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WEAKLY FACTORIAL PROPERTY OF A GENERALIZED REES RING $D[X, d/X]$

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ABSTRACT. Let D be an integral domain, X an indeterminate over D, $d \in D$, and $R = D[X, d/X]$ a subring of $D[X, 1/X]$. In this paper, we show that R is a weakly factorial domain if and only if D is a weakly factorial GCD-domain and $d = 0$, d is a unit of D or d is a prime element of D. We also show that, if D is a weakly factorial GCD-domain, p is a prime element of D, and $n \geq 2$ is an integer, then $D[X, p^n/X]$ is an almost weakly factorial domain with $Cl(D[X, p^n/X]) = \mathbb{Z}_n$.

1. Introduction. Let D be an integral domain, I a proper ideal of D and t an indeterminate over D. Then, $R = D[tI, t^{-1}]$ is a subring of $D[t, t^{-1}]$, called the *generalized Rees ring of D with respect to I*. In **[19]**, Whitman proved that, if I is finitely generated, then R is a unique factorization domain (UFD) if and only if D is a UFD and t^{-1} is a prime element of R. Also, in **[17,** Proposition 3], Mott showed that R is a GCD-domain if and only if D is a GCD-domain and t^{-1} is a prime element of R. In **[15,** Corollary 3.10], the authors proved that R is a Prüfer v-multiplication domain (PvMD) if and only if D is a PvMD, under the assumption that t^{-1} is a prime element of R and $\bigcap_{n=1}^{\infty} I^n = (0)$. Let $I = dD$ for some $d \in D$ and $t^{-1} = X$; thus, $R = D[X, d/X]$. In **[1]**, the authors studied several types of divisibility properties of R, including Krull domains, UFDs and GCD-domains.

An element $a \in D$ is said to be *primary* if the principal ideal aD of D is a primary ideal, and we say that D is a *weakly factorial domain* (WFD) if each nonzero nonunit of D can be written as a finite product of primary elements of D. Clearly, a prime element of D is primary,

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and D is a UFD if and only if every nonzero nonunit of D can be written as a finite product of prime elements of D. Hence, a UFD is a WFD, while a rank-one nondiscrete valuation domain is a WFD but not a UFD. It is known that $D[X]$ is a WFD if and only if D is a weakly factorial GCD-domain. More generally, if Γ is a torsionless, commutative, cancellative monoid whose quotient group satisfies the ascending chain condition on its cyclic subgroups, then the semigroup ring $D[\Gamma]$ is a WFD if and only if D is a weakly factorial GCD domain and Γ is a weakly factorial GCD-semigroup [**10,** Theorem 9]. In this paper, we study when $R = D[X, d/X]$ is a WFD.

1.1. Results. Let X be an indeterminate over D, $d \in D$, and $R = D[X, d/X]$. In this paper, among other things, we show that R is a WFD if and only if R is a weakly factorial GCD-domain. The latter condition holds if and only if D is a weakly factorial GCD-domain and $d = 0$, d is a unit of D or d is a prime element of D. We also prove that R is a ring of Krull type if and only if D is a ring of Krull type. We finally prove that, if D is a weakly factorial GCD-domain, p is a prime element of D, and $n \geq 2$ is an integer, then $D[X, p^n/X]$ is an almost weakly factorial domain with $Cl(D[X, p^n/X]) = \mathbb{Z}_n$.

