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SIGN-CHANGING SOLUTIONS TO A CLASS OF NONLINEAR EQUATIONS INVOLVING THE p-LAPLACIAN

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ABSTRACT. This paper deals with a class of nonlinear problems

$$-(r^{n-1}|u'|^{p-2}u')' + r^{n-1}q(r)|u|^{p-2}u = r^{n-1}w(r)f(u)$$

in $(0,1)$, where $1 \leq n < p < \infty$ and $' = d/dr$. We study the existence of nodal solutions to this nonautonomous system. We give necessary and sufficient conditions for the existence of sign-changing solutions and also observe an application related to the case of multi-point boundary conditions. Methods used here are energy function control, shooting arguments and Prüfer-type substitutions.

1. Introduction. The aim of this paper is to study the properties of existence and nonexistence of sign-changing solutions to a class of nonlinear equations

$$-(r^{n-1}|u'|^{p-2}u')' + r^{n-1}q(r)|u|^{p-2}u = r^{n-1}w(r)f(u), \tag{1.1}$$

$$u'(0) = u(1) = 0, \tag{1.2}$$

where $r = |x|$, $' = d/dr$, $p > 1$ and $n \geq 1$. It is known that (1.1)–(1.2) is a radial version of the following p-Laplacian problem considered on the unit ball in $\mathbb{R}^n$,

$$-\triangle_p u = f(|x|, u), \tag{1.3}$$

where $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The p-Laplacian operator has attracted much attention and arises in various fields, such as non-Newtonian fluids and nonlinear diffusion problems. The quantity p is a characteristic

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of the medium. Media with $p > 2$ are called *dilatant fluids*, and those with $p < 2$ are called *pseudoplastics*. If $p = 2$, they are *Newtonian fluids*. For the above topics, one may refer to **[9, 10, 17, 20, 24, 28, 29]** and their bibliographies. Recently, some results for radial solutions related to (1.3) have been obtained **[7, 13, 14, 23, 26, 31]**. For the radial eigenvalues related to (1.3) in $\mathbb{R}^2$, one may refer to **[1]**.

In **[15, 16]**, the author dealt with the case of $n = 1$ (non-radially symmetric case) for the 2-Laplacian operator. Here, we consider the general case $n > 1$, $p > 1$. Furthermore, the purpose of this paper is to extend our previous work **[31]** to a more general p-Laplacian equation with restrictions (C_1)–(C_4) mentioned below. Throughout the paper, we assume that the following conditions hold:

- (C_1) $p > n$;
- (C_2) $w,\ q \in C^1(\mathbb{R}^+)$, and $w \geq \delta_1$ on $[0, \infty)$ for some $\delta_1 > 0$;
- (C_3) $f \in C^1(\mathbb{R})$, $f(s) > 0$ for $s > 0$ and $f(-s) = -f(s)$ for $s \neq 0$;
- (C_4) there exist extended real numbers $0 \leq f_0,\ f_\infty \leq \infty$ such that $\lim_{s\to 0^+} f(s)/s^{p-1} = f_0$ and $\lim_{s\to\infty} f(s)/s^{p-1} = f_\infty$.

In fact, the eigenvalues are easy to analyze for one-dimensional p-Laplacian eigenvalue problems coupled with two-point boundary conditions **[2, 4, 5, 6, 11, 18]**, etc. In **[31]**, the authors considered (1.1)–(1.2) with $q \equiv 0$. Two sufficient conditions for the existence of sign-changing solutions with prescribed number of zeros are established by comparing the ratios $f(u)/|u|^{p-2}u$ at infinity and zero, respectively, with the eigenvalues of the radial p-Laplacian problem with Neumann-Dirichlet boundary conditions.

In **[8]**, del Pino and Manásevich studied the bifurcation of solutions from eigenvalues of the p-Laplacian. They also discussed the existence of sign-changing radial solutions for

$$-\Delta_p u = g(u) \quad \text{in } \Omega; \qquad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ of $C^{2,\beta}$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous with $g(0) = 0$. They showed **[8,** Theorem 4.2**]** that, if

$$\sup_{s\in\mathbb{R}} \left| \frac{g(s)}{|s|^{p-2}s} \right| < \infty \quad \text{and} \quad \lim_{s\to 0} \frac{g(s)}{|s|^{p-2}s} < \lambda_k \leq \lambda_n < \liminf_{|s|\to\infty} \frac{g(s)}{|s|^{p-2}s},$$

then, for each $k \leq j \leq n$, there is a radial solution with exactly $j-1$ nodes, where λ_k is the kth eigenvalue of $-(r^{n-1}|u'|^{p-2}u')' = \mu r^{n-1}|u|^{p-2}u$ coupled with $u'(0)=0$ and $u(1)=0$.

Motivated by **[8, 31]**, we study in this paper (1.1)–(1.2) with a nonautonomous nonlinear term which is different from [**8,** Theorem 4.2] and extend the result in [**31**] to the general case; see Theorems 1.1, 1.2 and Corollary 1.3. Corollary 1.3 gives necessary and sufficient conditions for the existence and nonexistence of sign-changing solutions with the prescribed number of zeros to this problem. However, this extension is not trivial due to the fact that two energy functions (mentioned in Section 2) with a general q may not be nonnegative. More subtle arguments are needed in the proofs even when one develops the elementary property to the initial value problem. Under the derivation of the results of Theorems 1.1, 1.2 and Corollary 1.3, we observe an application to the case of *multi-point* boundary conditions. The existence of solutions, especially positive solutions, of boundary value problems with multi-point boundary conditions have been studied extensively, see, for example, **[12, 21, 22, 25, 32]** and the references therein. Moreover, to the best of the authors' knowledge, there is nothing so far on the existence of nodal solutions to problems with the multi-point boundary conditions related to (1.1). Essentially, the methods used in this work are the energy function control, shooting arguments and Prüfer-type substitutions.

Now, in order to discuss the main result, we introduce the following eigenvalue problem

$$-(r^{n-1}|y'|^{p-2}y')' = r^{n-1}(\lambda w(r) - q(r))|y|^{p-2}y, \tag{1.4}$$

$$y'(0) = y(1) = 0. \tag{1.5}$$

It is well known that, cf., **[2, 27, 30]**, (1.4)–(1.5) has a countable number of eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ satisfying

$$-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \lambda_{k+1} < \cdots \longrightarrow \infty,$$

and the corresponding eigenfunction $y_k(r)$ has exactly $k-1$ zeros in $(0,1)$. In the sequel, we assume λ_m is the first positive eigenvalue of (1.4)–(1.5) and define $\mathbb{N}_m = \{n \in \mathbb{N} : \ n \geq m\}$. Similarly, (1.4) with the Neumann boundary conditions

$$y'(0) = y'(1) = 0, \tag{1.6}$$

has a countable number of eigenvalues $\{\mu_i\}_{i\in\mathbb{N}}$ satisfying

$$-\infty < \mu_1 < \mu_2 < \mu_3 < \cdots < \mu_k < \mu_{k+1} < \cdots \longrightarrow \infty,$$

and its corresponding kth eigenfunction has exactly k zeros in $(0,1)$.

Our main result is the following.

Theorem 1.1.

(i) *For all* $u \in (0,\infty)$*, if* $f(u)/u^{p-1} < \lambda_k$ *for some* $k \in \mathbb{N}$*, then* (1.1)–(1.2) *has no solution with exactly* i *zeros in* $(0,1)$ *for any* $i \geq k-1$.

