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ON WEAK CONTINUITY OF THE MOSER FUNCTIONAL IN LORENTZ-SOBOLEV SPACES

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ABSTRACT. Let $B(R) \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, be an open ball. By a result from **[1]**, the Moser functional with the borderline exponent from the Moser-Trudinger inequality fails to be sequentially weakly continuous on the set of radial functions from the unit ball in $W_0^{1,n}(B(R))$, only in the exceptional case of sequences acting like a concentrating Moser sequence.

We extend this result into the Lorentz-Sobolev space $W_0^1 L^{n,q}(B(R))$, with $q \in (1, n]$, equipped with the norm

$$||\nabla u||_{n,q} := ||t^{1/n-1/q} |\nabla u|^*(t)||_{L^q((0,|B(R)|))}.$$

We also consider the case of a nontrivial weak limit and the corresponding Moser functional with the borderline exponent from the concentration-compactness alternative.

1. Introduction. Throughout the paper, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, $\mathcal{L}_n$ is the n-dimensional Lebesgue measure and $|\Omega|$ stands for $\mathcal{L}_n(\Omega)$. By ∇u, we denote the generalized gradient of a function u, and u^* is its non-increasing rearrangement. The space $W_0^{1,n}(\Omega)$ or $W_0^1 L^{n,q}(\Omega)$, with $q \in (1, \infty)$, stands for the closure of $C_0^\infty(\Omega)$ in $W^{1,n}(\Omega)$ or $W^1 L^{n,q}(\Omega)$, respectively. We use the standard notation $q' = q/(q-1)$ (with the convention that $\infty' = 1$ and $1' = \infty$).

For functions from $W_0^{1,n}(\Omega)$, the famous Moser-Trudinger inequality **[11]** concerning a classical embedding theorem by Trudinger **[13]** states that

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(1.1)
$$\sup_{||\nabla u||_{L^n(\Omega)}\le 1}\int_\Omega \exp((K|u(x)|)^{n'})\,dx\begin{cases}\le C(n,K,|\Omega|) & \text{when } K\le n\omega_n^{1/n},\\ =\infty & \text{when } K> n\omega_n^{1/n}.\end{cases}$$

The proof in the case of $K > n\omega_n^{1/n}$ easily follows from the properties of the Moser functions $m_s \in W_0^{1,n}(B(R))$, $s\in(0,1)$, defined by
(1.2)
$$m_s(x)=\begin{cases} n^{-1/n}\omega_n^{-1/n}\log^{1/n'}(1/t) & \text{for } |x|\in[0,sR],\\ n^{-1/n}\omega_n^{-1/n}\log^{-1/n}(1/t)\log(R/|x|) & \text{for } |x|\in[sR,R].\end{cases}$$

From (1.1) and the Vitali convergence theorem, see e.g., [**7,** page 187], it follows that, if $p<1$, then the functional

$$J_p(u)=\int_\Omega \exp((n\omega_n^{1/n}p|u(x)|)^{n'})\,dx$$

is sequentially weakly continuous on the unit ball in $W_0^{1,n}(\Omega)$, that is,

$$u_k\rightharpoonup u \quad\text{and}\quad ||\nabla u_k||_{L^n(\Omega)}\le 1 \Longrightarrow J_p(u_k)\longrightarrow J_p(u).$$

If $p\ge 1$, then it is well known and easy to check that the above implication is not true. Indeed, if $p>1$, and Ω contains the origin, we fix $R>0$ such that $B(R)\subset\Omega$, and we obtain $J_p(m_s)\to\infty$ as $s\to 0_+$, while, for every sequence $s_k\subset(0,1)$ such that $s_k\to 0$, we have $m_{s_k}\rightharpoonup 0$ and $J_p(0)=\mathcal{L}_n(\Omega)<\infty$ (in the case of $0\notin\Omega$, we use translated Moser functions). If $p=1$, fix $R>0$, set $\Omega=B(R)$ and check that there are $C_0>\mathcal{L}_n(B(R))=J_1(0)$ and $t_0\in(0,1)$ such that $J_1(m_s)\ge C_0$ for every $s\in(0,t_0)$.

The following characterization of the sequential weak continuity of the functional J_p for $p=1$ and $u_k\rightharpoonup 0$, where u_k are radial functions from $W_0^{1,n}(B(R))$, is given in the recent paper [**1**].

Theorem 1.1. *Let $n\in\mathbb{N}$, $n\ge 2$ and $R>0$. Suppose that $\{u_k\}\subset W_0^{1,n}(B(R))$ are radial functions such that $||\nabla u_k||_{L^n(B(R))}\le 1$ and $u_k\rightharpoonup 0$ in $W_0^{1,n}(B(R))$. If*

$$\limsup_{k\to\infty} J_1(u_k)>J_1(0),$$

then there are $\{u_{k_m}\} \subset \{u_k\}$ *and* $\{s_m\} \subset (0,1)$, $s_m \to 0$, *such that*

$$u_{k_m} - m_{s_m} \overset{m\to\infty}{\longrightarrow} 0 \quad in\ W_0^{1,n}(B(R)).$$

In fact, Theorem 1.1 gives some information only in the case of $u = 0$ almost everywhere; otherwise, i.e., when u is nontrivial, Theorem 1.2 and the Vitali convergence theorem imply that $\lim_{k\to\infty} J_1(u_k) = J_1(u)$.

Note that, in **[1]**, a more difficult version of Theorem 1.1 for non-radial functions on an open set $\Omega \subset \mathbb{R}^2$ is given. In that case, one must consider translated Moser sequences. It is an open problem whether some analogue of Theorem 1.1 for non-radial functions in the general dimension $n \geq 2$ holds.

If $p > 1$ and $u_k \rightharpoonup u$ (we do not mind whether u is trivial or not), then there are many sequences distant from $\{m_{s_k}\}$ such that $J_p(u_k) \to \infty$, while we always have $J_p(u) < \infty$ by the Trudinger embedding, (for example, if we fix any $\varrho \in [1, p)$ and consider $u_k = \varrho^{-(n-1)/n} m_{s_k}$, with $s_k \to 0$, then we can observe that $u_k \rightharpoonup 0$ in $W_0^{1,n}(B(R))$).

A natural question to ask is, "what occurs if the limit function u in Theorem 1.1 is nontrivial?" This question was answered in **[3]**. The result is as follows. If $0 \leq ||\nabla u||_{L^n(B(R))} < 1$, then there is a $P > 1$ depending on $||\nabla u||_{L^n(B(R))}$ such that the functional J_P behaves in a similar way as that in Theorem 1.1 while, for every $p < P$, we have $J_p(u_k) \to J_p(u)$ and, for every $p > P$, we generally do not have that $\{J_p(u_k)\}$ is a bounded sequence. On the other hand, if $||\nabla u||_{L^n(B(R))} = 1$, it is easy to see that $u_k \to u$ (in norm) and $J_p(u_k) \to J_p(u)$ for every $p \in \mathbb{R}$.

The above-mentioned constant P is the borderline exponent corresponding to the following result from **[5]** and **[9,** Theorem I.6, Remark I.18] which concerns one of the cases in the concentration-compactness alternative for the Moser-Trudinger inequality.

Theorem 1.2. *Let* $n \in \mathbb{N}$, $n \geq 2$, *and let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. Let* $\{u_k\} \subset W_0^{1,n}(\Omega)$ *be a sequence satisfying*

$$||\nabla u_k||_{L^n(\Omega)} \leq 1, \qquad u_k \rightharpoonup u \quad in\ W_0^{1,n}(\Omega)$$

and $u_k \to u$ *almost everywhere in* Ω, *for some non-trivial function*

$u \in W_0^{1,n}(\Omega)$. Set

$$\theta = ||\nabla u||^n_{L^n(\Omega)} \in (0,1] \quad \text{and} \quad P = (1-\theta)^{-1/n} \tag{1.3}$$

(where $P = \infty$ if $\theta = 1$). Then, for every $p < P$, there is a $C > 0$ such that

$$\int_\Omega \exp((n\omega_n^{1/n} p\,|u_k(x)|)^{n'})\,dx \leq C.$$

Moreover, such an upper bound for p is sharp.

It is natural to work with the functional J_p with $p = P$ in the version of Theorem 1.1 with a nontrivial weak limit. Indeed, if $p < P$, we can again use the Vitali convergence theorem. Furthermore, it is shown in **[5]** that, if we take a suitable function $u \in W_0^{1,n}(B(3R))$, and if we set

$$u_k = u + (1-\theta)^{1/n} m_{1/k},$$

then we have $||\nabla u_k||_{L^n(B(3R))} = 1$, $u_k \rightharpoonup u$ and $J_p(u_k) \to \infty$ for every $p > P$. Hence, for $p > P$, we can again construct many sequences such that $u_k \rightharpoonup u$ and $J_p(u_k) \to \infty$, while $J_p(u) < \infty$.

Now, let us recall the full statement of the main result of **[3]**.