1.2. Definitions. Let K be the quotient field of D and $F(D)$ the set of nonzero fractional ideals of D. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \bigcup\{J_v \mid J \subseteq I$ and $J \in F(D)$ is finitely generated$\}$. We say that $I \in F(D)$ is a t-ideal if $I_t = I$, and a t-ideal of D is a maximal t-ideal if it is maximal among proper integral t-ideals of D. Let t-Max(D) be the set of maximal t-ideals of D. It is known that t-Max$(D) \neq \emptyset$ if D is not a field; each ideal in t-Max(D) is a prime ideal; each prime ideal minimal over a t-ideal is a t-ideal; and $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. We say that t-dim$(D) = 1$ if D is not a field and each prime t-ideal of D is a maximal t-ideal. An $I \in F(D)$ is said to be t-invertible if $(II^{-1})_t = D$. Let $T(D)$ be the set of t-invertible fractional t-ideals of D. Then, $T(D)$ is an abelian group under $I * J = (IJ)_t$. Clearly, Prin(D), the set of nonzero principal fractional ideals of D, is a subgroup of $T(D)$, and $Cl(D) = T(D)/\text{Prin}(D)$ is called the (t-)class group of D. Note that, if D is a Krull (respectively, Prüfer) domain, then $Cl(D)$ is the usual divisor (respectively, ideal) class group of D.

Let $X^1(D)$ be the set of height-one prime ideals of D. We say that D is a *weakly Krull domain* if $D = \bigcap_{P \in X^1(D)} D_P$, and this intersection has finite character, i.e., each nonzero nonunit of D is a unit in D_P except finitely many prime ideals in $X^1(D)$. In this case, t-Max$(D) = X^1(D)$, and thus, t-dim$(D) = 1$ when D is a weakly Krull domain. Clearly, Krull domains are weakly Krull domains. An *almost weakly factorial domain* (AWFD) is an integral domain D in which, for each $0 \neq d \in D$, there is an integer $n = n(d) \geq 1$ such that d^n can be written as a finite product of primary elements of D. It is known that D is a WFD (respectively, an AWFD) if and only if D is a weakly Krull domain with $Cl(D) = \{0\}$ (respectively, $Cl(D)$ torsion) [**5,** Theorem] (respectively, [**4,** Theorem 3.4]); hence,

$$\text{WFD} \Longrightarrow \text{AWFD} \Longrightarrow \text{ weakly Krull domain.}$$

It is easy to see that, if N is a multiplicative subset of a weakly Krull domain (respectively, WFD, an AWFD) D, then D_N satisfies the corresponding property. We say that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. It is known that D is a PvMD if and only if $D[X]$ is a PvMD [**18,** Corollary 4] or, equivalently, $D[X, 1/X]$ is a PvMD [**16,** Theorem 3.10]. In addition, D is a GCD-domain if and only if D is a PvMD with $Cl(D) = \{0\}$ [**9,** Proposition 2].

2. Main results. Let D be an integral domain, X an indeterminate over D, $d \in D$, $R = D[X, d/X]$, $S = \{X^k \mid k \geq 0\}$, and $T = \{(d/X)^k \mid k \geq 0\}$; thus, $R_S = D[X, 1/X]$. Clearly, if $d = 0$ (respectively, d is a unit of D), then $R = D[X]$ (respectively, $R = D[X, 1/X]$). Also, if $d \neq 0$, then $R_T = D[X/d, d/X]$, $R = R_S \cap R_T$, [**1,** Lemma 7(b)], and $R_T \cong D[y, y^{-1}]$ for an indeterminate y over D.

Lemma 2.1. *Let A be a commutative ring with identity and X an indeterminate over A. Then, the zero ideal of A is primary if and only if the zero ideal of $A[X]$ is primary.*

Proof. Let $N(B)$ (respectively, $Z(B)$) be the set of nilpotent elements (respectively, zero divisors) of a ring B. Clearly, $N(B) \subseteq Z(B)$, and the zero ideal of B is primary if and only if $N(B) = Z(B)$. Hence, it suffices to show that $Z(A) \subseteq N(A)$ if and only if $Z(A[X]) \subseteq N(A[X])$. Assume that $Z(A) \subseteq N(A)$, and let $f \in Z(A[X])$. Then, $fg = 0$ for some $0 \neq g \in A[X]$. Hence, if $c(h)$ is the ideal of A generated by the coefficients of a polynomial $h \in A[X]$, then by the Dedekind-Mertens lemma [**12,** Theorem 28.1], there is an integer $n \geq 1$ such that

$$c(f)^{n+1}c(g) = c(f)^n c(fg) = (0).$$

Note that $c(f) \subseteq Z(A)$ since $g \neq 0$, and $c(f)$ is finitely generated. Hence, by assumption, $c(f)^m = (0)$ for some integer $m \geq 1$. Thus,

$$f^m \in (c(f)[X])^m = c(f)^m[X] = (0),$$

and hence, $f \in N(A[X])$. The converse follows since $Z(A) \subseteq Z(A[X])$. $\square$