(ii) *For all* $u \in (0,\infty)$*, if* $\lambda_k < f(u)/u^{p-1}$ *for some* $k \in \mathbb{N}$*, then* (1.1)–(1.2) *has no solution with exactly* i *zeros in* $(0,1)$ *for any* $i \leq k-1$.

(iii) *For all* $u \in (0,\infty)$*, if* $f(u)/u^{p-1} \neq \lambda_k$ *for any* $k \in \mathbb{N}$*, then* (1.1)–(1.2) *has no nontrivial solution.*

Theorem 1.2. *Let* f_0 *and* f_∞ *be defined as in* (C_4)*. Assume that there exists* $k \in \mathbb{N}_m$ *such that either* $\lambda_k \in (f_0, f_\infty)$ *or* (f_∞, f_0)*. Then* (1.1)–(1.2) *has a solution with exactly* $k-1$ *zeros in* $(0,1)$.

The combination of Theorems 1.1 and 1.2 immediately leads to the following necessary and sufficient conditions.

Corollary 1.3. *Assume* $f(u)/u^{p-1} \in (f_0, f_\infty)$ *or* (f_∞, f_0) *for all* $u \in (0,\infty)$*. Then, for* $k \in \mathbb{N}_m$*,* (1.1)–(1.2) *has a solution with exactly* $k-1$ *zeros in* $(0,1)$ *if and only if* $\lambda_k \in (f_0, f_\infty)$ *or* (f_∞, f_0).

As a byproduct from the derivation of the above result, we intend to extend the similar result to the case of (1.1) coupled with the multi-point boundary conditions

$$u'(0) = 0, \qquad u'(1) - \sum_{i=1}^{d} k_i r_i^{n-1/p-1} u'(r_i) = 0, \tag{1.7}$$

or

$$u'(0) = 0, \qquad u(1) - \sum_{i=1}^{d} k_i u(r_i) = 0, \tag{1.8}$$

where $d \in \mathbb{N}$ and $r_i \in (0,1)$ for $i = 1, 2, 3, \ldots, d$. Note that, by definition, the following property for f_0 and f_∞ is valid. Assume that

$$f_0,\ f_\infty < \infty. \tag{1.9}$$

By (C_4) and (1.9), i.e., for every $\epsilon > 0$, there exists an $M > 0$ such that $|f(u)| \le (f_\infty + \epsilon)|u|^{p-1}$ for $|u| \ge M$. Then, for this M, there exists an $N_1 > 0$ satisfying $|f(u)| \le N_1 |u|^{p-1}$ for $|u| < M$. Thus, for every u, there exists an $N > 0$ which is independent of u such that

$$|f(u)| \le N|u|^{p-1}. \tag{1.10}$$

Now, we define two constants

$$\overline{N} \equiv \max\{w(r)N + |q(r)| :\ r \in [0,1]\} \quad \text{and} \quad \widehat{N} \equiv \frac{p-1}{p-n} + \overline{N}. \tag{1.11}$$

Then, we obtain the next result for the multi-point boundary value problem.

Theorem 1.4.

(i) *Consider problems* (1.1) and (1.7), *and denote by* $\{\lambda_i\}_{i\in\mathbb{N}}$ *the eigenvalues of* (1.4) *and* (1.5). *Assume that* $f_0 < \lambda_k$ *and* $\lambda_{k+1} < f_\infty < \infty$ *for some* $k \in \mathbb{N}$. *Suppose that the following additional condition is valid*

$$1 - e^{\hat{N}/(p-1)} \sum_{i=1}^{d} |k_i| > 0. \tag{1.12}$$

Then, (1.1) *and* (1.7) *has a solution with* k *zeros in* $(0,1)$.

(ii) *Consider problems* (1.1) *and* (1.8), *and denote by* $\{\mu_i\}_{i\in\mathbb{N}}$ *the eigenvalues of* (1.4) *and* (1.6). *Assume that* $f_0 < \mu_k$ *and* $\mu_{k+1} < f_\infty < \infty$ *for some* $k \in \mathbb{N}$. *Suppose that condition* (1.12) *holds. Then,* (1.1) *and* (1.8) *have a solution with* k *or* $k+1$ *zeros in* $(0,1)$.

The outline of this paper is as follows. In Section 2, some elementary properties for the initial value problem are first developed. A version of the Pohozaev identity is also discussed in this section. In Section 3, some technical lemmas are represented. Finally, the proofs of Theorems 1.1, 1.2 and 1.4 are given in Section 4.

2. Preliminaries. In this section, we focus on elementary properties for solutions to the initial value problem consisting of (1.1) coupled with

$$u(0) = \alpha, \qquad u'(0) = 0, \tag{2.1}$$

where α is a positive parameter. The local existence and uniqueness of the solution to (1.1) and (2.1) is valid and can be quoted from **[23, 26]**; it will be stated in Theorem 2.1. Then, we will show global existence to the local solution by dividing the arguments into two cases.

Theorem 2.1. **[23,** Theorem EUCD], **[26,** Theorems 1, 4]. *Assume that conditions* (C_1) *and* (C_2) *hold. Then, there exists a local solution* $u(r;\alpha)$ *of* (1.1) *and* (2.1). *Moreover, this solution is unique in a neighborhood* $J = [0, a]$ *for some* $a > 0$.

By local existence in J, assume here that $J = [0, a]$ is the maximal interval of the existence of solution $u(r;\alpha)$.

Note that (1.1) may be rewritten as follows, for $r \neq 0$:

$$(|u'|^{p-1}u')' + \frac{n-1}{r}|u'|^{p-2}u' + wf(u) - q|u|^{p-2}u = 0;$$

i.e., for $r \neq 0$:

$$-(n-1)\frac{|u'|^{p-2}u'}{r} = -q|u|^{p-2}u + wf(u) + (p-1)|u'|^{p-2}u''. \tag{2.2}$$

As (2.2) is multiplied by $u'(r)$, then

$$-(n-1)\frac{|u'|^{p}}{r} = -q|u|^{p-2}uu' + wf(u)u' + (p-1)|u'|^{p-2}u'u''. \tag{2.3}$$

Since the local solution uniquely exists in J by Theorem 2.1, all terms on the right-hand side of (2.2) are bounded in J. Then $(|u'|^{p-2}u')/r$ is also bounded in J. Similarly, in (2.3), the boundedness of $(|u'(r)|^p)/r$ in J is obtained. By (2.3) and the initial conditions (2.1), $\lim_{r\to 0}(|u'(r)|^p)/r = 0$.

Now, we show the global existence of $u(r,\alpha)$ in $[0,1]$. Here two versions of energy are employed to achieve this goal.