Theorem 1.3. *Let $n \in \mathbb{N}$, $n \geq 2$ and $R > 0$. Let $\{u_k\} \subset W_0^{1,n}(B(R))$ be radial functions such that $||\nabla u_k||_{L^n(B(R))} \leq 1$ and $u_k \rightharpoonup u$ in $W_0^{1,n}(B(R))$. Let $\theta \in [0,1]$ and $P \in [1,\infty]$ be defined by (1.3). If $\theta < 1$ and*

$$\limsup_{k\to\infty} J_P(u_k) > J_P(u),$$

then there are $\{u_{k_m}\} \subset \{u_k\}$ and $\{s_m\} \subset (0,1)$, $s_m \to 0$, such that

$$u_{k_m} - u - (1-\theta)^{1/n} m_{s_m} \overset{m\to\infty}{\longrightarrow} 0 \quad \text{in } W_0^{1,n}(B(R)).$$

1.1. The Lorentz-Sobolev case. The aim of this paper is to extend Theorem 1.3 into Lorentz-Sobolev spaces $W_0^1 L^{n,q}(\Omega)$, with $q \in (1,n]$, equipped with the norm

$$||\nabla u||_{n,q} := ||t^{1/n-1/q}|\nabla u|^*(t)||_{L^q((0,|\Omega|))}. \tag{1.4}$$

Recall that the above quantity is not a norm but a quasi-norm for $q > n$.

The Moser-type inequality for Lorentz-Sobolev spaces $W_0^1L^{n,q}(\Omega)$ was obtained in **[2]**, and it has the following form. If $q \in (1,\infty)$, then

$$\sup_{||\nabla u||_{n,q}\le 1} \int_\Omega \exp((K|u(x)|)^{q'})\,dx \begin{cases} \le C(n,K,q,|\Omega|) & \text{when } K \le n\omega_n^{1/n}, \\ = \infty & \text{when } K > n\omega_n^{1/n} \end{cases}$$

and, if $q = \infty$, then

$$\sup_{||\nabla u||_{n,\infty}\le 1} \int_\Omega \exp(K|u(x)|)\,dx \begin{cases} \le C(n,K,|\Omega|) & \text{when } K < n\omega_n^{1/n}, \\ = \infty & \text{when } K \ge n\omega_n^{1/n}. \end{cases}$$

Note that, since $\infty' = 1$, the main difference between cases $q \in (1,\infty)$ and $q = \infty$ is the uniform boundedness of the integrals in the case $K = n\omega_n^{1/n}$ for $q \in (1,\infty)$. There is no Moser-type inequality for $q = 1$ since $W_0^1L^{n,1}(\Omega)$ is embedded into $L^\infty(\Omega)$.

We define the Moser functionals as

$$J_p(u) = \int_\Omega \exp((n\omega_n^{1/n}p|u(x)|)^{q'})\,dx \tag{1.5}$$

and, for a fixed $R > 0$, and every $s \in (0,1)$, we define the Moser function $m_s \in W_0^1L^{n,q}(B(R))$ by

$$m_s(x) = \begin{cases} n^{-1/q}\omega_n^{-1/n}\log^{(q-1)/q}(1/s) & \text{for } 0 \le |x| \le sR, \\ n^{-1/q}\omega_n^{-1/n}\log^{-1/q}(1/s)\log(R/|x|) & \text{for } sR \le |x| \le R. \end{cases} \tag{1.6}$$

Now, let us recall the result from **[4]** concerning the improvement of the Moser-Trudinger inequality in the case of a nontrivial weak limit.

Theorem 1.4. *Let $n \in \mathbb{N}$, $n \ge 2$, $q \in (1,\infty)$, and let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Let $u \in W_0^1L^{n,q}(\Omega)$ be a non-trivial function, and let $\{u_k\} \subset W_0^1L^{n,q}(\Omega)$ be a sequence such that*

$$||\nabla u_k||_{n,q} \le 1, \qquad u_k \rightharpoonup u \text{ in } W_0^1L^{n,q}(\Omega)$$

and $u_k \to u$ almost everywhere in Ω.

Set

$$P := \begin{cases} (1 - ||\nabla u||_{n,q}^q)^{-1/q} & \text{for } ||\nabla u||_{n,q} < 1, \\ \infty & \text{for } ||\nabla u||_{n,q} = 1. \end{cases}$$

If $q \in (1, n]$, *then, for every* $p < P$, *there is a* $C > 0$ *such that*

$$\int_\Omega \exp((n\omega_n^{1/n} p|u_k(x)|)^{q'})\,dx \le C \quad \textit{for every } k \in \mathbb{N}. \tag{1.7}$$

Moreover, the assumption $p < P$ *is sharp.*

If $q \in (n, \infty)$, *then there is a* $\widetilde{P} \in (1, P\,]$ *such that* (1.7) *holds for every* $p < \widetilde{P}$ *but not for* $\widetilde{P} = P$ *in general.*

Note that, in the case of $q \in (1, n]$ (in this case, the quantity (1.4) is a norm) the result is of the same type as Theorem 1.2. On the other hand, when $q \in (n, \infty)$, the fact that the quantity (1.4) is not weakly lower semicontinuous, entails some loss of integrability, see [**4,** Lemma 3.1].

Now, we shall state our new result concerning the sequential weak continuity of the functional J_P.

Theorem 1.5. *Let* $n \in \mathbb{N}$, $n \ge 2$, $q \in (1, n]$, *and let* $R > 0$. *Let* $\{u_k\} \subset W_0^1 L^{n,q}(B(R))$ *be a sequence of radial functions satisfying*

$$||\nabla u_k||_{n,q} \le 1 \quad \textit{and} \quad u_k \rightharpoonup u \textit{ in } W_0^1 L^{n,q}(B(R))$$

for some $u \in W_0^1 L^{n,q}(B(R))$. *Set*

$$\theta := ||\nabla u||_{n,q}^q \in [0, 1] \quad \textit{and} \quad P = (1-\theta)^{-1/q} \in [1, \infty]. \tag{1.8}$$

If $\theta < 1$ *and*

$$\limsup_{k\to\infty} J_P(u_k) > J_P(u),$$

then there are $\{u_{k_m}\} \subset \{u_k\}$ *and* $\{s_m\} \subset (0, 1)$, $s_m \to 0$, *such that*

$$u_{k_m} - u - (1-\theta)^{1/q} m_{s_m} \overset{m\to\infty}{\longrightarrow} 0 \quad \textit{in } W_0^1 L^{n,q}(B(R)).$$

It can easily be seen (again, with the aid of the Vitali convergence theorem) that, if $\theta = 1$, then $J_p(u_k) \to J_p(u)$ for every $p \in \mathbb{R}$.

We do not study the case of $q > n$ for two reasons. On one hand, we do not know the value of the borderline parameter $\widetilde{P}$ from Theorem 1.4. On the other hand, for $q > n$, the quantity (1.4) is not a norm and we lose such tools as the uniform convexity.

The paper is organized as follows. After Section 2, Preliminaries, we show that, if $q \leq n$, then the norm (1.4) is uniformly convex. Such a result was already proven by Halperin **[8]**; however, his definition of uniform convexity slightly differs from the classical one by Clarkson **[6]**, which is the definition that is useful for our purposes.

In Section 4, we derive some properties of the Moser functions from (1.6).

Section 5 contains construction and properties of a collection of auxiliary linear functionals that are used to estimate the distance from the Moser functions. Recall that, in paper **[1]** (the Sobolev case), the suitable functionals were

$$L_s(u) = \int_{B(R)} |\nabla m_s|^{n-2} \nabla m_s \cdot \nabla u \, dx, \quad s \in (0,1),$$

(for $n = 2$, it is merely a scalar product of the gradients) satisfying, in addition, an important property,

$$L_s(u) = \frac{h(Rs)}{g_s(Rs)},$$

where $h, g_s \colon (0, R) \mapsto \mathbb{R}$ are the one-dimensional representatives of radial functions u and m_s, respectively (the above identity is easily obtained using the definition of Moser functions (1.2) and the Newton formula). In the case of Lorentz-Sobolev spaces, the functionals must be modified so that they correspond to the norm (1.4). The resulting functionals are given in (5.1) (we also had to overcome the fact that the weight $t \mapsto t^{(q/n)-1}$ has a bit of wild behavior near the origin).

Note that our functionals do not use the non-increasing rearrangement (surprisingly, as it is involved in norm (1.4)). This defect is repaired by the fact that $-g_s'$ is positive and decreasing on (sR, R), and the Hardy-Littlewood inequality ensures that $L_s(u)$ is large only if $-h'$ behaves in a similar manner.

In Section 6, we present the conclusion of the proof of Theorem 1.5. The basic strategy of the proof is inspired by **[1]**; the problems arising when dealing with nontrivial limit functions are solved in the same way as **[3]**. However, some problems related to the non-increasing rearrangement involved in the norm (1.4) occurred, and the solution to these problems required some new ideas.

2. Preliminaries.

2.1. Notation. If u is a measurable function on Ω, then, by $u = 0$ (or $u \neq 0$), we mean that u is equal (or not equal) to the zero function almost everywhere on Ω.

By $B(x, R)$, we denote an open Euclidean ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $R > 0$. If $x = 0$, we simply write $B(R)$.

We write that $u_k \rightharpoonup u$ in $W_0^1 L^{n,q}(\Omega)$, $q \in (1, \infty)$, if

$$\int_\Omega \frac{\partial u_k}{\partial x_i}\, v\, dx \longrightarrow \int_\Omega \frac{\partial u}{\partial x_i}\, v\, dx,$$

for every $v \in L^{n',q'}(\Omega)$ and $i = 1, \ldots, n$.

