), \quad y \in \Big[-\frac{1}{2}, \frac{1}{2}\Big].$$

For smoothness properties, integral representations, or different expansions of these functions, cf. also [**1**], [**2**], [**5**]. The quadrature rule and these functions are then combined in functions characterizing stability of the qualocation method:

$$\begin{aligned} D_\pm(y) := &\sum_{j=1}^J w_j \big(1 + y^{r-\beta} G_{r-\beta}^\sigma(\xi_j;y)\big)\big(1 + y^{r'} G_{r'}^{\sigma'}(\xi_j;y)\big) \\ &+ y^{r-\beta+r'} \sum_{j=1}^J w_j H_{r'}^{\sigma'}(\xi_j;y) H_{r-\beta}^\sigma(\xi_j;y), \quad y \in \Big[0, \frac{1}{2}\Big], \end{aligned}$$

where

$$\sigma = \left\{ \begin{array}{ll} + & \text{if } r \text{ is even} \\ - & \text{if } r \text{ is odd} \end{array} \right\} \quad \text{for the } D_+ \text{ case,}$$

$$\sigma = \left\{ \begin{array}{ll} - & \text{if } r \text{ is even} \\ + & \text{if } r \text{ is odd} \end{array} \right\} \quad \text{for the } D_- \text{ case,}$$

and

$$\sigma' = \left\{ \begin{array}{ll} + & \text{if } r' \text{ is even} \\ - & \text{if } r' \text{ is odd} \end{array} \right\}.$$

Then the Sloan and Wendland-stability result, [**9**, Theorem 3] for constant coefficients reads as follows

Theorem. *Consider the qualocation method with a symmetric quadrature rule having positive weights.*

1. *Assume that r and r' are of the same parity, and if $J = 1$ also that $\xi_1 \neq \frac{1}{2}$ if r and r' are even, and $\xi_1 \neq 0$ if r and r' are odd. The method is stable for all strongly elliptic operators L if and only if*

$$D_+(y) \geq |D_-(y)| \quad \text{for all } y \in \left[0, \frac{1}{2}\right].$$

2. *Assume that r and r' are of opposite parity, and if $J = 1$ also that $\xi_1 \neq 0$ if r is even and r' is odd, and that $\xi_1 \neq \frac{1}{2}$ if r is odd and r' is even. The method is stable for all oddly elliptic operators L, if and only if*

$$D_-(y) \geq |D_+(y)| \quad \text{for all } y \in \left[0, \frac{1}{2}\right].$$

With the next theorem we will prove that the absolute values can be omitted in the previous theorem, the first step in our analysis yielding a simpler criterion for stability. According to the stability result, we have to show that $D_\pm \geq 0$ if $\sigma \cdot \sigma' = -$. Therefore, we will study for those cases an arbitrary summand appearing in the definition of $D_\pm$.

Theorem. *Let $a > \frac{1}{2}$, $b > \frac{1}{2}$, $0 \leq x < 1$ and $x > 0$ if $a \leq 1$ or $b \leq 1$. Define for $y \in [0, \frac{1}{2}]$*

$$\begin{aligned} \tilde{D}(y; x, a, b) := 1 &+ y^a G_a^-(x; y) + y^b G_b^+(x; y) \\ &+ y^{a+b}\big(G_a^-(x; y) G_b^+(x; y) + H_a^-(x; y) H_b^+(x; y)\big). \end{aligned} \tag{3.1}$$

Then

1) $\tilde{D}(0;x,a,b)=1$,
2) $\tilde{D}\left(\frac{1}{2};x,a,b\right)=0$,
3) $\tilde{D}(y;x,a,b)\geq 0$ *for all* $y\in[0,\frac{1}{2}]$.

Proof. Property 1) is obvious. The second one follows after a short calculation proving that

$$\begin{aligned}
\left(\frac{1}{2}\right)^{a}H_a^{+}\left(x;\frac{1}{2}\right)&=-2\sin(\pi x)\sum_{l=0}^{\infty}\cos((2l+1)\pi x)/(2l+1)^{a},\\
\left(\frac{1}{2}\right)^{a}H_a^{-}\left(x;\frac{1}{2}\right)&=2\cos(\pi x)\sum_{l=0}^{\infty}\sin((2l+1)\pi x)/(2l+1)^{a},\\
1+\left(\frac{1}{2}\right)^{a}G_a^{+}\left(x;\frac{1}{2}\right)&=2\cos(\pi x)\sum_{l=0}^{\infty}\cos((2l+1)\pi x)/(2l+1)^{a},\\
1+\left(\frac{1}{2}\right)^{a}G_a^{-}\left(x;\frac{1}{2}\right)&=2\sin(\pi x)\sum_{l=0}^{\infty}\sin((2l+1)\pi x)/(2l+1)^{a}.
\end{aligned}$$