(i) Consider the case of $\lim_{s\to\infty} f(s)/s^{p-1} = f_\infty = \infty$. Let u be a solution of (1.1) and (2.1), and define the functional $E[u](r,\alpha)$ by:

$$E[u](r,\alpha) \equiv \frac{|u'(r)|^p}{p} + w(r)F(u(r)) - \frac{1}{p(p-1)}q(r)|u(r)|^p, \tag{2.4}$$

with

$$E[u](0,\alpha) = w(0)F(\alpha) - \frac{q(0)}{p(p-1)}\,\alpha^p, \tag{2.5}$$

where $F(s) \equiv 1/(p-1)\int_0^s f(t)\,dt$. Then, it follows from (1.1), (2.3) and the boundedness of $(|u'(r)|^p)/r$ that, for $r \in (0,a]$,

$$\begin{aligned}
\frac{d}{dr}E[u](r,\alpha) &\equiv E[u]'(r,\alpha)\\
&= -\frac{n-1}{p-1}\cdot\frac{|u'(r)|^p}{r} + w'(r)F(u(r)) - \frac{q'(r)}{p(p-1)}|u(r)|^p \qquad (2.6)\\
&\le kw(r)F(u(r)) - \frac{(k+1)}{p(p-1)}q(r)|u(r)|^p\\
&\quad + \frac{1}{p(p-1)}\left[(k+1)q(r) - q'(r)\right]|u(r)|^p, \qquad (2.7)
\end{aligned}$$

where

$$k = \max\left\{\frac{|w'(r)|}{w(r)} : r \in [0,1]\right\}.$$

Since $w(r) > 0$ is continuous and q, q' are bounded on $[0,1]$, we can find some positive constant h such that

$$\frac{h}{p(p-1)}[(k+1)q(r) - q'(r)] \le w(r) \quad \text{and} \quad \frac{h}{p(p-1)}|q(r)| \le w(r). \tag{2.8}$$

In this case, $f_\infty = \infty$, there exists an $M > 0$ such that $|u(r)|^p \le hF(u(r))$ for $|u(r)| \ge M$. For $r \in J_M \equiv \{r \in J:\ |u(r)| \le M\}$, there exists $N_M > 0$ satisfying $E'[u](r,\alpha) \le N_M$ on J_M. Furthermore, for

$r \in J \backslash J_M$, by (2.7), (2.8), we have

$$\begin{aligned} E[u]'(r,\alpha) &\le kw(r)F(u(r)) - \frac{k+1}{p(p-1)} q(r)|u(r)|^p + \frac{w(r)}{h} \cdot hF(u(r)) \\ &= (k+1)\left\{ w(r)F(u(r)) - \frac{1}{p(p-1)} q(r)|u(r)|^p \right\} \\ &\le (k+1)E[u](r,\alpha). \end{aligned}$$

Note that the second term of (2.8) implies that $E[u](r,\alpha) \ge 0$ on $J \backslash J_M$. Hence, for $r \in J$,

$$E[u]'(r,\alpha) \le N_M + (k+1)E[u](r,\alpha).$$

Integrating the above inequality, we can obtain that, for $r \in J$,

$$E[u](r,\alpha) \le E[u](0,\alpha) + aN_M + \int_0^r (k+1)E[u](t,\alpha)\,dt.$$

By Gronwall's inequality and (2.5), for $r \in J$,

$$E[u](r,\alpha) \le \left(w(0)F(\alpha) - \frac{q(0)}{p(p-1)}\alpha^p + aN_M \right) e^{(k+1)a}. \tag{2.9}$$

(ii) Let $f_\infty < \infty$. Then, for any $\epsilon > 0$, there exists an $M' > 0$ such that $|f(u)| \le (f_\infty + \epsilon)|u|^{p-1}$ for $|u(r)| \ge M'$. Now, rewrite (1.1) and (2.1) as:

(2.10)

$$\begin{cases} u'(r) = |v(r)|^{p^*-2}v(r), \\ v'(r) = q(r)|u(r)|^{p-2}u(r) - w(r)f(u(r)) - \dfrac{n-1}{r}|u'(r)|^{p-2}u'(r), \end{cases}$$

with $u(0) = \alpha$ and $v(0) = 0$, where $p^* = p/(p-1)$ is the conjugate exponent of p. Note that $(n-1)[|u'(r)|^{p-2}u'(r)]/r$ can be bounded by some positive constant k_α from (2.2) and the argument below (2.3).

Integrating (2.10) over $[0, r]$ for $r \in J$, we have

$$u(r) = \alpha + \int_0^r |v(t)|^{p^*-2} v(t)\, dt, \tag{2.11}$$

$$\begin{aligned} v(r) = &\int_0^r q(t)|u(t)|^{p-2}u(t)\, dt \\ &- \int_0^r w(t) f(u(t))\, dt \\ &- (n-1) \int_0^r \frac{|u'(t)|^{p-2}u'(t)}{t}\, dt. \end{aligned} \tag{2.12}$$

For $|u(r)| < M'$ and by (2.12), it is easy to obtain that $|v(r)| \le N_{M'} + k_\alpha$, where $N_{M'}$ is some constant dependent upon M'. On the other hand, by the above for $|u(r)| \ge M'$ and $r \in J$, we obtain

$$\begin{aligned} |v(r)| \le &\int_0^r |q(t)||u(t)|^{p-1} dt \\ &+ \int_0^r w(t)(f_\infty + \epsilon)|u(t)|^{p-1} dt + k_a \\ \le &\, k_\alpha + c_1 \int_0^r |u(t)|^{p-1} dt, \end{aligned}$$

where $c_1 > 0$ is some constant. Thus, for $r \in J$,

$$|v(r)| \le (N_{M'} + k_\alpha) + c_1 \int_0^r |u(t)|^{p-1} dt.$$

Then, it follows from the Hölder inequality that

$$|v(r)| \le (N_{M'} + k_\alpha) + c_2 \bigg(\int_0^r |u(t)|^p dt \bigg)^{(p-1)/p}$$

for some $c_2 > 0$, i.e.,

$$|v(r)|^{p/(p-1)} \le [N_{M'} + k_\alpha]^{p/(p-1)} + \bigg(\int_0^r |u(t)|^p dt \bigg)(c_2^{p/(p-1)} + \overline{N}_\alpha),$$

where $\overline{N}_\alpha$ is another constant. Thus, for $r \in J$,

$$|v(r)|^{p/(p-1)} \le c_3 + c_4 \int_0^r |u(t)|^p dt, \tag{2.13}$$

where c_3 and c_4 are positive constants. Similarly, for some constant $c_5 > 0$,

$$|u(r)|^p \leq \alpha^p + c_5 \int_0^r |v(t)|^{p/(p-1)} dt. \tag{2.14}$$

From (2.13)–(2.14), we obtain

$$|u(r)|^p + |v(r)|^{p^*} \leq (\alpha^p + c_3) + c_6 \int_0^r (|u(t)|^p + |v(t)|^{p^*})\, dt,$$

where $c_6 = \max\{c_4,\ c_5\}$. Note that the constants c_i, $i = 1, 2, \dots, 6$, depend on α, a and M'. By Gronwall's inequality, for $r \in J$ and fixed $\alpha > 0$,

$$|u(r)|^p + |v(r)|^{p^*} \leq (\alpha^p + c_3) \exp(c_6 a) < \infty. \tag{2.15}$$

From the above two cases, we derive the following result.

Proposition 2.2. *For $\alpha > 0$, the solution $u(r;\alpha)$ of the initial value problem* (1.1), *and* (2.1) *exists over the whole interval* $[0, 1]$.