By C, we denote a generic positive constant which may depend upon n, p, q and R. This constant may vary from expression to expression, as usual. Occasionally we use that, for every $\epsilon > 0$, something is true. Then, the constant C in such a case may also depend upon a fixed $\epsilon > 0$.

2.2. Non-increasing rearrangement. The non-increasing rearrangement f^* of a measurable function f on Ω is

$$f^*(t) = \sup\{s \geq 0 : |\{x \in \Omega : |f(x)| > s\}| > t\} \quad \text{for } t \in (0, \infty).$$

We shall use the Hardy-Littlewood inequality for measurable functions

$$\int_\Omega |f(x)g(x)|\, dx \leq \int_0^{|\Omega|} f^*(t) g^*(t)\, dt.$$

When dealing with a radial function u on $B(R)$, it is often convenient to work with its one-dimensional representative $h : (0, R) \mapsto [0, \infty)$, defined by

$$h(|x|) := u(x) \quad \text{for } 0 < |x| < R. \tag{2.1}$$

Remark 2.1. For every radial function $u \in W_0^{1,1}(\Omega)$, its one-dimensional representative h from (2.1) is locally absolutely continuous on $(0, R)$, and thus, differentiable almost everywhere.

Proof. The proof easily follows from the fact that every function from $W^{1,1}(\Omega)$ satisfies ACL, i.e., it is absolutely continuous on almost all lines parallel to coordinate axes, see **[10,** subsection 1.1.3]. □

Finally, let us recall an inequality obtained in **[12]**. If Ω is open and $u \in W_0^{1,1}(\Omega)$, then, for every $t \in (0, |\Omega|)$, we have

(2.2)
$$u^*(t) \leq \frac{1}{n\omega_n^{1/n}} = \left(t^{-1/n'}\int_0^t |\nabla u|^*(s)\,ds + \int_t^{|\Omega|} |\nabla u|^*(s)s^{-1/n'}\,ds\right).$$

If Ω is bounded, combining (2.2) with Hölder's inequality,

$$||\nabla u||_{n,q} \leq 1 \quad \text{and} \quad ((1/q - 1/n)q' + 1)(1/q') = (n-1)/n,$$

we obtain

(2.3)
$$\begin{aligned}
u^*(t) \leq \frac{1}{n\omega_n^{1/n}} \Bigg(t^{-1/n'} & \int_0^t |\nabla u|^*(s)s^{1/n-1/q}s^{1/q-1/n}\,ds \\
& + \int_t^{|\Omega|} |\nabla u|^*(s)s^{1/n-1/q}s^{(1/q)-1}\,ds\Bigg) \\
\leq \frac{1}{n\omega_n^{1/n}} \Bigg(t^{-1/n'} & \left(\int_0^t (|\nabla u|^*(s)s^{1/n-1/q})^q ds\right)^{1/q} \\
& \times \left(\int_0^t (s^{1/q-1/n})^{q'} ds\right)^{1/q'} \\
& + \left(\int_t^{|\Omega|} (|\nabla u|^*(s)s^{1/n-1/q})^q\,ds\right)^{1/q} \\
& \times \left(\int_t^{|\Omega|} (s^{1/q-1})^{q'} ds\right)^{1/q'}\Bigg) \\
\leq \frac{1}{n\omega_n^{1/n}} \Bigg(||\nabla u||_{n,q} & t^{-(n-1)/n} \left(\frac{1}{(1/q-1/n)q'+1}\left[s^{(1/q-1/n)q'+1}\right]_0^t\right)^{1/q'} \\
& + ||\nabla u||_{n,q}\left(\left[\log(s)\right]_t^{|\Omega|}\right)^{1/q'}\Bigg) \\
\leq C + \frac{1}{n\omega_n^{1/n}} & \log^{1/q'}\left(\frac{|\Omega|}{t}\right).
\end{aligned}$$

Note that (2.3) implies that, for any $\varepsilon > 0$, we have

$$u^*(t) \le \begin{cases} (1+\varepsilon)n^{-1}\omega_n^{-1/n}\log^{1/q'}(|\Omega|/t) & \text{for } t \text{ sufficiently small,} \\ C & \text{otherwise.} \end{cases}$$

3. Uniform convexity. Clarkson [**6**] defined uniform convexity as follows.

Definition 3.1. A Banach space is uniformly convex if, for every $\varepsilon > 0$, there is a $\delta > 0$ with the property that, if $||f|| = ||g|| = 1$ and $||f - g|| > \varepsilon$, then $||(f+g)/2|| < 1 - \delta$.

In this paper, uniform monotonicity is also used.

Definition 3.2. A Banach space is uniformly monotone if, for every $\varepsilon > 0$, there is an $\eta > 0$ with the property that, if $0 \le g \le f$, $||f|| = 1$ and $||g|| > \varepsilon$, then $||f - g|| < 1 - \eta$.

It is an easy exercise to show that uniform convexity implies uniform monotonicity.

Halperin [**8**] proved that Lorentz spaces have the following property.

Theorem 3.3. *Let $1 < q \le p < \infty$. For every $\varepsilon > 0$ and $\eta \in (0,1)$, there is a $\delta > 0$ with the property that, whenever two non-negative Lorentz functions satisfy $||u||_{p,q} = ||v||_{p,q} = 1$ and $(1-\eta)u(x) \ge v(x)$ in some set G with $||u\chi_G||_{p,q} > \varepsilon$, then $||(u+v)/2||_{p,q} < 1 - \delta$.*

Our aim is to prove that the Halperin property implies uniform convexity.

Corollary 3.4. *If $1 < q \le p < \infty$, then the Lorentz norm is uniformly convex.*

Proof. Step 1. *Uniform convexity for non-negative functions.* Fix $\varepsilon > 0$, set $\eta = \varepsilon$, and let $\delta > 0$ be the constant given by Theorem 3.3. Let u and v be two non-negative Lorentz functions satisfying $||u||_{p,q} = ||v||_{p,q} = 1$ and $||(u+v)/2||_{p,q} \ge 1-\delta$. Hence, the set $G_u := \{(1-\varepsilon)u \ge$

$v\}$ satisfies $||u\chi_{G_u}||_{p,q} \le \varepsilon$, and the set $G_v := \{(1-\varepsilon)v \ge u\}$ satisfies $||v\chi_{G_v}||_{p,q} \le \varepsilon$. Thus,

$$\begin{aligned} ||u-v||_{p,q} = {} & ||(u-v)\chi_{G_u}||_{p,q} + ||(v-u)\chi_{G_v}||_{p,q} \\ & + ||(u-v)\chi_{\Omega\setminus(G_u\cup G_v)}||_{p,q} \\ \le {} & ||u\chi_{G_u}||_{p,q} + ||v\chi_{G_v}||_{p,q} + ||\varepsilon u||_{p,q} + ||\varepsilon v||_{p,q} \\ \le {} & 4\varepsilon. \end{aligned}$$

Step 2. *Uniform monotonicity.* Fix $\varepsilon \in (0,1)$, let $\delta > 0$ be the number corresponding to $\varepsilon/2$ in the definition of uniform convexity (for non-negative functions), and let f and g be the same as in Definition 3.2. Set $u = f$ and $v = (f-g)/||f-g||_{p,q} \ge 0$. We assume that $||f-g||_{p,q} > 1 - (\varepsilon/2)$; otherwise, we complete the proof of the uniform monotonicity, setting $\eta = \varepsilon/2$.

Hence,

$$\begin{aligned} ||u-v||_{p,q} &= \left\| \left(1 - \frac{1}{||f-g||_{p,q}}\right) f + \frac{1}{||f-g||_{p,q}}\, g \right\|_{p,q} \\ &= \frac{1}{||f-g||_{p,q}}\, ||g - (1 - ||f-g||_{p,q})f||_{p,q} \\ &\ge ||g - (1 - ||f-g||_{p,q})f||_{p,q} \\ &\ge ||g||_{p,q} - (1 - ||f-g||_{p,q})||f||_{p,q} \\ &> \varepsilon - \frac{1}{2}\,\varepsilon = \frac{1}{2}\,\varepsilon. \end{aligned}$$

Thus, by uniform convexity for non-negative functions,

$$\begin{aligned} 1 - \delta > \left\| \frac{1}{2}\,(u+v) \right\|_{p,q} &= \frac{1}{||f-g||_{p,q}} \left\| \frac{||f-g\|_{p,q}+1}{2}\, f - \frac{1}{2}\, g \right\|_{p,q} \\ &\ge \frac{1}{||f-g||_{p,q}} \left\| \frac{||f-g||_{p,q}+1}{2}\, f - \frac{||f-g||_{p,q}+1}{2}\, g \right\|_{p,q} \\ &= \frac{||f-g||_{p,q}+1}{2}. \end{aligned}$$

Therefore, $||f-g||_{p,q} < 1 - 2\delta$, and we can set $\eta = 2\delta$.

Step 3. *Uniform convexity for general functions.* Fix $\varepsilon > 0$, and let $\eta > 0$ be the number from Step 2 corresponding to $\varepsilon/3$. Let u

and v be two Lorentz functions satisfying $||u||_{p,q} = ||v||_{p,q} = 1$ and $||u-v||_{p,q} > \varepsilon$. We distinguish between two cases.