In order to show property 3), we compute the derivative of $\tilde{D}$ with respect to y by term-wise differentiation

$$\begin{aligned}
\frac{\partial}{\partial y}\tilde{D}(y;x,a,b)=ay^{a-1}G_a^{-}(x;y)&+by^{b-1}G_b^{+}(x;y)\\
&+(a+b)y^{a+b-1}\big(G_a^{-}(x;y)G_b^{+}(x;y)+H_a^{-}(x;y)H_b^{+}(x;y)\big)\\
&-a\big(y^{a}G_{a+1}^{+}(x;y)+y^{a+b}G_{a+1}^{+}(x;y)G_b^{+}(x;y)\\
&\qquad+y^{a+b}H_{a+1}^{+}(x;y)H_b^{+}(x;y)\big)\\
&-b\big(y^{b}G_{b+1}^{-}(x;y)+y^{a+b}G_a^{-}(x;y)G_{b+1}^{-}(x;y)\\
&\qquad+y^{a+b}H_a^{-}(x;y)H_{b+1}^{-}(x;y)\big).
\end{aligned}$$

After adding and subtracting $ay^{b-1}G_b^{+}(x;y)$, $by^{a-1}G_a^{-}(x;y)$, and 1 as well, we have

$$\begin{aligned}
\frac{\partial}{\partial y}\tilde{D}(y;x,a,b)=\frac{1}{y}\big(&(a+b)\tilde{D}(y;x,a,b)\\
&-a\tilde{D}^{+}(y;x,a+1,b)-b\tilde{D}^{-}(y;x,a,b+1)\big),
\end{aligned}\tag{3.2}$$

where for $\tau \in \{+,-\}$

$$\begin{aligned}\tilde{D}^\tau(y;x,a,b) := 1 &+ y^a G_a^\tau(x;y) + y^b G_b^\tau(x;y)\\ &+ y^{a+b}\big(G_a^\tau(x;y)G_b^\tau(x;y) + H_a^\tau(x;y)H_b^\tau(x;y)\big).\end{aligned}$$

We know from [**1**] that $\tilde{D}^\tau(y;x,a,b) \geq 0$ with equality if and only if $\tau = +$ and $(y,x) = (\frac{1}{2},\frac{1}{2})$ or $\tau = -$ and $(y,x) = (\frac{1}{2},0)$. Therefore, formula (3.2) shows that $\tilde{D}$ remains negative if it becomes negative at some $0 < y < \frac{1}{2}$. Hence, the first two parts of the theorem complete the proof that $\tilde{D}$ cannot change sign on $[0,\frac{1}{2}]$. □

Remark. Property 2) of $\tilde{D}$ is the reason why qualocation is not stable for strongly (oddly) elliptic operators if r and r' have opposite (the same) parity.

The theorem shows that the Sloan and Wendland-stability results still hold if $|D_\tau|$ is replaced by D_τ because the function D_τ is just a positively weighted sum of $\tilde{D}-$functions. Hence, the stability analysis will become (slightly) easier. We only have to study the *stability function* S depending on $y, r', r-\beta$, and the quadrature rule Q_J:

r' even:

$$S(y;r') := \sum_{j=1}^{J} w_j\big(\tilde{D}^+(y;\xi_j,r-\beta,r') - \tilde{D}(y;\xi_j,r-\beta,r')\big), \tag{3.3}$$

where r has to be even in the strongly elliptic case and odd in the oddly elliptic case, and

r' odd:

$$S(y;r') := \sum_{j=1}^{J} w_j\big(\tilde{D}^-(y;\xi_j,r-\beta,r') - \tilde{D}(y;\xi_j,r',r-\beta)\big), \tag{3.4}$$

where r has to be odd in the strongly elliptic case and even in the oddly elliptic case.