Proof. Suppose that $u(r;\alpha)$ does not exist on the whole interval. Without loss of generality, we assume that $u(r;\alpha)$ exists on a maximal right interval $[0, c)$ for $0 < c < 1$. Then, $u(r;\alpha)$ is unbounded on $[0, c)$, $\lim_{r\to c^-} |u(r;\alpha)| = \infty$. Otherwise, if $u(r;\alpha)$ is bounded on $[0, c)$, then $\lim_{r\to c^-} u'(r;\alpha)$ exists by integrating (1.1) over the interval $[0, c)$. This implies that $u(r;\alpha)$ can be extended through c.

On the other hand, since $\lim_{r\to c^-} |u(r;\alpha)| = \infty$, there exists a sequence $r_n \to c^-$ such that $|u(r_n;\alpha)| \to \infty$. For the case of $f_\infty = \infty$, this implies that $\lim_{s\to\infty} (F(s))/|s|^p = f_\infty = \infty$. Thus, by (2.4), we have

$$E[u](r_n, \alpha) \geq \left(w(r_n) \frac{F(u(r_n))}{|u(r_n)|^p} - \frac{1}{p(p-1)} q(r_n) \right) \cdot |u(r_n)|^p \longrightarrow \infty \quad \text{as } n \to \infty.$$

This provides a contradiction to the argument in (2.9). For the case of $f_\infty < \infty$, it also contradicts the argument in (2.15). Therefore, the solution $u(r;\alpha)$ exists on $[0, 1]$. $\square$

Now, we represent a variant of the well-known Pohozaev identity for (1.1) and (2.1) in Lemma 2.3 for the sake of independent interest. The proof may easily be verified by differentiating (2.16) and applying (1.1). Therefore, it is omitted here.

Lemma 2.3. (*Pohozaev identity*). *Any solution $u = u(r;\alpha)$ of* (1.1) *and* (2.1) *satisfies the identity*

$$\frac{d}{dr}P(r;u) = \frac{n}{p}r^{n-1}\left(pw(r)\overline{F}(u)\left[1+\frac{rw'(r)}{nw(r)}\right] - |u|^p\left[q(r)+\frac{rq'(r)}{n}\right]\right),$$

where

(2.16)
$$P(r;u) = \frac{p-1}{p}r^n|u'|^p + \frac{n-p}{p}\int_0^r s^{n-1}|u'|^p ds + r^n w\overline{F}(u) - \frac{1}{p}r^n q(r)|u|^p$$

with $\overline{F}(u) = \int_0^u f(t)\,dt$. Furthermore, if the solution u is nontrivial and has infinitely many zeros in $[0,\infty)$, then there exists a sequence $\{r_j\}$ such that $r_j \to \infty$ and $(d/dr)P(r_j;u) = 0$ for every j.

3. Prüfer-type substitutions and some technical lemmas. At the beginning of this section, we introduce a Prüfer-type substitution for the solution $u(r;\alpha)$ of (1.1) and (2.1) by using the generalized sine function $S_p(r)$. The generalized sine function S_p has been extensively studied in the literature, see Lindqvist **[19]** or **[2, 11, 27]** with a minor difference in setting. Note that

$$\pi_p \equiv 2\int_0^{(p-1)^{1/p}} \frac{dt}{1-(t^p/(p-1))^{1/p}} = \frac{2(p-1)^{1/p}\pi}{p\sin(\pi/p)}$$

is the first zero of S_p in the positive real axis. With the help of the generalized sine function, we introduce phase-plane coordinates $\rho > 0$ and θ for a solution $u(r;\alpha)$ of (1.1) and (2.1) as follows:

(3.1)
$$\begin{cases} |u(r;\alpha)|^{p-2}u(r;\alpha) = \rho(r;\alpha)|S_p(\theta(r;\alpha))|^{p-2}S_p(\theta(r;\alpha)), \\ r^{n-1}|u'(r;\alpha)|^{p-2}u'(r;\alpha) = \rho(r;\alpha)|S_p'(\theta(r;\alpha))|^{p-2}S_p'(\theta(r;\alpha)), \end{cases}$$

with

$$\theta(0;\alpha) = \frac{\pi_p}{2} \quad \text{and} \quad \rho(0;\alpha) = \alpha^{p-1}. \tag{3.2}$$

Then

$$\rho^{p/(p-1)}(r;\alpha) = |u(r;\alpha)|^p + \frac{r^{p(n-1)/(p-1)}}{p-1}|u'(r;\alpha)|^p \tag{3.3}$$

and

$$\frac{r^{n-1}|u'|^{p-2}u'}{|u|^{p-2}u} = \frac{|S_p'|^{p-2}S_p'}{|S_p|^{p-2}S_p}.$$

Differentiating both sides with respect to r and employing (1.1), we obtain

$$\begin{aligned}\theta'(r;\alpha) = {} & \frac{r^{n-1}}{p-1}\left(w(r)\frac{f(u(r;\alpha))}{|u(r;\alpha)|^{p-2}u(r;\alpha)} - q(r)\right) \\ & \cdot |S_p(\theta(r;\alpha))|^p + r^{1-n/p-1}|S_p'\theta(r;\alpha)|^p \equiv G(r;\alpha;\theta),\end{aligned} \tag{3.4}$$

$$\begin{aligned}\frac{\rho'(r;\alpha)}{\rho(r;\alpha)} = {} & \left[r^{1-n/p-1} - r^{n-1}\left(w(r)\frac{f(u(r;\alpha))}{|u(r;\alpha)|^{p-2}u(r;\alpha)} - q(r)\right)\right] \\ & \cdot |S_p(\theta(r;\alpha))|^{p-2}S_p(\theta(r;\alpha))S_p'(\theta(r;\alpha)).\end{aligned} \tag{3.5}$$

It is clear that $u(r;\alpha)$ is a solution to (1.1) and (2.1) if and only if $\{\theta(r;\alpha), \rho(r;\alpha)\}$ satisfies (3.4)–(3.5) with conditions (3.2). Similarly, the Prüfer phase function for (1.4)–(1.5) with $\lambda = \lambda_k$ satisfies

$$\begin{cases}\phi_k'(r;\lambda_k) = \dfrac{r^{n-1}}{p-1}(\lambda_k w(r) - q(r))|S_p(\phi_k(r;\lambda_k))|^p \\ \qquad\qquad\qquad + r^{1-n/p-1}|S_p'(\phi_k(r;\lambda_k))|^p \equiv F(r;\lambda_k;\phi_k), \\ \phi_k(0;\lambda_k) = \dfrac{\pi_p}{2}, \quad \phi_k(1;\lambda_k) = k\pi_p.\end{cases}$$

Note that a similar method may be found in **[5, 6, 27, 31]**.

The following lemmas are necessary for the proof of Theorem 1.2.

Lemma 3.1.

(i) *Assume $f_0 < \lambda_k$ for some $k \in \mathbb{N}_m$. Then, there exists an $\alpha_* > 0$ such that $\theta(1;\alpha) < k\pi_p$ for all $\alpha \in (0,\alpha_*)$.*

(ii) *Assume $f_0 > \lambda_k$ for some $k \in \mathbb{N}_m$. Then, there exists an $\alpha_* > 0$ such that $\theta(1;\rho) > k\pi_p$ for all $\alpha \in (0,\alpha_*)$.*

Remark 3.2. The proof of Lemma 3.1 is basically the same as that of [**31,** Lemma 4.1] by applying the comparison lemma [**3,** page 30]. Since no further argument is necessary, the proof of this lemma is omitted.