Case 1. If $||\,|u|-|v|\,||_{p,q} > \min\{\varepsilon/6, \eta/2\}$, then, by Step 1, we can use the uniform convexity for non-negative functions $|u|, |v|$ to obtain $\delta > 0$ such that

$$1-\delta > \left\|\frac{1}{2}(|u|+|v|)\right\|_{p,q} \geq \left\|\frac{1}{2}(u+v)\right\|_{p,q}$$

This completes the first case.

Case 2. Let $||\,|u|-|v|\,||_{p,q} \leq \min\{\varepsilon/6, \eta/2\}$. We simply suppose that $u \geq 0$ and $v = v_+ - v_-$, where $v_+, v_- \geq 0$. To simplify notation, write $u = u_1 + u_2$, where $u_1 = u\chi_{\{v\geq 0\}}$ and $u_2 = u\chi_{\{v<0\}}$. Now, as $||\,|u|-|v|\,||_{p,q} \leq \varepsilon/6$, we have

$$\begin{aligned}\varepsilon &< ||u-v||_{p,q}\\ &= ||u_1+u_2-v_++v_-||_{p,q}\\ &\leq ||u_1-v_+||_{p,q} + ||u_2-v_-||_{p,q} + 2||v_-||_{p,q}\\ &\leq \frac{\varepsilon}{6}+\frac{\varepsilon}{6}+2||v_-||_{p,q}.\end{aligned}$$

Hence, $||v_-||_{p,q} > \varepsilon/3$, and thus, uniform monotonicity implies $||v_+||_{p,q} \leq 1-\eta$, see the first line of Step 3. Therefore, as $||\,|u|-|v|\,||_{p,q} \leq \eta/2$,

$$\begin{aligned}\left\|\frac{1}{2}\,(u+v)\right\|_{p,q} &= \frac{1}{2}\,||u_1+u_2+v_+-v_-||_{p,q}\\ &\leq \frac{1}{2}\,(||u_1-v_+||_{p,q}+2||v_+||_{p,q}+||u_2-v_-||_{p,q})\\ &\leq \frac{1}{2}\left(\frac{\eta}{2}+2(1-\eta)+\frac{\eta}{2}\right)\\ &= 1-\frac{1}{2}\,\eta.\end{aligned}$$

This completes the second case. $\square$

Remark 3.5. If $1 \leq p < q \leq \infty$, then the Lorentz quasi-norm is not uniformly convex.

Proof. Fix $\delta > 0$ very small. Define $u, v : [0,1) \mapsto [0,\infty)$ by

$$u(t) = \begin{cases} 1+\delta & \text{for } t \in [0,1/2), \\ 1-\delta & \text{for } t \in [1/2,1) \end{cases}$$

and

$$v(t) = \begin{cases} 1-\delta & \text{for } t \in [0,1/2), \\ 1+\delta & \text{for } t \in [1/2,1). \end{cases}$$

Straightforwardly, we have $(u+v)/2 \equiv 1$.

Now, let us estimate the quasi-norms.

Case $q \in (p,\infty)$. We have:

$$\left\| \frac{u+v}{2} \right\|_{p,q}^q = \int_0^1 t^{(q/p)-1}\, dt = \frac{p}{q}\, [t^{q/p}]_0^1 = \frac{p}{q}$$

and

$$\begin{aligned} ||u||_{p,q}^q = ||v||_{p,q}^q &= \int_0^{1/2} t^{(q/p)-1}(1+\delta)^q dt + \int_{1/2}^1 t^{(q/p)-1}(1-\delta)^q dt \\ &= \frac{p}{q}\,(1+\delta)^q [t^{q/p}]_0^{1/2} + \frac{p}{q}\,(1-\delta)^q [t^{q/p}]_{1/2}^1 \\ &= \frac{p}{q}\left((1+\delta)^q \left(\frac{1}{2}\right)^{q/p} + (1-\delta)^q \left(1 - \left(\frac{1}{2}\right)^{q/p} \right) \right). \end{aligned}$$

Hence, if $\delta > 0$ is small enough, we have

$$\begin{aligned} &\frac{q}{p}\left(||u||_{p,q}^q - \left\| \frac{u+v}{2} \right\|_{p,q}^q \right) \\ &\qquad = (1+\delta)^q \left(\frac{1}{2}\right)^{q/p} + (1-\delta)^q \left(1 - \left(\frac{1}{2}\right)^{q/p} \right) - 1 \\ &\qquad = \left(\frac{1}{2}\right)^{q/p} (1 + q\delta - 1 + q\delta + \mathrm{o}(\delta)) + 1 - q\delta + \mathrm{o}(\delta) - 1 \\ &\qquad = q\delta \left(\left(\frac{1}{2}\right)^{(q/p)-1} - 1 + \mathrm{o}(1) \right). \end{aligned}$$

Now, since $(1/2)^{(q/p)-1} < 1$, setting $\widetilde{u} = u/||u||_{p,q}$ and $\widetilde{v} = v/||v||_{p,q}$, we obtain $\|\widetilde{u}\| = \|\widetilde{v}\| = 1$ and $||(\widetilde{u}+\widetilde{v})/2|| > 1$. Nevertheless, uniform convexity requires the last number to be bounded away from 1 from below.

Case $q=\infty$. In this case, we easily see that

$$\left\|\frac{u+v}{2}\right\|_{p,\infty}=\sup_{t\in(0,1)} t^{q/p}=1,$$

and, if δ is small enough, we obtain

$$||u||_{p,\infty}=||v||_{p,\infty}=\max\left\{\sup_{t\in(0,1/2)} t^{1/p}(1+\delta),\ \sup_{t\in(1/2,1)} t^{1/p}(1-\delta)\right\}=1-\delta.$$

Thus, the proof is finished. □

It is a well-known fact that, if a sequence converges weakly in a uniformly convex Banach space, that is, $u_k\rightharpoonup u$ and $||u_k||\to||u||$ (where $\|\cdot\|$ is a norm in this space), then $u_k\to u$ (strong norm convergence). We shall need a slight modification of this property.

Lemma 3.6. *In every uniformly convex Banach space the following assertion holds. For every $\varepsilon>0$, there is a $\delta\in(0,1)$ such that*

$$u_k\rightharpoonup u,\qquad ||u||=1,\qquad ||u_k||\le 1+\delta \text{ for every } k$$
$$\Longrightarrow ||u_k-u||<\varepsilon \text{ for every } k \text{ sufficiently large.}$$

Proof. The proof is straightforward. □

4. Moser functions. In this section, we study properties of the Moser functions defined by (1.6). We begin with the estimate of the Dirichlet norm. We have

$$(4.1)\quad |\nabla m_s|(x)=\begin{cases}0 & \text{for } 0\le|x|<sR,\\ n^{-1/q}\omega_n^{-1/n}\log^{-1/q}(\frac{1}{s})1/|x| & \text{for } sR<|x|<R,\end{cases}$$

and thus,

$$(4.2)\quad |\nabla m_s|^*(t)=\begin{cases}n^{-1/q}\omega_n^{-1/n}\log^{-1/q}(\frac{1}{s})1/((t/\omega_n)+s^nR^n)^{1/n}\\ \qquad \text{for } 0<t<\omega_nR^n-\omega_ns^nR^n,\\ 0 \qquad \text{for } \omega_nR^n-\omega_ns^nR^n<t<\omega_nR^n;\end{cases}$$

indeed, the value of $|\nabla m_s|^*(t)$ corresponds to the value of $|\nabla m_s|$ on the sphere $\partial B(\varrho)$, where $\varrho>0$ satisfies $t=|B(\varrho)|-|B(sR)|=$

$\omega_n(\varrho^n - s^n R^n)$. Hence,

$$\begin{aligned}\|\nabla m_s\|_{n,q}^q &= \int_0^{\omega_n R^n} t^{q/n}(|\nabla m_s|^*(t))^q \frac{dt}{t} \\ &= \frac{1}{n}\log^{-1}\left(\frac{1}{s}\right)\int_0^{\omega_n R^n-\omega_n s^n R^n} t^{q/n}\left(\frac{1}{t+\omega_n s^n R^n}\right)^{q/n}\frac{dt}{t}.\end{aligned}$$

Applying the change of variables $t = \omega_n s^n R^n y$ we infer for $s > 0$ small enough such that $\log(\log(1/s)) > 0$,

$$\begin{aligned}\|\nabla m_s\|_{n,q}^q &= \log^{-1}\left(\frac{1}{s^n}\right)\int_0^{s^{-n}-1}\left(\frac{y}{y+1}\right)^{q/n}\frac{dy}{y} \\ &= \log^{-1}\left(\frac{1}{s^n}\right)\left(\int_0^{\log(\log(1/s))} + \int_{\log(\log(1/s))}^{s^{-n}-1}\right) \\ &= \log^{-1}\left(\frac{1}{s^n}\right)(I_1+I_2).\end{aligned} \tag{4.3}$$

Next,

$$0 < I_1 < \int_0^{\log(\log(1/s))} y^{(q/n)-1}\,dy = \frac{n}{q}\log^{q/n}\left(\log\left(\frac{1}{s}\right)\right),$$

and thus, $I_1 \log^{-1}(1/s^n) \to 0$ as $s \to 0_+$.