Proposition 2.2. *The following statements are equivalent for $R = D[X, d/X]$ with $0 \neq d \in D$.*

(i) *X is irreducible (respectively, prime, primary) in R;*
(ii) *d/X is irreducible (respectively, prime, primary) in R;*
(iii) *d is a nonunit (respectively, prime, primary) in D.*

Proof. The properties of irreducible and prime appear in [**1,** Proposition 1]. For the primary property, note that

$$(D/dD)[X] \cong D\left[X, \frac{d}{X}\right]/(X) \cong D\left[X, \frac{d}{X}\right]\Big/\left(\frac{d}{X}\right).$$

Also, note that an ideal I of a ring A is primary if and only if $Z(A/I) = N(A/I)$. Thus, the result follows directly from Lemma 2.1. $\square$

Corollary 2.3. *Let $d \in D$ be a nonzero nonunit and $R = D[X, d/X]$. If dD is primary in D, then $\sqrt{XR}$ is a maximal t-ideal of R and $(X, d/X)_v = R$.*

Proof. By Proposition 2.2, XR is a primary ideal of R, and thus, $\sqrt{XR}$ is a maximal t-ideal [**8,** Lemma 2.1]. Next, note that

$$R\left[\frac{1}{X}\right] = D\left[X, \frac{1}{X}\right]$$

and $dD[X, 1/X]$ is primary. Hence, if $Q = dD[X, 1/X] \cap R$, then Q is primary. In addition, $d = X \cdot d/X$ and $X \notin \sqrt{Q}$ since $QR[1/X] \subsetneq R[1/X]$. Thus, $d/X \in Q$, and, since d/X is primary by Proposition 2.2, $(X, d/X)_v = R$. $\square$

A nonzero prime ideal Q of $D[X]$ is called an *upper to zero* in $D[X]$ if $Q \cap D = (0)$, and we say that D is a UMT-*domain* if each upper to zero in $D[X]$ is a maximal t-ideal of $D[X]$. It is known that D is a PvMD if and only if D is an integrally closed UMT-domain [**14,** Proposition 3.2]. In addition, $D[X]$ is a weakly Krull domain if and only if $D[X, 1/X]$ is a weakly Krull domain, which is exactly when D is a weakly Krull UMT-domain [**3,** Propositions 4.7, 4.11].

Proposition 2.4. *Let $d \in D$ be a nonzero nonunit and $R = D[X, d/X]$. Then, R is a weakly Krull domain if and only if D is a weakly Krull* UMT-*domain.*

Proof. Let $S = \{X^k \mid k \geq 0\}$ and $T = \{(d/X)^k \mid k \geq 0\}$. If R is a weakly Krull domain, then $R_S = D[X, 1/X]$ is a weakly Krull domain, and thus, D is a weakly Krull UMT-domain. Conversely, assume that D is a weakly Krull UMT-domain. Then, both

$$R_S = D\left[X, \frac{1}{X}\right] \quad \text{and} \quad R_T = D\left[\frac{d}{X}, \frac{X}{d}\right]$$

are weakly Krull domains. Note that $R = R_S \cap R_T$. Thus, R is a weakly Krull domain. $\square$

Let S be a saturated, multiplicative set of D and

$$N(S) = \{d \in D \mid (s, d)_v = D \text{ for all } s \in S\}.$$

Clearly, $D = D_S \cap D_{N(S)}$. We say that S is a *splitting set* if, for each $0 \neq d \in D$, we have $d = st$ for some $s \in S$ and $t \in N(S)$. It is known that, if S is a splitting set of D generated by a set of prime elements, then $D_{N(S)}$ is a UFD and $Cl(D) = Cl(D_S)$ [**2,** Theorem 4.2].