Lemma 3.3. *Let M, $\alpha > 0$, and define $I_{M,\alpha} = \{r \in (0,1] : |u(r;\alpha)| < M\}$. For any $L > 0$ and sufficiently large M, there exists an $\alpha^* > 0$ such that $|u'(r;\alpha)| > L$ for $\alpha > \alpha^*$ and $r \in I_{M,\alpha}$.*

Proof. The proof is divided into two cases of $f_\infty < \infty$ and $f_\infty = \infty$, individually. Here, the significance is to analyze two versions of energy.

(i) Let $f_\infty < \infty$. By (3.5) and the Prüfer-type substitution (3.1), for $r \in I_{M,\alpha}$, we obtain the boundedness:

$$\bigg|\bigg(w(r)\frac{f(u(r;\alpha))}{|u(r;\alpha)|^{p-2}u(r;\alpha)} - q(r)\bigg) \\ \cdot |S_p(\theta(r;\alpha))|^{p-2}S_p(\theta(r;\alpha))S_p'(\theta(r;\alpha))\bigg| \le M_1$$

for some $M_1 > 0$. Thus, for $r \in I_{M,\alpha}$,

$$\frac{\rho'(r;\alpha)}{\rho(r;\alpha)} \ge -r^{1-n/p-1} - M_1.$$

For $r \in (0,1]\backslash I_{M,\alpha}$, there exists an $M_2 > 0$ such that

$$w(r)\bigg|\frac{f(u(r;\alpha))}{|u(r;\alpha)|^{p-2}u(r;\alpha)}\bigg| + |q(r)| \le M_2.$$

Hence, for $r \in (0,1]$ and $\overline{M} = \max\{M_1, M_2\}$, we obtain

$$\frac{\rho'(r;\alpha)}{\rho(r;\alpha)} \ge -r^{1-n/p-1} - \overline{M}. \tag{3.6}$$

Integrating (3.6) over $[0,r]$ for $r \in (0,1]$, by (C_1) we obtain

$$\begin{aligned} \ln\frac{\rho(r;\alpha)}{\rho(0;\alpha)} &\ge -\frac{p-1}{p-n}r^{p-n/p-1} - \overline{M}r \\ &\ge -\frac{p-1}{p-n} - \overline{M} \equiv -\widehat{M}. \end{aligned} \tag{3.7}$$

Then, by (3.2) for $r \in (0,1]$,

$$\rho(r;\alpha) \ge \rho(0;\alpha)e^{-\hat{M}} = \alpha^{p-1}e^{-\hat{M}} \longrightarrow \infty \quad \text{as } \alpha \to \infty.$$

By (3.3), for any $L > 0$, there exists an $\alpha^* > 0$ such that $\alpha > \alpha^*$ and $r \in I_{M,\alpha}$,

$$\begin{aligned}\left(M^p + \frac{r^{p(n-1)/p-1}}{p-1}|u'(r;\alpha)|^p\right)^{(p-1)/p} &\geq \rho(r;\alpha) \\ &> \left(M^p + \frac{r^{p(n-1)/p-1}}{p-1}L^p\right)^{(p-1)/p}.\end{aligned}$$

This leads to the inequality $|u'(r;\alpha)| > L$.

(ii) Let $f_\infty = \infty$. For any fixed $\alpha > 0$, recall (2.3) and $\lim_{r\to 0}|u'(r)|^p/r = 0$, i.e., for every α, $\epsilon > 0$ there exists a $\delta_{\alpha,\epsilon} > 0$ such that $n-1/p-1 \cdot |u'(r)|^p/r < \epsilon$ whenever $0 < r < \delta_{\alpha,\epsilon}$. Now, returning to (2.6), for $r \in (0, \delta_{\alpha,\epsilon})$, we obtain

$$\begin{aligned}\frac{d}{dr}E[u](r,\alpha) &\equiv E[u]'(r,\alpha) \\ &= -\frac{n-1}{p-1}\cdot\frac{|u'(r)|^p}{r} + w'(r)F(u(r)) - \frac{q'(r)}{p(p-1)}|u(r)|^p \\ &\geq -\epsilon + w'(r)F(u(r)) - \frac{q'(r)}{p(p-1)}|u(r)|^p.\end{aligned}$$

For $r \in [\delta_{\alpha,\epsilon}, 1]$,

$$E[u]'(r,\alpha) \geq -\frac{p(n-1)}{\delta_{\alpha,\epsilon}(p-1)}\cdot\frac{|u'(r)|^p}{p} + w'(r)F(u(r)) - \frac{q'(r)}{p(p-1)}|u(r)|^p.$$

Then, for $r \in (0,1]$, we get

$$\begin{aligned}E[u]'(r,\alpha) &\geq -\epsilon - \frac{p(n-1)}{\delta_{\alpha,\epsilon}(p-1)}\cdot\frac{|u'(r)|^p}{p} \\ &\quad + w'(r)F(u(r)) - \frac{q'(r)}{p(p-1)}|u(r)|^p \qquad (3.8)\\ &\geq -\epsilon - \frac{p(n-1)}{\delta_{\alpha,\epsilon}(p-1)}\cdot\frac{|u'(r)|^p}{p} \\ &\quad - kw(r)F(u(r)) + \frac{(k+1)}{p(p-1)}q(r)|u(r)|^p \\ &\quad - \frac{1}{p(p-1)}[(k+1)q(r) + q'(r)]|u(r)|^p. \qquad (3.9)\end{aligned}$$

For $r \in I_{M,\alpha}$, by (3.8), there exists a $\overline{k} > 0$ such that

$$
\begin{aligned}
E[u]'(r,\alpha) &\geq -\epsilon - \frac{p(n-1)}{\delta_{\alpha,\epsilon}(p-1)} \\
&\qquad \cdot \frac{|u'(r)|^p}{p} - \overline{k}\left(w(r)F(u(r)) - \frac{1}{p(p-1)}q(r)|u(r)|^p\right) \\
&\geq -\epsilon - \widehat{k}E[u](r,\alpha),
\end{aligned} \tag{3.10}
$$

where $\widehat{k} = \max\{p(n-1)/(\delta_{\alpha,\epsilon}(p-1), \overline{k})\}$. Applying a similar argument as that in (2.8), we rewrite (3.9) as

$$
\begin{aligned}
E[u]'(r,\alpha) \geq -\epsilon &- \frac{p(n-1)}{\delta_{\alpha,\epsilon}(p-1)} \cdot \frac{|u'(r)|^p}{p} \\
&- kw(r)F(u(r)) + \frac{(k+1)}{p(p-1)}q(r)|u(r)|^p \\
&- \frac{w(r)}{h}|u(r)|^p,
\end{aligned} \tag{3.11}
$$

for some $h > 0$. In the case $f_\infty = \infty$, there exists a sufficiently large $M > 0$ such that $|u(r)|^p \leq hF(u(r))$ for $|u(r)| \geq M$. Hence, for $r \in (0,1]\backslash I_{M,\alpha}$, (3.11) becomes

$$
\begin{aligned}
E[u]'(r,\alpha) \geq -\epsilon &- \frac{p(n-1)}{\delta_{\alpha,\epsilon}(p-1)} \cdot \frac{|u'(r)|^p}{p} \\
&- (k+1)\left(w(r)F(u(r)) - \frac{1}{p(p-1)}q(r)|u(r)|^p\right) \\
\geq -\epsilon &- \widetilde{k}E[u](r,\alpha),
\end{aligned} \tag{3.12}
$$

where $\widetilde{k} = \max\{p(n-1)/(\delta_{\alpha,\epsilon}(p-1)), k+1\}$. By (3.10) and (3.12), for $r \in (0,1]$, we obtain

$$
E[u]'(r,\alpha) \geq -\epsilon - KE[u](r,\alpha), \tag{3.13}
$$

where $K = \max\{\widehat{k}, \widetilde{k}\}$. Solving (3.13) for $r \in (0,1]$ and sufficiently large M, we have

$$
E[u](r,\alpha) \geq -\frac{\epsilon}{K} + \left(E[u](0,\alpha) + \frac{\epsilon}{K}\right)e^{-K}. \tag{3.14}
$$