Inserting the definition of the $\tilde{D}-$, $\tilde{D}^- -$, and $\tilde{D}^+-$functions finally unifies both formulas:

$$\sigma' := \left\{\begin{array}{ll} + & \text{if } r' \text{ is even}\\ \text{-} & \text{if } r' \text{ is odd}\end{array}\right\} \Longrightarrow$$

$$
\begin{aligned}
S(y;r') = y^{r-\beta} \sum_{j=1}^{J} w_j \big(G_{r-\beta}^{\sigma'}(\xi_j;y) - G_{r-\beta}^{-\sigma'}(\xi_j;y)\big)\big(1 + y^{r'} G_{r'}^{\sigma'}(\xi_j;y)\big) \\
+ y^{r-\beta+r'} \sum_{j=1}^{J} w_j H_{r'}^{\sigma'}(\xi_j;y)\big(H_{r-\beta}^{\sigma'}(\xi_j;y) - H_{r-\beta}^{-\sigma'}(\xi_j;y)\big).
\end{aligned}
$$

The *stability conditions of Sloan and Wendland* are then equivalent to the nonnegativity of the appropriate stability function for all $y \in [0, \frac{1}{2}]$. Property 1) in the last theorem, which also holds for $\tilde{D}^\tau$, and $w_j > 0$ indicates that stability may easily be destroyed in a neighborhood of 0.

4. The stability function at $y = 0$. Fortunately, the behavior at $y = 0$ can be derived from [**1**] or [**9**, p. 464], respectively. With these results we get the expansion

(4.5)

$$
\begin{aligned}
&S(y;r') \\
&= (-1)^{r'} \binom{\beta-r}{b} (-1)^b \left(\sum_{j=1}^{J} w_j G_{r-\beta+b}(\xi_j) \right) y^{r-\beta+b} + \mathcal{O}(y^{r-\beta+b+1}) \\
&\quad + \sum_{j=1}^{J} w_j \left\{ \begin{matrix} G_{r'}(\xi_j)\, G_{r-\beta}(\xi_j), & r' \text{ even} \\ H_{r'}(\xi_j)\, H_{r-\beta}(\xi_j), & r' \text{ odd} \end{matrix} \right\} y^{r-\beta+r'} + \mathcal{O}(y^{r-\beta+r'+1}),
\end{aligned}
$$

if $r' \geq b$, where b is the additional order achieved by the method. This result immediately implies the following necessary condition for stability because $\binom{\beta-r}{b}(-1)^b = \binom{r-\beta+b-1}{b} > 0$.

Remark. Whenever we say in this paper that a method in *not* stable then it always means that the method does not satisfy the stability conditions of Sloan and Wendland.

Theorem. *If $r' > b$ then a method is not stable, if*

$$
(-1)^{r'} \operatorname{sgn} \left(\sum_{j=1}^{J} w_j G_{r-\beta+b}(\xi_j) \right) < 0. \tag{4.6}
$$

Remark . If we set $J = 1$ and $b = 0$, then midpoint collocation ($\xi_1 = \frac{1}{2}, r-\beta > \frac{1}{2}$) and collocation in the meshpoints ($\xi_1 = 0$, $r-\beta > 1$) are covered by this theorem and the stability results to come as well.

A first application of the theorem yields instability results for the simplest qualocation rules with $J = 2$, $b = 1$. They were introduced in [**7**], [**2**] and [**8**]. Indeed, stability of the rules was shown under the assumption that either L is purely odd or purely even. In these cases symmetry even yields $b = 2$. But in our general setting (L is the sum of an even and an odd part) stability for all such L does not hold.

Lemma 1. 1) *The $l_{2,1,r-\beta}$–rule is not stable for even r' ($r-\beta > 1$).*
2) *The $g_{2,1,r-\beta}$-rule is not stable for odd $r' > 1$ ($r - \beta > \frac{1}{2}$).*

Proof. Let $\alpha := r - \beta$.

1) $J = 2$, $\xi_1 = 0$, $\xi_2 = \frac{1}{2}$, $b = 1$, $w_1 = (2^{\alpha-1} - 1)/(2^\alpha - 1)$, $w_2 = 2^{\alpha-1}/(2^\alpha - 1)$, and $G_{\alpha+1}(0) = 2\zeta(\alpha+1)$, $G_{\alpha+1}(\frac{1}{2}) = 2\zeta(\alpha+1)(2^{-\alpha} - 1)$ imply

$$(-1)^{r'}\mathrm{sgn}\left(\sum_{j=1}^{2} w_j G_{\alpha+1}(\xi_j)\right)$$
$$= (-1)^{r'}\mathrm{sgn}\left(\zeta(\alpha+1)\Big(2^{\alpha-1} - 1 + \frac{1}{2} - 2^{\alpha-1}\Big)/\Big(2^\alpha - 1\Big)\right) = -(-1)^{r'}.$$