Lemma 2.5. *Let S be a splitting set of an integral domain D generated by a set of prime elements in D. Then, D is a* WFD *if* (*and only if*) *D_S is a* WFD.

Proof. Since $Cl(D) = Cl(D_S) = \{0\}$, it suffices to show that D is a weakly Krull domain. Note that $D_{N(S)}$ is a UFD; thus, $D_{N(S)}$ is a weakly Krull domain. Hence, D is a weakly Krull domain since D_S is a weakly Krull domain, by assumption, and $D = D_S \cap D_{N(S)}$. $\square$

We next give the main result of this paper for which we recall from [**10,** Theorem 9] that the following three conditions are equivalent:

(i) $D[X]$ is a WFD;
(ii) $D[X, 1/X]$ is a WFD; and
(iii) D is a weakly factorial GCD-domain.

Note also that D is a PvMD if and only if D_P is a valuation domain for all $P \in t$-Max(D) [**13,** Theorem 5].

Theorem 2.6. *Let $R = D[X, d/X]$ with $d \in D$. Then, the following statements are equivalent.*

(i) *R is a* WFD;
(ii) *R is a weakly factorial* GCD-*domain*;
(iii) *D is a weakly factorial* GCD-*domain, and $d = 0$, d is a unit of D or d is a prime element of D.*

Proof.

(i) $\Rightarrow$ (ii). If $d = 0$ or d is a unit of D, then $R = D[X]$ or $R = D[X, 1/X]$, and hence, R is a weakly factorial GCD-domain.

Now, assume that d is a nonzero nonunit. It suffices to show that R is a PvMD since a GCD-domain is a PvMD with trivial class group. Let $Q \in t$-Max(R). Then, ht$Q = 1$. If $X \notin Q$, then $Q_S \subsetneq R_S$, where $S = \{X^n \mid n \geq 0\}$. Note that R_S is a WFD by (i) and $R_S = D[X, 1/X]$; thus, D is a weakly factorial GCD-domain, and hence, R_S is a PvMD. Thus, $R_Q = (R_S)_{Q_S}$ is a rank-one valuation domain. Next, assume that $X \in Q$. Then, $d/X \notin Q$ by Corollary 2.3, and hence, if $T = \{(d/X)^n \mid n \geq 0\}$, then $R_T = D[d/X, X/d]$ and $Q_T \subsetneq R_T$. Note that D is a PvMD; thus, $D[d/X, X/d]$ is a PvMD. Note also that Q_T is a prime t-ideal of R_T since ht$(Q_T) = 1$. Hence, $R_Q = (R_T)_{Q_T}$ is a rank-one valuation domain. Thus, R is a PvMD.

(ii) $\Rightarrow$ (iii). Note that $R[1/X] = D[X, 1/X]$ is a WFD. Hence, D is a weakly factorial GCD-domain. Assume that d is a nonzero nonunit. Then, X is irreducible in R by Proposition 2.2, and, since R is a GCD-domain, X is a prime in R. Thus, again by Proposition 2.2, d is a prime element of D.

(iii) $\Rightarrow$ (i). If $d = 0$, then $R = D[X]$, and hence, R is a WFD. Next, if d is a unit, then $R = D[X, 1/X]$. Thus, R is a WFD. Finally, assume that d is a prime element of D. Then, X is a prime element of R by Proposition 2.2. In addition, $\bigcap_{n=0}^{\infty} X^n R = \{0\}$ since dD is a height-one prime ideal of D; thus, $S = \{X^n \mid n \geq 0\}$ is a splitting set of R [**2,** Proposition 2.6]. Note that $R_S = D[X, 1/X]$. Hence, R_S is a WFD. Thus, R is a WFD by Lemma 2.5. $\square$

An integral domain D is a *ring of Krull type* if there is a set $\{V_\alpha\}$ of valuation overrings of D such that

(i) each $V_\alpha = D_P$ for some prime ideal P of D;
(ii) $D = \bigcap_\alpha V_\alpha$; and
(iii) this intersection has finite character.