In this case, by (2.5) and (C_2),

$$E[u](0,\alpha) = \alpha^p\left(w(0)\frac{F(\alpha)}{\alpha^p} - \frac{q(0)}{p(p-1)}\right) \longrightarrow \infty \quad \text{as } \alpha \to \infty. \tag{3.15}$$

Note that, in (2.4), the term

$$\left| -\frac{1}{p(p-1)}q(r)|u(r;\alpha)|^p + w(r)F(u(r;\alpha))\right|$$

can be uniformly bounded by some $K_1 > 0$ for all $\alpha > 0$ and $r \in I_{M,\alpha}$. For any $L > 0$, by (3.14)–(3.15), we choose α^* sufficiently large such that

$$E[u](r,\alpha) > \frac{L^p}{p} + K_1 \quad \text{for } \alpha > \alpha^* \text{ and } r \in (0,1].$$

Then, for $\alpha > \alpha^*$ and $r \in I_{M,\alpha}$,

$$\frac{|u'(r)|^p}{p} + K_1 \geq E[u](r,\alpha) > \frac{L^p}{p} + K_1.$$

This concludes $|u'(r,\alpha)| > L$ for $\alpha > \alpha^*$ and $r \in I_{M,\alpha}$. □

Note that the next corollary is useful to the proof of Lemma 3.5. For the sake of convenience, it is represented independently.

Corollary 3.4. *For sufficiently large α, assume that the number of zeros of $u(r;\alpha)$ are uniformly bounded in $(0,1)$. Then, the measure of $I_{M,\alpha}$ tends to zero as $\alpha \to \infty$.*

Proof. The proof of this corollary is similar to that of [**31,** Corollary 3.4 (ii)] by applying Lemma 3.3. Hence, it can be omitted. □

Lemma 3.5.

(i) *Assume $f_\infty > \lambda_k$ for some $k \in \mathbb{N}_m$. Then, there exists an $\alpha^* > 0$ such that $\theta(1;\alpha) > k\pi_p$ for all $\alpha \in (\alpha^*, \infty)$.*
(ii) *Assume $f_\infty < \lambda_k$ for some $k \in \mathbb{N}_m$. Then, there exists an $\alpha^* > 0$ such that $\theta(1;\alpha) < k\pi_p$ for all $\alpha \in (\alpha^*, \infty)$.*

Remark 3.6. The idea of the proof of this lemma is similar to that used in [**31,** Lemma 4.2 (ii)]. For the sake of convenience, we provide the details here.

Proof.

(i) Assume the contrary. Then, there exists an α_l with $\alpha_l \to \infty$ such that $\theta(1;\alpha_l) \le k\pi_p$. This implies that $u(r;\alpha_l)$ has at most $k-1$ zeros in $(0,1)$. By Corollary 3.4, we have that

$$\lim_{\alpha_l \to \infty} \|I_{M,\alpha_l}\| = 0, \tag{3.16}$$

where $\|\cdot\|$ is the Lebesgue measure. Since $f_\infty > \lambda_k$, we choose $\lambda > 0$ such that $\lambda_k < \lambda < f_\infty$ and take $M > 0$ such that

$$\frac{f(u(r;\alpha))}{|u(r;\alpha)|^{p-2}u(r;\alpha)} \ge \lambda \quad \text{for } |u(r;\alpha)| \ge M.$$

For each $\alpha > 0$, let $\phi(r;\alpha)$ and $\phi_k(r;\alpha)$ be Prüfer angles of the solutions of (1.4) and (2.1) with λ and λ_k, respectively. Then, $\phi_k(1;\alpha) = k\pi_p$, and hence, by the comparison theorem, $\phi(1;\alpha) = k\pi_p + \epsilon$ for some $\epsilon > 0$. Here, recall that $\phi(r;\alpha)$ satisfies

$$\begin{aligned} \phi'(r;\alpha) = {} & \frac{r^{n-1}}{p-1}(\lambda w(r) - q(r))|S_p(\phi(r;\alpha))|^p \\ & + r^{1-n/p-1}|S_p'(\phi(r;\alpha))|^p \equiv F(r;\alpha;\phi), \end{aligned} \tag{3.17}$$

from (3.6). On the other hand, define

$$g(r;\alpha) = \begin{cases} \dfrac{f(u(r;\alpha))}{|u(r;\alpha)|^{p-2}u(r;\alpha)} & |u(r;\alpha)| < M, \\ \lambda & |u(r;\alpha)| \ge M. \end{cases}$$

By (3.4),

$$\begin{aligned} \theta'(r;\alpha) \ge {} & \frac{r^{n-1}}{p-1}(w(r)g(r;\alpha) - q(r))|S_p(\theta(r;\alpha))|^p \\ & + r^{1-n/p-1}|S_p'\theta(r;\alpha)|^p \equiv H(r;\alpha;\theta). \end{aligned} \tag{3.18}$$

Let $\psi(r;\alpha)$ be the solution of equation

$$\psi'(r;\alpha) = H(r;\alpha;\psi), \tag{3.19}$$

satisfying $\psi(0;\alpha) = \pi_p/2$. Now, from (3.17)–(3.19), we obtain, for

$\alpha = \alpha_l$ and $r \in (0, 1]$,

$$\begin{aligned}\psi(r;\alpha) - \phi(r;\alpha) &= \int_0^r (H(s;\alpha;\psi) - F(s;\alpha;\phi))\, ds \\ &= \int_0^r [(H(s;\alpha;\psi) - F(s;\alpha;\psi)) \\ &\qquad + (F(s;\alpha;\psi) - F(s;\alpha;\phi))]\, ds \\ &= \int_0^r \frac{s^{n-1}}{p-1} w(s)[g(s;\alpha) - \lambda]|S_p(\psi(s;\alpha))|^p ds \\ &\quad + \int_0^r \frac{\partial}{\partial\phi} F(s;\alpha;\xi)[\psi(s;\alpha) - \phi(s;\alpha)]\, ds,\end{aligned}$$

where $\xi(s;\alpha)$ is between $\psi(s;\alpha)$ and $\phi(s;\alpha)$. By (3.16), we have

$$\begin{aligned}&\left| \int_0^r \frac{s^{n-1}}{p-1} w(s)[g(s;\alpha) - \lambda]|S_p(\psi(s;\alpha))|^p ds \right| \\ &\qquad\qquad \le \int_{I_{M,\alpha}} \frac{s^{n-1}}{p-1} w(s)|g(s;\alpha) - \lambda|\, ds \longrightarrow 0,\end{aligned}$$

as $\alpha = \alpha_l \to \infty$. Thus, for any $\delta > 0$, we choose α^* sufficiently large such that, for $\alpha = \alpha_l > \alpha^*$,

$$\left| \int_0^r \frac{s^{n-1}}{p-1} w(s)[g(s;\alpha) - \lambda]|S_p(\psi(s;\alpha))|^p ds \right| < \delta.$$