2) $J = 2$, $w_1 = w_2 = \frac{1}{2}$, $b = 1$, $\xi_1 = 1 - \xi_2 =$ unique zero of G_α in $(0, \frac{1}{2})$. Therefore, $G_{\alpha+1}(\xi_1) = G_{\alpha+1}(\xi_2) > 0$, cf. [**1**] and

$$(-1)^{r'}\mathrm{sgn}\left(\sum_{j=1}^{2} w_j G_{\alpha+1}(\xi_j)\right) = (-1)^{r'}. \qquad \square$$

The criterion (4.6) is easy to check for a given rule but obviously more difficult to apply to general $l_{J,b,r-\beta}$-rules, e.g., because then the weights and knots are not given explicitly. Nevertheless, a generalization of Lemma 1 can be derived.

Theorem. *Let $r' > b$.*

1) *The $l_{J,b,r-\beta}$-rule is not stable for even $J+r'$, $r-\beta>1$.*

2) *The $g_{J,b,r-\beta}$-rule is not stable for odd $J+r'$, $r-\beta>\frac{1}{2}$.*

Proof. We have to study the term $\sum_{j=1}^{J} w_j G_{r-\beta+b}(\xi_j)$, which is simply $Q_J(G_{r-\beta+b})$, $Q_J \in \{l_{J,b,r-\beta}, g_{J,b,r-\beta}\}$. But the integral over $G_{r-\beta+b}$ vanishes. Therefore, $-Q_J(G_{r-\beta+b})$ is just the error of the quadrature rule. And that rule has been constructed from a quadrature rule $Q_{\tilde{J}}$ for a Chebyshev-system over $[0,\frac{1}{2}]$, namely the system $\{1,-G_{\alpha_1},\dots,-G_{\alpha_{J-1}}\}$, where $\alpha_1<\cdots<\alpha_{J-1}<r-\beta+b$ are the indices appearing in the definition of Q_J in Section 2, cf. [**3**]. Indeed, Q_J has been derived from $Q_{\tilde{J}}$ by a symmetric extension. Going backwards now, $Q_{\tilde{J}}(f):=\sum_{j=1}^{\tilde{J}} \tilde{w}_j f(\xi_j)$ is reconstructed by

(i) $\tilde{J}$ is the largest index such that $\xi_1<\cdots<\xi_{\tilde{J}}\le\frac{1}{2}$ and

(ii) $\tilde{w}_j := w_j$ if $\xi_j\in(0,\frac{1}{2})$ and $\tilde{w}_j := \frac{1}{2}w_j$ otherwise, $j=1(1)\tilde{J}$. Then obviously $Q_J(f)=2Q_{\tilde{J}}(f)$ for all functions satisfying $f(t)=f(1-t)$, $t\in[0,1]$. On the other hand, $\{1,-G_{\alpha_1},\dots,-G_{\alpha_{J-1}},-G_{r-\beta+b}\}$ is also a Chebyshev-system over $[0,\frac{1}{2}]$. Hence, there exists a best one-sided L_1-approximation of $-G_{r-\beta+b}$ in $\langle 1,-G_{\alpha_1},\dots,-G_{\alpha_{J-1}}\rangle$, say $\Pi_J(-G_{r-\beta+b})$, interpolating $-G_{r-\beta+b}$ at the knots $\xi_i\in[0,\frac{1}{2}]$ and satisfying either $-G_{r-\beta+b}(t)-\Pi_J(-G_{r-\beta+b})(t)\le 0$, if $\frac{1}{2}$ is a knot, or $-G_{r-\beta+b}(t)-\Pi_J(-G_{r-\beta+b})(t)\ge 0$, if $\frac{1}{2}$ is not a knot, $t\in[0,\frac{1}{2}]$ (see [**6**], proof of Theorem 5.9, for the second case and [**3**] for the first one). Furthermore, $\Pi_J(-G_{r-\beta+b})$ is integrated exactly by $Q_{\tilde{J}}$, $2\int_0^{\frac{1}{2}}\Pi_J(-G_{r-\beta+b})(t)\,dt = 2Q_{\tilde{J}}(\Pi_J(-G_{r-\beta+b})) = \sum_{j=1}^{J} w_j(-G_{r-\beta+b}(\xi_j))$, and $0=\int_0^1 G_{r-\beta+b}(t)\,dt = 2\int_0^{\frac{1}{2}} G_{r-\beta+b}(t)\,dt$ for symmetry reasons.