Then D is a ring of Krull type if and only if D is a PvMD of finite t-character (i.e., each nonzero nonunit of D is contained in only finitely many maximal t-ideals) [**13,** Theorem 7]. It is known that D is a ring of Krull type if and only if $D[X]$ is a ring of Krull type or, equivalently, $D[X, 1/X]$ is a ring of Krull type (cf., [**13,** Propositions 9, 12]).

Theorem 2.7. *Let $R = D[X, d/X]$ with $d \in D$. Then, R is a ring of Krull type if and only if D is a ring of Krull type.*

Proof. If $d = 0$ (respectively, d is a unit of D), then $R = D[X]$ (respectively, $R = D[X, 1/X]$). Hence, we may assume that d is a nonzero nonunit of D. Let

$$S = \{X^n \mid n \geq 0\} \quad \text{and} \quad T = \left\{\left(\frac{d}{X}\right)^n \mid n \geq 0\right\}.$$

Then

$$R_S = D\left[X, \frac{1}{X}\right], \qquad R_T = D\left[\frac{X}{d}, \frac{d}{X}\right],$$

and $R = R_S \cap R_T$. Also, $D[X/d, d/X]$ is isomorphic to $D[y, 1/y]$ for an indeterminate y over D.

If R is a ring of Krull type, then R_S is a ring of Krull type [**13,** Proposition 12], and thus, D is a ring of Krull type. Conversely, assume that D is a ring of Krull type. Then, both R_S and R_T are rings of Krull type. Let $A = \{P \in t\text{-Spec}(R) \mid P_S \in t\text{-Max}(R_S)\}$ and $B = \{P \in t\text{-Spec}(R) \mid P_T \in t\text{-Max}(R_T)\}$. Then, R_P is a valuation domain for each $P \in A \cup B$,

$$R = R_S \cap R_T = \Big(\bigcap_{P \in A} R_P \Big) \cap \Big(\bigcap_{P \in B} R_P \Big),$$

and this intersection has finite character. Thus, R is a ring of Krull type. $\square$

A *generalized Krull domain* is an integral domain D such that

(i) D_P is a valuation domain for all $P \in X^1(D)$;
(ii) $D = \bigcap_{P \in X^1(D)} D_P$; and
(iii) this intersection has finite character.

Then, we have the following implications:

• Generalized Krull domain $\Leftrightarrow$ ring of Krull type + t-dimension one $\Leftrightarrow$ weakly Krull domain + PvMD $\Rightarrow$ weakly Krull domain,

•
$$\begin{tikzcd}
\text{Generalized Krull domain} \arrow[r, Rightarrow] & \text{ring of Krull type} \arrow[d, Rightarrow] & \\
\text{Krull domain} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{P}v\text{MD} \arrow[r, Rightarrow] & \text{UMT-domain.}
\end{tikzcd}$$

However, in general, the reverse implications do not hold.

Corollary 2.8. *Let $R = D[X, d/X]$ with $d \in D$. Then, R is a generalized Krull domain if and only if D is a generalized Krull domain.*

Proof. Let the notation be as in the proof of Theorem 2.7, and note that a generalized Krull domain is merely a ring of Krull type with t-dimension one. Note also that R and D are PvMDs by Theorem 2.7; hence, $t\text{-dim}(R) = 1 \Leftrightarrow t\text{-dim}(R_S) = t\text{-dim}(R_T) = 1 \Leftrightarrow t\text{-dim}(D) = 1$. Thus, again by Theorem 2.7, R is a generalized Krull domain if and only if D is a generalized Krull domain. $\square$