For the second integral, we have

$$I_2 \le \int_{\log(\log(1/s))}^{s^{-n}-1}\frac{dy}{y} = \log(s^{-n}-1) - \log\left(\log\left(\log\left(\frac{1}{s}\right)\right)\right)$$

and contrariwise,

$$\begin{aligned}I_2 &\ge \left(\frac{\log(\log(1/s))}{\log(\log(1/s))+1}\right)^{q/n}\int_{\log(\log(1/s))}^{s^{-n}-1}\frac{dy}{y} \\ &= \left(1-\frac{1}{\log(\log(1/s))+1}\right)^{q/n}\left(\log(s^{-n}-1)-\log\left(\log\left(\log\left(\frac{1}{s}\right)\right)\right)\right).\end{aligned}$$

Thus, $I_2 \log^{-1}(1/s^n) \to 1$ as $s \to 0_+$. Hence, we obtain from (4.3),

$$\|\nabla m_s\|_{n,q} \overset{s\to 0_+}{\longrightarrow} 1. \tag{4.4}$$

Note that, by a minor modification of the above procedure, it can be shown that

(4.5)
$$\begin{aligned}\int_{\omega_n s^n \log^n(1/s)R^n-\omega_n s^n R^n}^{\omega_n R^n-\omega_n s^n R^n} & t^{q/n}(|\nabla m_s|^*(t))^q \frac{dt}{t} \\ &= \log^{-1}\left(\frac{1}{s^n}\right)\int_{\log^n(1/s)-1}^{s^{-n}-1}\left(\frac{y}{y+1}\right)^{q/n}\frac{dy}{y} \overset{s\to 0_+}{\longrightarrow} 1.\end{aligned}$$

Furthermore, as $|\nabla m_s|^*$ is uniformly bounded for s bounded away from zero, see (4.2), we obtain from (4.4),

$$\|\nabla m_s\|_{n,q} \le C \quad \text{for every } s \in (0,1). \tag{4.6}$$

It can also be seen from (1.6) and (4.1) that the Moser functions concentrate at the origin in the following sense:

$$\eta > 0 \Longrightarrow \sup_{\eta<|x|<R} |\nabla m_s(x)| \overset{s\to 0_+}{\longrightarrow} 0 \quad \text{and} \quad \sup_{\eta<|x|<R} |m_s(x)| \overset{s\to 0_+}{\longrightarrow} 0.$$

We also have, for any sequence $s_k \in (0,1)$, $s_k \to 0$,

$$m_{s_k} \longrightarrow 0 \quad \text{on } B(R)\setminus\{0\} \quad \text{and} \quad m_{s_k} \rightharpoonup 0 \quad \text{in } W_0^1 L^{n,q}(B(R)).$$

5. Linear functionals. In this section, we use the following notation. The function $h\colon (0,R)\mapsto \mathbb{R}$ is the one-dimensional representative of a radial function $u \in W_0^1 L^{n,q}(B(R))$, and functions $g_s\colon (0,R)\mapsto\mathbb{R}$, $s\in(0,1)$, represent m_s, that is,

$$h(|x|) = u(x) \quad \text{and} \quad g_s(|x|) = m_s(x)$$
$$\text{for } x \in B(R)\setminus\{0\}.$$

For every $s \in (0,1/e)$, define a linear functional L_s acting on the radial function $u \in W_0^1 L^{n,q}(B(R))$ by

(5.1)
$$\begin{aligned} L_s(u) = \int_{\omega_n s^n \log^n(1/s)R^n-\omega_n s^n R^n}^{\omega_n R^n-\omega_n s^n R^n} & t^{q/n}\left|g_s'\left(\left(\frac{t}{\omega_n}+s^n R^n\right)^{1/n}\right)\right|^{q-2} \\ & \times g_s'\left(\left(\frac{t}{\omega_n}+s^n R^n\right)^{1/n}\right) h'\left(\left(\frac{t}{\omega_n}+s^n R^n\right)^{1/n}\right)\frac{dt}{t}.\end{aligned}$$

Changing the variables so that $z = ((t/\omega_n) + s^n R^n)^{1/n}$ (hence, $t = \omega_n z^n - \omega_n s^n R^n$ and $dt/dz = n\omega_n z^{n-1}$) and using (1.6) and (4.1), we infer

(5.2)
$$
\begin{aligned}
L_s(u) &= \int_{s\log(1/s)R}^{R} (\omega_n z^n - \omega_n s^n R^n)^{q/n-1} |g_s'(z)|^{q-2} g_s'(z) h'(z) n\omega_n z^{n-1} dz \\
&= -\int_{s\log(1/s)R}^{R} (\omega_n z^n - \omega_n s^n R^n)^{q/n-1} \\
&\quad \times \left(n^{-1/q} \omega_n^{-1/n} \log^{-1/q}\left(\frac{1}{s}\right) \frac{1}{z} \right)^{q-1} h'(z) n\omega_n z^{n-1} dz \\
&= -\omega_n^{1/n} n^{1/q} \log^{-(q-1)/q}\left(\frac{1}{s}\right) \int_{s\log(1/s)R}^{R} \left(1 - \frac{s^n R^n}{z^n}\right)^{q/n-1} h'(z)\, dz \\
&= -\frac{1}{g_s(sR)} \int_{s\log(1/s)R}^{R} \left(1 - \frac{s^n R^n}{z^n}\right)^{q/n-1} h'(z)\, dz.
\end{aligned}
$$

Hence, we have (recall that $q \le n$ and $h(z) \to 0$ as $z \to R_-$)

(5.3)
$$
\begin{aligned}
h\left(s\log\left(\frac{1}{s}\right) R \right) &= -\int_{s\log(1/s)R}^{R} h'(z)\, dz \\
&\le L_s(u) g_s(sR) = L_s(u) n^{-1/q} \omega_n^{-1/n} \log^{(q-1)/q}\left(\frac{1}{s}\right).
\end{aligned}
$$

On the other hand, from (5.2), we also have

$$
\begin{aligned}
h\left(s\log\left(\frac{1}{s}\right) R \right) &\ge \left(1 - \frac{s^n R^n}{(s\log(1/s)R)^n}\right)^{1-q/n} L_s(u) g_s(sR) \\
&\ge \left(1 - \frac{1}{\log^n(1/s)}\right) L_s(u) g_s(sR).
\end{aligned} \tag{5.4}
$$

Let ψ be the inverse function to $s \mapsto s\log(1/s)$ on $(0, 1/e)$. From (5.3), we obtain

(5.5)

$$\begin{aligned} h(sR) &\le L_{\psi(s)}(u) n^{-1/q} \omega_n^{-1/n} \log^{(q-1)/q}\left(\frac{1}{\psi(s)}\right) \\ &\le L_{\psi(s)}(u) n^{-1/q} \omega_n^{-1/n} \log^{(q-1)/q}\left(\frac{1}{s}\right)\left(1 + \frac{C}{\log^{1/2}(1/s)}\right). \end{aligned}$$

Indeed, for $s > 0$ very small, we have

$$s\log^{-2}(1/s) < \psi(s) < s\log^{-1}(1/s),$$

and thus,

$$\begin{aligned} \log\left(\frac{1}{\psi(s)}\right) &\le \log\left(\frac{1}{s\log^{-2}(1/s)}\right) = \log\left(\frac{1}{s}\right) + \log\left(\log^2\left(\frac{1}{s}\right)\right) \\ &= \log\left(\frac{1}{s}\right) + 2\log\left(\log\left(\frac{1}{s}\right)\right) \\ &= \log\left(\frac{1}{s}\right)\left(1 + \frac{2\log(\log(1/s))}{\log(1/s)}\right) \\ &\le \log\left(\frac{1}{s}\right)\left(1 + \frac{1}{\log^{1/2}(1/s)}\right). \end{aligned}$$

Lemma 5.1. *$L_s(u) \le C\|\nabla u\|_{n,q}$ and $L_s(u) \le (1 + \mathrm{o}(s))\|\nabla u\|_{n,q}$ as $s \to 0_+$.*

Proof. Using (4.4), (5.1), Hölder's inequality, $q \le n$, the relation between $|\nabla m_s|$ and $|\nabla m_s|^*$ (compare (4.1) and (4.2)) and the Hardy-Littlewood inequality, we obtain

$$\begin{aligned} L_s(u) \le & \int_0^{\omega_n R^n} t^{(q/n)-1} \left| g_s'\left(\left(\frac{t}{\omega_n} + s^n R^n\right)^{1/n}\right)\right|^{q-1} \\ & \times \left| h'\left(\left(\frac{t}{\omega_n} + s^n R^n\right)^{1/n}\right)\right| dt \\ = & \int_0^{\omega_n R^n} \left(t^{q/n-1}\right)^{(q-1)/q} (|\nabla m_s|^*(t))^{q-1} \left(t^{q/n-1}\right)^{1/q} \end{aligned}$$

$$
\begin{aligned}
&\times \left| h'\left(\left(\frac{t}{\omega_n} + s^n R^n \right)^{1/n} \right) \right| dt \\
&\leq \|\nabla m_s\|_{n,q}^{q-1} \left(\int_0^{\omega_n R^n} t^{q/n-1} \left| h'\left(\left(\frac{t}{\omega_n} + s^n R^n \right)^{1/n} \right) \right|^q dt \right)^{1/q} \\
&\leq \|\nabla m_s\|_{n,q}^{q-1} \left(\int_0^{\omega_n R^n} t^{q/n-1} ((|\nabla u|\chi_{B(R)\setminus B(sR)})^*(t))^q dt \right)^{1/q} \\
&= \|\nabla m_s\|_{n,q}^{q-1} \|\nabla u\|_{n,q}.
\end{aligned}
$$