Hence, $2\int_0^{\frac{1}{2}}(-G_{r-\beta+b}(t)-\Pi_J(-G_{r-\beta+b})(t))\,dt=\sum_{j=1}^{J} w_j G_{r-\beta+b}(\xi_j)$. Therefore $\sum_{j=1}^{J} w_j G_{r-\beta+b}(\xi_j)$ is negative if $\frac{1}{2}$ is a knot and positive otherwise. Taking into account that $\frac{1}{2}$ is a knot of an l-rule for an even J and a knot of a g-rule for an odd J completes the proof. □

Remark. If $-G_{r-\beta+b}$ would be replaced by $-G_\tau$, $\tau>\alpha_{J-1}$, then all arguments of the proof still apply showing that indeed

$$\operatorname{sgn}(Q_J(G_\tau)) = \operatorname{sgn}(Q_J(G_{r-\beta+b})) \quad \text{for all } \tau>\alpha_{J-1}. \tag{4.7}$$

With these results we are now in a position to prove the stability theorem.

5. Stable qualocation methods.

Here we assume constant coefficients in the operator L. The general case will be discussed in the next chapter.

Theorem. *Let $r' \geq b$ and r, r' of the same parity in the strongly elliptic case and of opposite parity in the oddly elliptic case.*

1) *Qualocation with the $l_{J,b,r-\beta}$-rule is stable for odd $J+r'$ $(r-\beta > 1)$.*

2) *Qualocation with the $g_{J,b,r-\beta}$-rule is stable for even $J+r'$ $(r-\beta > \frac{1}{2})$.*

Proof. The functions f to be discussed here are satisfying $f(t) = f(1-t)$. Therefore we will use the $Q_{\tilde{J}}$ from the previous proof instead of Q_J.

If r' is odd, then we have to consider the stability function (3.4):

$$2y^{r-\beta}\sum_{j=1}^{\tilde{J}} \tilde{w}_j \big(G^-_{r-\beta}(\xi_j;y) - G^+_{r-\beta}(\xi_j;y)\big)\big(1 + y^{r'}G^-_{r'}(\xi_j;y)\big)$$
$$+2y^{r-\beta+r'}\sum_{j=1}^{\tilde{J}} \tilde{w}_j H^-_{r'}(\xi_j;y)\big(H^-_{r-\beta}(\xi_j;y) - H^+_{r-\beta}(\xi_j;y)\big).$$

As $H^-_{r'}(x;y)\big(H^-_{r-\beta}(x;y) - H^+_{r-\beta}(x;y)\big) \geq 0$ for all $x \in [0,1]$, see [**1**], it remains to study

$$Q_{\tilde{J}}\big(\big(G^-_{r-\beta}(\cdot;y) - G^+_{r-\beta}(\cdot;y)\big)\big(1 + y^{r'}G^-_{r'}(\cdot;y)\big)\big).$$

According to [**1**], see also [**9**, p. 464], this expression is equal to

$$\sum_{k=0}^{\infty}(-1)^k \binom{\beta - r}{k} y^k\big(-Q_{\tilde{J}}\big(G_{r-\beta+k}\big(1 + y^{r'}G^-_{r'}(\cdot;y)\big)\big)\big).$$

From [**1**] we also know that $\big(1 + y^{r'}G^-_{r'}(\cdot;y)\big)$ is nonnegative and monotonically increasing in $[0,\frac{1}{2}]$ whereas G_τ is monotonically decreasing for all $\tau > 0$. All this implies that

$$-Q_{\tilde{J}}\big(G_{r-\beta+k}\big(1 + y^{r'}G^-_{r'}(\cdot;y)\big)\big) > -Q_{\tilde{J}}\big(G_{r-\beta+k}\big)\big(1 + y^{r'}G^-_{r'}(\lambda_k;y)\big),$$

where λ_k is the smallest knot of $Q_{\tilde{J}}$ such that $G_{r-\beta+k}(\lambda_k) \leq 0$. Such a knot λ_k exists, because for the rules we are just discussing $Q_{\tilde{J}}(G_{r-\beta+k}) = 0$ holds for $k = 0(1)b-1$ and $Q_{\tilde{J}}(G_{r-\beta+k}) < 0$ holds for $k \geq b$, see (4.7) and the Proof before that remark. Stability then follows with the positivity of $(-1)^k \binom{\beta-r}{k}$.