A Krull domain D is called an *almost factorial domain* if $Cl(D)$ is torsion. It is known that a Krull domain D is an almost factorial domain if and only if D is an AWFD [**11,** Proposition 6.8]. In addition, if D is a UFD, p is a prime element of D, and $n \geq 1$ is an integer, then $D[X, p^n/X]$ is a Krull domain with $Cl(D[X, p^n/X]) = \mathbb{Z}_n$ [**1,** Theorems

8 and 16]. The next result is an AWFD analog for which we first note that, if $R = D[X,d/X]$ with $d \in D$, then R is a $\mathbb{Z}$-graded integral domain with $\deg(aX^n) = n$ and $\deg(a(d/X)^n) = -n$ for $0 \neq a \in D$ and the integer $n \geq 0$. Let

$$H = \{aX^k \mid 0 \neq a \in D \text{ and } k \geq 0\} \cup \left\{ a\left(\frac{d}{X}\right)^k \mid 0 \neq a \in D \text{ and } k \geq 0 \right\}.$$

Then, H is the set of nonzero homogeneous elements of R, and, if Q is a maximal t-ideal of R with $Q \cap H \neq \emptyset$, then Q is generated by $Q \cap H$ [**7,** Lemma 1.2].

Corollary 2.9. *Let D be a weakly factorial* GCD-*domain, p a prime element of D and $n \geq 2$ an integer. Then, $R = D[X, p^n/X]$ is an* AWFD *with $Cl(R) = \mathbb{Z}_n$.*

Proof. A generalized Krull domain is a weakly Krull PvMD, and thus, R is a weakly Krull PvMD by Corollary 2.8. Hence, it suffices to show that $Cl(R) = \mathbb{Z}_n$.

Let $Q = \sqrt{XR}$, $S = \{X^k \mid k \geq 0\}$, and note that Q is a unique maximal t-ideal of R with $Q \cap S \neq \emptyset$ since X is primary by Proposition 2.2. Let $\Lambda = \{P \in t\text{-Max}(R) \mid P \cap S = \emptyset\}$. Then, t-Max$(R) \setminus \Lambda = \{Q\}$, and $R_S = D[X, 1/X]$ is a WFD, so $Cl(R_S) = \{0\}$. Thus, $Cl(R)$ is generated by the classes of t-invertible Q-primary t-ideals of R [**3,** Theorem 4.8].

Note that X is homogeneous, $(X, p^n/X)_v = R$, and $(a,p)_v = R$ for all $a \in D \setminus pD$. Hence, $Q = (X,p)_v$, and, since R is a PvMD, Q is a t-invertible prime t-ideal. Note that

$$(Q^n)_t = ((X,p)^n)_t = (X^n, p^n)_t = \left(X^n, X\frac{p^n}{X}\right)_v = XR$$

(see [**6,** Lemma 3.3] for the second equality), while $(Q^k)_t$ is not principal for $k = 1, \ldots, n-1$. Note also that, if A is a Q-primary t-ideal of R, then $A = (Q^m)_t$ for some $m \geq 1$, and thus, $(A^n)_t = X^m R$. Hence, $Cl(R) = \mathbb{Z}_n$. □

We conclude this paper with some examples of weakly factorial GCD domains with prime elements.

Example 2.10.

(1) Let V be a rank-one nondiscrete valuation domain, y an indeterminate over V and $D = V[y]$. Then, D is a weakly factorial GCD-domain, and y is a prime of D. Thus, $R = D[X, y/X]$ is a WFD, and $D[X, y^n/X]$ is an AWFD with

$$Cl\left(D\left[X, \frac{y^n}{X}\right]\right) = \mathbb{Z}_n$$

for all integers $n \geq 2$.

(2) Let D be a weakly factorial GCD-domain, X an indeterminate over D, and $S = \{f \in D[X] \mid f \neq X,\ f \text{ a prime in } D[X]\}$. Then, $D[X]_S$ is a one-dimensional weakly factorial GCD-domain with a prime element X. (For, if Q is a prime ideal of $D[X]$ with ht$Q \geq 2$, then Q contains a nonconstant prime polynomial since D is a GCD-domain.)

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