Now, the results follow from (4.4) and (4.6), respectively. □

Lemma 5.2. *For every fixed radial Lorentz-Sobolev function u, we have $L_s(u) \to 0$ as $s \to 0_+$.*

Proof. Fix $\varepsilon > 0$. By the absolute continuity of the Lebesgue integral, we can find $\tau \in (0,1)$ so small that

$$
\int_0^\tau t^{(q/n)-1} (|\nabla u|^*(t))^q dt < \varepsilon^q. \tag{5.6}
$$

Next, from (4.1) and (5.1),

$$
\begin{aligned}
|L_s(u)| &\leq \int_{\omega_n s^n \log^n(1/s) R^n - \omega_n s^n R^n}^{\omega_n R^n - \omega_n s^n R^n} t^{(q/n)-1} \left(\frac{n^{-1/q} \omega_n^{-1/n} \log^{-1/q}(1/s)}{(t/\omega_n + s^n R^n)^{1/n}} \right)^{q-1} \\
&\quad \times \left| h'\left(\left(\frac{t}{\omega_n} + s^n R^n \right)^{1/n} \right) \right| dt \leq C \log^{-(q-1)/q} \left(\frac{1}{s} \right) \\
&\quad \times \int_{\omega_n s^n \log^n(1/s) R^n - \omega_n s^n R^n}^{\omega_n R^n - \omega_n s^n R^n} t^{(1/n)-1} \left| h'\left(\left(\frac{t}{\omega_n} + s^n R^n \right)^{1/n} \right) \right| dt \\
&= C \log^{-(q-1)/q} \left(\frac{1}{s} \right) \left(\int_{\omega_n s^n \log^n(1/s) R^n - \omega_n s^n R^n}^{\tau} + \int_\tau^{\omega_n R^n - \omega_n s^n R^n} \right) \\
&= C \log^{-(q-1)/q} \left(\frac{1}{s} \right) (I_1 + I_2).
\end{aligned}
$$

From (5.6), Hölder's inequality and the fact that the non-increasing rearrangement of the function $|h'((\cdot/\omega_n + s^n R^n)^{1/n})|$ is $(|\nabla u|\chi_{B(R)\setminus B(sR)})^*$, we obtain for s sufficiently small,

$$
\begin{aligned}
I_1 &\le \int_{\omega_n s^n \log^n(1/s) R^n - \omega_n s^n R^n}^{\tau} t^{1/n-1} |\nabla u|^*(t)\, dt \\
&\le \left(\int_0^{\tau} (|\nabla u|^*(t) t^{(1/n)-(1/q)})^q dt \right)^{1/q} \\
&\quad \times \left(\int_{\omega_n s^n \log^n(1/s) R^n - \omega_n s^n R^n}^{\tau} (t^{1/q-1})^{q'} dt \right)^{1/q'} \\
&\le \varepsilon ([\log(t)]_{\omega_n s^n \log^n(1/s) R^n - \omega_n s^n R^n}^{\tau})^{1/q'} \le C\varepsilon \log^{1/q'} \left(\frac{1}{s}\right).
\end{aligned}
$$

For the second integral, use finiteness of the Dirichlet norm of u and Hölder's inequality in order to obtain

$$
\begin{aligned}
I_2 &\le \int_{\tau}^{\omega_n R^n - \omega_n s^n R^n} t^{1/n-1} |\nabla u|^*(t)\, dt \\
&\le \left(\int_0^{\omega_n R^n} (|\nabla u|^*(t) t^{1/n-(1/q)})^q dt \right)^{1/q} \left(\int_{\tau}^{\omega_n R^n} (t^{1/q-1})^{q'} dt \right)^{1/q'} \\
&\le C([\log(t)]_{\tau}^{\omega_n R^n})^{1/q'} = C \log^{1/q'} \left(\frac{1}{\tau}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|L_s(u)| &\le C \log^{-(q-1)/q} \left(\frac{1}{s}\right) (I_1 + I_2) \\
&\le C\varepsilon + C \log^{-(q-1)/q} \left(\frac{1}{s}\right) \log^{1/q'} \left(\frac{1}{\tau}\right),
\end{aligned}
$$

and the result follows easily. □

Lemma 5.3. *If $u_k \rightharpoonup u$, $\|\nabla u_k\|_{L^{n,q}} \le 1$ and $\limsup_{k\to\infty} J_P(u_k) > J_P(u)$, then there is a subsequence $\{u_{k_m}\} \subset \{u_k\}$ and a sequence $\{s_m\} \subset (0,1)$ such that $s_m \to 0$ and (recall that $\theta = ||\nabla u||_{n,q}^q$)*

$$
\liminf_{m\to\infty} L_{s_m}(u_{k_m}) \ge (1-\theta)^{1/q}.
$$

Proof. Proceed by contradiction. Suppose that there are $\delta > 0$, $k_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$
L_s(u_k) \le (1-2\varepsilon)(1-\theta)^{1/q} \tag{5.7}
$$

for every $s < 2\delta$ and every $k \geq k_0$. Furthermore, suppose that δ is small enough such that

$$(1-2\varepsilon)\left(1+\frac{C}{\log^{1/2}(1/\delta)}\right) \leq 1-\varepsilon, \tag{5.8}$$

where the constant C is a fixed number obtained from (5.5).

Passing to a subsequence, also suppose that $u_k \to u$ almost everywhere in $B(R)$. By (1.5) and (1.8), we have

$$\begin{aligned}
J_P(u_k) &= \int_{B(R)} \exp\left((n\omega_n^{1/n}P|u_k|)^{q'}\right) dx \\
&= n\omega_n \int_0^R \exp\left((n\omega_n^{1/n}(1-\theta)^{-1/q}h_k(y))^{q'}\right)y^{n-1}dy \\
&= n\omega_n R^n \int_0^1 \exp\left((n\omega_n^{1/n}(1-\theta)^{-1/q}h_k(Rz))^{q'}\right)z^{n-1}dz \\
&= n\omega_n R^n \left(\int_0^\delta + \int_\delta^1\right).
\end{aligned}$$

We shall obtain a common majorant for the above integrands. First, from (5.5) and Lemma 5.1, we obtain

$$\exp((n\omega_n^{1/n}(1-\theta)^{-1/q}h_k(Rz))^{q'})\, z^{n-1} \leq C \quad \text{for } z \in (\delta, 1).$$

Thus, we have obtained an integrable majorant on $(\delta, 1)$ in a straightforward manner.

The next computation based on (5.5), (5.7) and (5.8) gives us an integrable majorant on $(0, \delta)$, and it also proves the integrability of the majorant. We have

$$\begin{aligned}
&\int_0^\delta \exp\left((n\omega_n^{1/n}(1-\theta)^{-1/q}h_k(Rz))^{q'}\right)z^{n-1}dz \\
&\quad \leq \int_0^\delta \exp\left(\left(n\omega_n^{1/n}(1-\theta)^{-1/q}L_{\psi(z)}(u_k)n^{-1/q}\omega_n^{-1/n}\log^{(q-1)/q}\left(\frac{1}{z}\right)\right.\right. \\
&\qquad \left.\left.\times\left(1+\frac{C}{\log^{1/2}(1/z)}\right)\right)^{q'}\right)z^{n-1}dz \\
&\quad = \int_0^\delta \exp\left(n(1-\theta)^{-q'/q}L_{\psi(z)}^{q'}(u_k)\log\left(\frac{1}{z}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\times \left(1 + \frac{C}{\log^{1/2}(1/z)}\right)^{q'}\Bigg) z^{n-1} dz \\
&\le \int_0^\delta \exp\left(n(1-2\varepsilon)^{q'} \log\left(\frac{1}{z}\right)\left(1 + \frac{C}{\log^{1/2}(1/z)}\right)^{q'}\right) z^{n-1} dz \\
&\le \int_0^\delta \exp\left(n(1-\varepsilon)^{q'} \log\left(\frac{1}{z}\right)\right) z^{n-1} dz \\
&\le \int_0^\delta z^{n-1-n(1-\varepsilon)^{q'}} z \le C.
\end{aligned}
$$

Hence, an integrable majorant has been obtained, and thus, the Lebesgue dominated convergence theorem can be used in order to obtain $J_P(u_k) \to J_P(u)$, a contradiction. □

Lemma 5.4. *Let* $\{s_k\} \subset (0,1)$, $s_k \to 0$, *and let* $\{u_k\} \subset W_0^1 L^{n,q}(B(R))$ *be radial functions satisfying* $||\nabla u_k||_{n,q} \le (1 + \mathrm{o}(1))$. *If* $L_{s_k}(u_k) \to 1$, *then*

$$u_k - m_{s_k} \longrightarrow 0 \quad \textit{in } W_0^1 L^{n,q}(B(R)).$$

Proof. The proof easily follows from uniform convexity of the norm $|| \cdot ||_{n,q}$ applied to gradients of functions u_k and m_{s_k}. We now give the details.