If r' is even, then the stability function is given by (3.3) and has the following form:

$$2y^{r-\beta} \sum_{j=1}^{\tilde{J}} \tilde{w}_j \big(G^+_{r-\beta}(\xi_j;y) - G^-_{r-\beta}(\xi_j;y)\big)\big(1 + y^{r'} G^+_{r'}(\xi_j;y)\big)$$
$$+2y^{r-\beta+r'} \sum_{j=1}^{\tilde{J}} \tilde{w}_j H^+_{r'}(\xi_j;y)\big(H^+_{r-\beta}(\xi_j;y) - H^-_{r-\beta}(\xi_j;y)\big).$$

Again $H^+_{r'}(x;y)\big(H^+_{r-\beta}(x;y) - H^-_{r-\beta}(x;y)\big) \geq 0$ holds for all $x \in [0,1]$. But now the nonnegative $\big(1+y^{r'}G^+_{r'}(\cdot;y)\big)$ is monotonically decreasing. On the other hand, meanwhile we have $\operatorname{sgn}(Q_{\tilde{J}}(G_\tau)) = +1$ for $\tau \geq r-\beta+b$, and

$$G^+_{r-\beta}(\xi_j;y) - G^-_{r-\beta}(\xi_j;y) = +\sum_{k=0}^{\infty} (-1)^k \binom{\beta-r}{k} y^k G_{r-\beta+k}(\xi_j).$$

Therefore, stability follows again with the monotonicity of $G_{r-\beta+k}$ on $[0,\frac{1}{2}]$:

$$\begin{aligned} Q_{\tilde{J}}\big(G_{r-\beta+k}\big(1+y^{r'}G^+_{r'}(\cdot;y)\big)\big) &> Q_{\tilde{J}}\big(G_{r-\beta+k}\big)\big(1+y^{r'}G^+_{r'}(\mu_k;y)\big) \\ &\geq 0, \end{aligned}$$

where μ_k is the greatest knot of $Q_{\tilde{J}}$ such that $G_{r-\beta+k}(\mu_k) \geq 0$. □

6. Conclusions. Hence, for all orders r' of test functions there exist stable J-point qualocation rules for strongly, and for oddly, elliptic operators. If the operators have constant coefficients, then an additional order $b = J-1$ is ensured. For nonconstant coefficients (cf. [**10**]) the qualocation method also needs $r > \beta+1$, $r' \geq 2$, and some polynomial exactness of the quadrature rule. Then, for $\beta \in \mathbf{Z}$, an additional order $b = J - 1 - \left[\frac{r-\beta-1}{2}\right]$ is achieved by the methods which are stable according to the last theorem. Therefore, stable rules with more than only two points are really useful. Furthermore, all stability results apply to the tolerant qualocation as has been shown in [**11**].

There remains one case not covered by the previous theorems, namely, $r' = b$ for those rules which are not stable according to (4.6) if $r' > b$. If $r' = b$, however, then one has to study (4.5) instead of (4.6) for $y = 0$. The first sum in (4.5) is negative in that case, but the second one is positive: $H_{r'} H_{r-\beta}$ is positive in $(0,1) \setminus \{\frac{1}{2}\}$ and $Q_J(G_{r'} G_{r-\beta}) = Q_J(G_{r-\beta}(G_{r'} - G_{r'}(\eta_{r-\beta})))$ with the unique root $\eta_{r-\beta}$ of $G_{r-\beta}$ in $[0, \frac{1}{2}]$ because those rules integrate exactly $G_{r-\beta}$. Furthermore, $G_{r-\beta}(G_{r'} - G_{r'}(\eta_{r-\beta}))$ is positive in $[0,1] \setminus \{\eta_{r-\beta}, 1 - \eta_{r-\beta}\}$ for monotonicity and symmetry reasons. Thus, the positive term may dominate. This happens indeed in the case $r' = b = 1$ for the $g_{2,1,r-\beta}$-rule, as we can show because the knots and weights are known. Unfortunately, that does not yet mean stability. Anyway, all computations lead to the:

Conjecture. *Methods which are not stable if $r' > b$ are stable if $r' = b$—and that was already part of a conjecture given in* [**9**]. *This means that an l- and a g-rule as well are stable if $r' = b$. Numerical experiments support the conjecture. They also indicate that the rule being stable for $r' > b$ yields smaller errors than the other one.*

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Johannes Gutenberg–Universität, Fachbereich Mathematik und Informatik, 55099 Mainz, Germany
E-mail address: `schneider@mathematik.uni-mainz.de`