Note that $|(\partial/\partial\phi)F(s;\alpha;\xi)|$ is uniformly bounded by some constant $K > 0$ for all $r \in (0, 1]$ and $\alpha > \alpha^*$. Then, we have

$$|\psi(r;\alpha) - \phi(r;\alpha)| < \delta + \int_0^r K|\psi(s;\alpha) - \phi(s;\alpha)|\, ds \quad \text{for } \alpha > \alpha^*.$$

By Gronwall's inequality, we obtain

$$|\psi(r;\alpha) - \phi(r;\alpha)| < \delta e^{Kr} < \epsilon,$$

if $\delta < \epsilon e^{-K}$. Therefore,

$$\psi(1;\alpha) > \phi(1;\alpha) - \epsilon \quad \text{on } (0, 1].$$

Combining (3.18)–(3.19) and the above, for $\alpha = \alpha_l > \alpha^*$ and $r \in (0, 1]$, we obtain

$$\theta(1;\alpha) \ge \psi(1;\alpha) > \phi(1;\alpha) - \epsilon = k\pi_p.$$

Here, we have reached a contradiction.

(ii) The proof of (ii) is quite similar to that of (i). Here, we merely give a sketch. By assumption, we choose $\lambda > 0$ satisfying $f_\infty < \lambda < \lambda_k$ and take $M > 0$ large enough such that

$$\frac{f(u(r;\alpha))}{|u(r;\alpha)|^{p-2}u(r;\alpha)} \le \lambda \quad \text{for } |u(r;\alpha)| \ge M.$$

By the comparison theorem, we assume that $\phi(1;\alpha) = k\pi_p - \epsilon$ for some $\epsilon > 0$. Then, we define $g(r;\alpha)$ as in (i) and obtain

$$\theta'(r;\alpha) \le H(r;\alpha;\theta).$$

For $|u(r;\alpha)| < M$,

$$g(r;\alpha)|S_p(\theta(r;\alpha))|^p = \frac{f(u(r;\alpha))}{\rho(r;\alpha)} S_p(\theta(r;\alpha))$$

is uniformly bounded for sufficiently large α by Lemma 3.3 and (3.3). Hence, $\theta(r;\alpha)$ is uniformly bounded for sufficiently large α. By Corollary 3.4, $\lim_{\alpha\to\infty} \|I_{M,\alpha}\| = 0$. Then, by a similar discussion as in the proof of (i), we also obtain

$$\theta(1;\alpha) \le \psi(1;\alpha) < \phi(1;\alpha) + \epsilon = k\pi_p.$$

We omit the details. $\square$

4. Proofs of the main results. In this section, we give the proofs of Theorems 1.1, 1.2 and 1.4.

Proof of Theorem 1.1.

(i) Assume, on the contrary, that (1.1)–(1.2) has a solution $u(r)$ with exactly i zeros in $(0,1)$ for some $i \ge k-1$. Let

$$\overline{w}(r) = w(r)\frac{f(u(r))}{|u(r)|^{p-2}u(r)}.$$

Then $\overline{w}(r)$ is continuous on $[0,1]$ by the continuous extension since $f_0 < \infty$. Recall that $\theta(r)$ is the Prüfer angle of $u(r)$ with $\theta(0) = \pi_p/2$ and satisfies (3.4). By the above and hypothesis, we have

$$\theta'(r) < F(r;\lambda_k;\theta),$$

by (3.6). Applying the comparison lemma, we conclude that $\theta(1) < \phi_k(1) = k\pi_p$. This provides a contradiction.

(ii) Assume, on the contrary, that (1.1)–(1.2) has a solution $u(r)$ with exactly i zeros in $(0,1)$ for some $i \leq k-1$. By assumption, we have $(f(u(r)))/(u(r)^{p-1}) > \lambda_k$ whenever $u(r) \neq 0$. By (3.4), if r_n satisfies $\theta(r_n) = n\pi_p$, i.e., $u(r_n) = 0$, then $\theta'(r_n) > 0$ is valid. This implies that $|u(r)| > 0$ when $r > r_n$ and r is close to r_n. This means that $u(r)$ cannot vanish on any nontrivial subintervals of $(0,1)$. Hence, by (3.4) and (3.6) we obtain

$$\theta'(r) > F(r;\lambda_k;\theta) \quad \text{almost everywhere on } [0,1].$$

By the comparison lemma and a similar argument as in (i), we obtain

$$\theta(1) > \phi_k(1) = k\pi_p.$$

This provides a contradiction.

(iii) The assumption implies that, either:

(a) $\lambda_k < (f(u(r)))/(u(r)^{p-1}) < \lambda_{k+1}$ for some $k \in \mathbb{N}$ for all $u \in (0,\infty)$, or

(b) $0 < (f(u(r)))/(u(r)^{p-1}) < \lambda_1$ for all $u \in (0,\infty)$ if $k = 1$.

By uniqueness, the conclusion immediately follows from (i) and (ii). □

Proof of Theorem 1.2. Assume that $f_0 < f_\infty$. By Lemma 3.1 (i), there exists an $\alpha_* > 0$ such that $\theta(1;\alpha) < k\pi_p$ for all $\alpha \in (0,\alpha_*)$. By Lemma 3.5 (i), there exists an $\alpha^* > \alpha_*$ such that $\theta(1;\alpha) > k\pi_p$ for all $\alpha \in (\alpha^*,\infty)$. Since $\theta(1;\alpha)$ is continuous in α on $(0,\infty)$, there exists an $\alpha_k \in [\alpha_*,\alpha^*]$ such that $\theta(1;\alpha_k) = k\pi_p$. This implies that $u(r;\alpha_k)$ is a solution of (1.1)–(1.2) with exactly $k-1$ zeros in $(0,1)$. The case of $f_\infty < \lambda_k < f_0$ may be proved by using Lemma 3.1 (ii) and Lemma 3.5 (ii). We omit the proof here. □

Proof of Theorem 1.4.

(i) By Lemma 3.1 (i) and Lemma 3.5 (i), there exists a $0 < \alpha_* < \alpha^* < \infty$ such that

$$\theta(1;\alpha) < k\pi_p \quad \text{for } \alpha \in (0,\alpha_*)$$

and

$$\theta(1;\alpha) > (k+1)\pi_p \quad \text{for } \alpha \in (\alpha^*,\infty).$$

By the continuity of $\theta(r;\alpha)$ in α, there exists an $\alpha_* \le \alpha_k < \alpha_{k+1} \le \alpha^*$ such that

$$\theta(1;\alpha_k) = k\pi_p \quad \text{and} \quad \theta(1;\alpha_{k+1}) = (k+1)\pi_p, \tag{4.1}$$

and

$$k\pi_p < \theta(1;\alpha) < (k+1)\pi_p \quad \text{for } \alpha_k < \alpha < \alpha_{k+1}. \tag{4.2}$$

By (3.5) and (1.10)–(1.11), for $r \in (0,1]$, we obtain

$$\frac{\rho'(r;\alpha)}{\rho(r;\alpha)} \ge -r^{1-n/p-1} - \overline{N}. \tag{4.3}$$