First, from Lemma 5.1, we infer that

$$||\nabla u_k||_{n,q} \longrightarrow 1.$$

Now, since $L_{s_k}(m_{s_k}) \to 1$ (see (4.5) and (5.1)) and $||\nabla m_{s_k}||_{n,q} \to 1$ (see (4.4)) from $L_{s_k}(u_k) \to 1$ and $||\nabla u_k||_{n,q} \to 1$, we obtain

$$
\begin{aligned}
&L_{s_k}\left(\frac{(m_{s_k}/||\nabla m_{s_k}||_{n,q}) + (u_k/||\nabla u_k||_{n,q})}{2}\right) \\
&\qquad = \frac{1}{2}\left(L_{s_k}\left(\frac{m_{s_k}}{||\nabla m_{s_k}||_{n,q}}\right) + L_{s_k}\left(\frac{u_k}{||\nabla u_k||_{n,q}}\right)\right) \longrightarrow 1.
\end{aligned}
$$

Combining this result with Lemma 5.1 and the triangle inequality, we obtain

$$\left\| \frac{(\nabla m_{s_k}/||\nabla m_{s_k}||_{n,q}) + (\nabla u_k/||\nabla u_k||_{n,q})}{2} \right\|_{n,q} \longrightarrow 1.$$

Therefore, uniform convexity of the norm $||\cdot||_{n,q}$ implies

$$\left\|\frac{\nabla m_{s_k}}{||\nabla m_{s_k}||_{n,q}} - \frac{\nabla u_k}{||\nabla u_k||_{n,q}}\right\|_{n,q} \longrightarrow 0.$$

Finally, since $||\nabla m_{s_k}||_{n,q} \to 1$ and $||\nabla u_k||_{n,q} \to 1$, we have

$$\begin{aligned}
&||\nabla m_{s_k} - \nabla u_k||_{n,q} \\
&\quad \le \left\|\nabla m_{s_k} - \frac{\nabla m_{s_k}}{||\nabla m_{s_k}||_{n,q}}\right\|_{n,q} + \left\|\frac{\nabla m_{s_k}}{||\nabla m_{s_k}||_{n,q}} - \frac{\nabla u_k}{||\nabla u_k||_{n,q}}\right\|_{n,q} \\
&\qquad + \left\|\frac{\nabla u_k}{||\nabla u_k||_{n,q}} - \nabla u_k\right\|_{n,q} \longrightarrow 0.
\end{aligned}$$

Thus, we are done. □

6. Proof of Theorem 1.5.

Proof of Theorem 1.5. The strategy of the proof was taken from the proof of [**3,** Theorem 1.3], which has three steps. However, there are still some technical difficulties that must be overcome. These occur in the case where the limit function u is nontrivial, and they are caused by the fact that, in the Lebesgue spaces for two functions with disjoint support, we have

$$||f+g||_p = (||f||_p^p + ||g||_p^p)^{1/p},$$

while a corresponding formula does not hold in Lorentz spaces in general. However, it was observed [**4**] that, if $\{f_k\}$ is a concentrating sequence and $\{g_k\}$ is a sequence with nice behavior, then we have

$$||f_k + g_k||_{n,q} - (||f_k||_{n,q}^q + ||g_k||_{n,q}^q)^{1/q} \overset{k\to\infty}{\longrightarrow} 0.$$

This principle is used in the proofs of inequalities (6.6) and (6.7) below. The proof is divided into five parts. Moreover, inequalities (6.6) and (6.7), which belong to Step 2, are proved separately, at the end of the proof of Theorem 1.5.

Assume that $\{u_k\}$ satisfies the assumptions of Theorem 1.5, $\theta \in [0,1)$ and $\limsup_{k\to\infty} J_P(u_k) > J_P(u)$. Passing to a subsequence, suppose that the limit exists and $\lim_{k\to\infty} J_P(u_k) > J_P(u)$. Again, passing to a subsequence, also suppose that $u_k \to u$ in $L^{n,q}(\Omega)$ and $u_k \to u$ almost everywhere in Ω.

Define the truncation operators T^L and T_L acting on any function $v \in W_0^1 L^{n,q}(\mathrm{B(R)})$ by

$$T^L(v) = \min\{|v|, L\} \operatorname{sign}(v) \quad \text{and} \quad T_L(v) = v - T^L(v).$$

Note that the weak convergence $u_k \rightharpoonup u$ implies $T^L(u_k) \rightharpoonup T^L(u)$ and $T_L(u_k) \rightharpoonup T_L(u)$ (indeed, $T^L(u_k)$ is bounded; hence, it has a weakly convergent subsequence, and the convergence almost everywhere implies that the weak limit must be $T^L(u)$, similarly for $T_L(u_k)$). We often use the following, simple observation. Since $q \le n$, we have that $t \mapsto t^{(q/n)-1}$ is non-increasing on $(0, \infty)$, and thus,

(6.1)
$$\begin{aligned}\int_0^{|B(R)|} t^{q/n-1}(|\nabla v|^*(t))^q dt \le \int_0^{|B(R)|} & t^{q/n-1}(|\nabla T_L(v)|^*(t))^q dt \\ & + \int_0^{|B(R)|} t^{q/n-1}(|\nabla T^L(v)|^*(t))^q dt.\end{aligned}$$

Step 1. Using Lemma 5.3, we find a sequence $\{s_k\} \subset (0,1)$, $s_k \to 0$, such that

$$\liminf_{k\to\infty} L_{s_k}(u_k) \ge (1-\theta)^{1/q}, \tag{6.2}$$

(passing to a subsequence of $\{u_k\}$, if necessary). Next, inequality (6.2) and Lemma 5.2 imply

$$\liminf_{k\to\infty} L_{s_k}(u_k - u) \ge (1-\theta)^{1/q}. \tag{6.3}$$

Step 2. Here, we prove

$$\limsup_{k\to\infty} ||\nabla(u_k - u)||_{n,q} \le (1-\theta)^{1/q}. \tag{6.4}$$

If $\theta = 0$, the proof trivially follows from the assumption $||\nabla u_k||_{n,q} \le 1$, $k \in \mathbb{N}$. Thus, let us suppose that $\theta \in (0,1)$ in the rest of this step, as well as in the proofs of inequalities (6.6) and (6.7).

Fix $\varepsilon > 0$. Also fix $L > 0$ large enough such that

$$\int_0^{|B(R)|} t^{q/n-1}(|\nabla T_L(u)|^*(t))^q dt = \tau, \tag{6.5}$$

where $\tau \in (0, \min\{\theta, 1-\theta\}/8)$ is a small number, specified below.

We have

$$
\begin{aligned}
||\nabla(u_k - u)||_{n,q} &\le ||\nabla(T^L(u_k) - T^L(u))||_{n,q} \\
&\quad + ||\nabla T_L(u_k)||_{n,q} + ||\nabla T_L(u)||_{n,q} \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

If τ is small enough, then (6.5) implies that $I_3 < \varepsilon$.

Next, we claim that, for k large enough, the following inequality holds

$$
\int_0^{|B(R)|} t^{q/n-1}(|\nabla T_L(u_k)|^*(t))^q dt \le 1 - \theta + 3\tau. \tag{6.6}
$$

We postpone the proof of (6.6). From (6.6), we observe that, if τ is small enough, we also have $I_2 < (1-\theta)^{1/q} + \varepsilon$.

We now proceed to the proof that $I_1 < \varepsilon$. This is based on Lemma 3.6 (recall that we have $T^L(u_k) \rightharpoonup T^L(u)$). Since the norm is homogeneous, we have by (1.8), (6.1) and (6.5),

$$
\theta^{1/q} = ||\nabla u||_{n,q} \ge ||\nabla T^L(u)||_{n,q} \ge (||\nabla u||_{n,q}^q - ||\nabla T_L(u)||_{n,q}^q)^{1/q} = (\theta - \tau)^{1/q},
$$

and, since τ is small enough, it remains to prove (so that Lemma 3.6 implies $||\nabla(T^L(u_k) - T^L(u))||_{n,q} < \varepsilon$)

$$
||\nabla T^L(u_k)||_{n,q} \le (\theta + \zeta)^{1/q}, \tag{6.7}
$$

where ζ is a small number dependent upon ε. We postpone the proof of (6.7).

Thus, when (6.6) and (6.7) are proved, we shall obtain

$$
I_1 + I_2 + I_3 \le \varepsilon + (1-\theta)^{1/q} + \varepsilon + \varepsilon,
$$

which concludes the proof of (6.4). □

Step 3. Our aim is to prove

$$
(1-\theta)^{-1/q}(u_k - u) - m_{s_k} \overset{k\to\infty}{\longrightarrow} 0 \quad \text{in } W_0^1 L^{n,q}(B(R)). \tag{6.8}
$$

Combining (6.3) and (6.4) with Lemma 5.1, we obtain

$$
L_{s_k}((1-\theta)^{-1/q}(u_k - u)) \overset{k\to\infty}{\longrightarrow} 1
$$

and

$$||(1-\theta)^{-1/q}\nabla(u_k - u)||_{n,q} \overset{k\to\infty}{\longrightarrow} 1.$$

Now, Lemma 5.4 concludes the proof of (6.8).

In order to complete the proof of Theorem 1.5, it remains to prove inequalities (6.6) and (6.7).