Integrating (4.3) over $[r_i,1]$ for $1 \le i \le d$, by (C_1) and (1.11), we obtain

$$\begin{aligned} \ln\frac{\rho(1;\alpha)}{\rho(r_i;\alpha)} &\ge -\frac{p-1}{p-n}(1-r_i^{p-n/p-1}) - \overline{N}(1-r_i) \\ &\ge -\frac{p-1}{p-n} - \overline{N} = -\widehat{N}. \end{aligned}$$

Then,

$$\rho(r_i;\alpha) \le e^{\hat{N}}\rho(1;\alpha), \quad i = 1,2,3,\ldots,d. \tag{4.4}$$

By (3.3), we observe that, for $\alpha = \alpha_k$ and $\alpha = \alpha_{k+1}$,

$$(p-1)\rho^{p/(p-1)}(r_i;\alpha) \ge r_i^{p(n-1)/p-1}|u'(r_i;\alpha)|^p, \quad 1 \le i \le d,$$

and

$$(p-1)\rho^{p/(p-1)}(1;\alpha) = |u'(1;\alpha)|^p.$$

Thus, for $\alpha = \alpha_k$, $\alpha = \alpha_{k+1}$ and $1 \le i \le d$,

$$r_i^{n-1/p-1}|u'(r_i;\alpha)| \le \sqrt[p-1]{(p-1)^{(p-1)/p}\rho(r_i;\alpha)} \tag{4.5}$$

and

$$|u'(1;\alpha)| = \sqrt[p-1]{(p-1)^{(p-1)/p}\rho(1;\alpha)}.$$

Define

$$\Gamma(\alpha) = u'(1;\alpha) - \sum_{i=1}^{d} k_i r_i^{n-1/p-1} u'(r_i;\alpha). \tag{4.6}$$

Assume that $k = 2n-1$ for $n \in \mathbb{N}$. Note that

(4.7)
$$u'(1;\alpha_{2n-1}) = u'(1;\alpha_k) < 0 \quad \text{and} \quad u'(1;\alpha_{2n}) = u'(1;\alpha_{k+1}) > 0,$$

by (4.1) and (3.1). Now, applying (1.12) and (4.4)–(4.7), we obtain

$$\begin{aligned}
\Gamma(\alpha_{2n-1}) &= u'(1;\alpha_{2n-1}) - \sum_{i=1}^{d} k_i r_i^{n-1/p-1} u'(r_i;\alpha_{2n-1}) \\
&\leq - \sqrt[p-1]{(p-1)^{(p-1)/p}\rho(1;\alpha_{2n-1})} \\
&\quad + \sum_{i=1}^{d} |k_i| \sqrt[p-1]{(p-1)^{(p-1)/p}\rho(r_i;\alpha_{2n-1})} \\
&\leq - \sqrt[p-1]{(p-1)^{(p-1)/p}\rho(1;\alpha_{2n-1})} \\
&\quad + \sum_{i=1}^{d} |k_i| \sqrt[p-1]{(p-1)^{(p-1)/p} e^{\hat{N}}\rho(1;\alpha_{2n-1})} \\
&= \sqrt[p-1]{(p-1)^{(p-1)/p}\rho(1;\alpha_{2n-1})}\left(-1 + e^{\hat{N}/(p-1)} \sum_{i=1}^{d} |k_i|\right) \\
&< 0
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(\alpha_{2n}) &= u'(1;\alpha_{2n}) - \sum_{i=1}^{d} k_i r_i^{n-1/p-1} u'(r_i;\alpha_{2n}) \\
&\geq \sqrt[p-1]{(p-1)^{(p-1)/p}\rho(1;\alpha_{2n})} \\
&\quad - \sum_{i=1}^{d} |k_i| \sqrt[p-1]{(p-1)^{(p-1)/p}\rho(r_i;\alpha_{2n})} \\
&\geq \sqrt[p-1]{(p-1)^{(p-1)/p}\rho(1;\alpha_{2n})} \\
&\quad - \sum_{i=1}^{d} |k_i| \sqrt[p-1]{(p-1)^{(p-1)/p} e^{\hat{N}}\rho(1;\alpha_{2n})} \\
&= \sqrt[p-1]{(p-1)^{(p-1)/p}\rho(1;\alpha_{2n})}\left(1 - e^{\hat{N}/(p-1)} \sum_{i=1}^{d} |k_i|\right) > 0.
\end{aligned}$$

By the continuity of $\Gamma(\alpha)$, there exists an $\overline{\alpha} \in (\alpha_{2n-1}, \alpha_{2n})$ such that $\Gamma(\overline{\alpha}) = 0$ and similarly for the case of $k = 2n$ with $n \in \mathbb{N}$. In both cases, from (4.2),

$$k\pi_p < \theta(1; \overline{\alpha}) < (k+1)\pi_p.$$

Hence, the solution $u(r; \overline{\alpha})$ has k zeros in $(0, 1)$ and satisfies the multi-point boundary condition (1.7). The proof of (i) is complete.

(ii) The idea of this proof is basically the same as that of (i). First, we outline the modification. For the case of Neumann eigenvalues, by Lemmas 3.1 and 3.5, we similarly obtain

$$\theta(1; \alpha) < \left(k + \frac{1}{2}\right)\pi_p \quad \text{for } \alpha \in (0, \alpha_*)$$

and

$$\theta(1; \alpha) > \left(k + \frac{3}{2}\right)\pi_p \quad \text{for } \alpha \in (\alpha^*, \infty),$$

where α_* and α^* are positive numbers satisfying $0 < \alpha_* < \alpha^* < \infty$. Then, there exist $\alpha_* \leq \alpha_k < \alpha_{k+1} \leq \alpha^*$ such that

$$\theta(1; \alpha_k) = \left(k + \frac{1}{2}\right)\pi_p \quad \text{and} \quad \theta(1; \alpha_{k+1}) = \left(k + \frac{3}{2}\right)\pi_p,$$

and

$$\left(k + \frac{1}{2}\right)\pi_p < \theta(1; \alpha) < \left(k + \frac{3}{2}\right)\pi_p \quad \text{for } \alpha_k < \alpha < \alpha_{k+1}.$$

Thus, for $\alpha = \alpha_k$, $\alpha = \alpha_{k+1}$ and $1 \leq i \leq d$,

$$|u(r_i; \alpha)| \leq \sqrt[p-1]{\rho(r_i; \alpha)} \quad \text{and} \quad |u(1; \alpha)| = \sqrt[p-1]{\rho(1; \alpha)}.$$

Define

$$\Gamma(\alpha) = u(1; \alpha) - \sum_{i=1}^{d} k_i u(r_i; \alpha).$$

Assume that $k = 2n - 1$ for $n \in \mathbb{N}$. Similarly to arguments in (i), there exists an $\overline{\alpha} \in (\alpha_{2n-1}, \alpha_{2n})$ such that $\Gamma(\overline{\alpha}) = 0$. Then, we obtain

$$\left(k + \frac{1}{2}\right)\pi_p < \theta(1; \overline{\alpha}) < \left(k + \frac{3}{2}\right)\pi_p.$$

Hence, the solution $u(r;\overline{\alpha})$ has k or $k+1$ zeros in $(0,1)$ and satisfies the multi-point boundary condition (1.8). Therefore, the proof is complete. $\square$

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