Proof of (6.6). This proof is based upon the method used in [**4,** Proof of Theorem 1.3 (iii)]. We omit several detailed computations; we recall the main ideas for the convenience of the reader.

First, by (6.1) and (6.5), we have $||\nabla T^L(u)||^q_{n,q} \geq \theta - \tau$. Thus, we can use absolute continuity of the Lebesgue integral to obtain $\sigma \in (0, |B(R)|)$ small enough such that

$$\int_\sigma^{|B(R)|} t^{(q/n)-1}(|\nabla T^L(u)|^*(t))^q dt \geq \theta - 2\tau. \tag{6.9}$$

Next, we decompose the interval $[\sigma, |B(R)|]$ into very short subintervals $[a_{j-1}, a_j]$, $j = 1, \ldots, m$, so that the function $t \mapsto t^{(q/n)-1}$ is extremely close to a constant on each subinterval. Furthermore, let G_j, $j = 1, \ldots, m$, be disjoint measurable subsets of $B(R)$ satisfying $|G_j| = a_j - a_{j-1}$ and chosen so that the values of $|\nabla T^L(u)|$ on G_j correspond to the values of $|\nabla T^L(u)|^*$ on $[a_{j-1}, a_j]$. Let G be the union of these sets. Now, using (6.9), the weak lower semicontinuity of the L^q-norm (since $q \leq n$, we have that $L^{n,q}$ is embedded into L^q) on each set G_j, the fact that $t \mapsto t^{(q/n)-1}$ is almost constant on each $[a_{j-1}, a_j]$ and the Hardy-Littlewood inequality, we obtain

$$\int_\sigma^{|B(R)|} t^{q/n-1}(|\nabla(T^L(u_k)|_G)|^*(t-\sigma))^q dt \geq \theta - 3\tau, \tag{6.10}$$

for k large enough.

Finally, by the Chebyshev inequality, if L is large enough, then $|\operatorname{supp} T_L(u_k)| < \sigma$ for every $k \in \mathbb{N}$. This property, the Hardy-Littlewood inequality, and the fact that $t \mapsto t^{(q/n)-1}$ is non-increasing, imply

$$\begin{aligned} \int_0^{|B(R)|} & t^{q/n-1}(|\nabla u_k|^*(t))^q dt \\ & \geq \int_0^{\sigma} t^{q/n-1}(|\nabla(T_L(u_k))|^*(t))^q dt \\ & \quad + \int_{\sigma}^{|B(R)|} t^{q/n-1}(|\nabla(T^L(u_k))|^*(t-\sigma))^q dt. \end{aligned} \tag{6.11}$$

Now, (6.6) follows from (6.10), (6.11) and the assumption $||\nabla u_k||_{n,q} \leq 1$. □

Proof of (6.7). We restrict to the case $q < n$ (we are not concerned with $q = n$ since the Sobolev case of Theorem 1.5 is part of Theorem 1.3).

Fix $\zeta > 0$. By uniform monotonicity of the Lorentz norm, we may find $\gamma > 0$ small enough such that

$$\begin{aligned} 0 \leq g \leq f, \qquad ||f||_{n,q} = 1, \qquad ||g||_{n,q} > \left(\frac{\zeta}{2}\right)^{1/q} \\ \Longrightarrow ||f-g||_{n,q} < (1-3\gamma)^{1/q}. \end{aligned} \tag{6.12}$$

Also suppose that

$$\gamma \leq \frac{\zeta}{4}. \tag{6.13}$$

Next, since we have (6.2), $T_L(u_k) \geq u_k - L$ and, since an additive constant is irrelevant for the behavior of L_s with s very small (observe (5.3) and (5.4)), we obtain, for k large enough,

$$L_{s_k}(T_L(u_k)) \geq (1-\theta-\tau)^{1/q}.$$

Thus, by Lemma 5.1, we have for k large enough,

$$||\nabla T_L(u_k)||_{n,q} \geq (1-\theta-2\tau)^{1/q}. \tag{6.14}$$

Recall that $\tau > 0$ is a very small number and can be made as small as desired.

Next, let $\xi > 0$ be small enough such that

$$\int_0^{|B(R)|} t^{q/n-1} \xi^q dt \leq \gamma,$$

and, for every $k \in \mathbb{N}$, set

$$\varrho_k := |\{|\nabla T_L(u_k)| > \xi\}|.$$

From the choice of ξ and (6.14), we may infer

$$\int_0^{\varrho_k} t^{q/n-1} (|\nabla T_L(u_k)|^*(t))^q dt \geq 1 - \theta - 2\tau - \gamma. \tag{6.15}$$

Next, we claim that, by passing to a subsequence, we obtain $\varrho_k \to 0$ as $k \to \infty$. We prove this claim by contradiction. Suppose that there are $\varrho_0 > 0$ and $k_0 \in \mathbb{N}$ such that $\varrho_k > \varrho_0$ for every $k > k_0$. This implies, for every $k > k_0$,

$$\int_{\varrho_0/2}^{|B(R)|} t^{q/n-1} (|\nabla T_L(u_k)|^*(t))^q dt \geq \int_{\varrho_0/2}^{\varrho_0} t^{q/n-1} \xi^q dt = C.$$

Thus, (6.6) yields

$$\int_0^{\varrho_0/2} t^{q/n-1} (|\nabla T_L(u_k)|^*(t))^q dt \leq 1 - \theta + 3\tau - C,$$

which means that, if τ is sufficiently small, then there is a $\beta \in (0, 1/2)$ such that, for every $k > k_0$,

$$\int_0^{\varrho_0/2} t^{q/n-1} (|\nabla T_L(u_k)|^*(t))^q dt \leq (1-\theta)(1-2\beta). \tag{6.16}$$

Now, follow the computation in (2.3), where the integral over (t, Ω) is decomposed into the integral over $(t, \varrho_0/2)$ and the integral over $(\varrho_0/2, |\Omega|)$, and apply estimate (6.16) after Hölder's inequality when estimating the integral over $(t, \varrho_0/2)$. We obtain, for t small enough,

$$\begin{aligned} u_k^*(t) &\leq L + (T_L(u_k))^*(t) \\ &\leq L + C + \frac{((1-\theta)(1-2\beta))^{1/q}}{n\omega_n^{1/n}} \log^{1/q'}\left(\frac{|\Omega|}{t}\right) + C \\ &= C + \frac{((1-\theta)(1-2\beta))^{1/q}}{n\omega_n^{1/n}} \log^{1/q'}\left(\frac{|\Omega|}{t}\right). \end{aligned}$$

Therefore, we have an integrable majorant of the integrand of $J_P(u_k)$. Indeed, for suitably small $t_0 > 0$, we have

$$\begin{aligned}
J_P(u_k) &= \int_{B(R)} \exp((n\omega_n^{1/n} P|u_k|)^{q'})\, dx \\
&= \int_0^{|B(R)|} \exp((n\omega_n^{1/n}(1-\theta)^{-1/q} u_k^*(t))^{q'})\, dt \\
&= \int_0^{|B(R)|} \exp\left(\left(C + (1-2\beta)^{1/q} \log^{1/q'}\left(\frac{|\Omega|}{t}\right)\right)^{q'}\right) dt \\
&\leq \int_0^{t_0} \exp\left(\left((1-\beta)^{1/q} \log^{1/q'}\left(\frac{|\Omega|}{t}\right)\right)^{q'}\right) dt + \int_{t_0}^{|B(R)|} \exp(C)\, dt \\
&= C\int_0^{t_0} t^{-(1-\beta)^{q'/q}}\, dt + C.
\end{aligned}$$

Thus, $J_P(u_k) \to J_P(u)$ by the Lebesgue dominated convergence theorem. This is a contradiction, and thus, we can pass to a subsequence in order to obtain $\varrho_k \to 0$ as $k \to \infty$.

Next, fix $D > 1$ large enough such that

$$\left(\frac{D}{D+1}\right)^{q/n-1} \leq 1 + \tau. \tag{6.17}$$

Now, use (6.10), (6.15) and the Hardy-Littlewood inequality to obtain

$$\begin{aligned}
1 \geq ||\nabla u_k||_{n,q}^q \geq \int_0^{\varrho_k} & t^{q/n-1}(|\nabla T_L(u_k)|^*(t))^q dt \\
&+ \int_\sigma^{|B(R)|} t^{q/n-1}(|\nabla T^L(u_k)|^*(t-\sigma))^q dt \geq 1 - 5\tau - \gamma.
\end{aligned} \tag{6.18}$$

Next, a trivial estimate

$$\int_0^{D\varrho_k} t^{q/n-1}(|\nabla T^L(u_k)|^*(t))^q dt \leq \int_0^{|B(R)|} t^{q/n-1}(|\nabla u_k|^*(t))^q dt \leq 1$$

implies, for k large enough (note that $(q/n)-1 < 0$; in our case, $q < n$ and $D\varrho_k$ is much smaller than σ for k large),

$$\int_\sigma^{\sigma + D\varrho_k} t^{q/n-1}(|\nabla T^L(u_k)|^*(t-\sigma))^q dt < \gamma. \tag{6.19}$$

Hence, if $\tau < \gamma/6$, we obtain, from (6.18) and (6.19),

$$1 \geq \int_0^{\varrho_k} t^{q/n-1}(|\nabla T_L(u_k)|^*(t))^q dt + \int_{\sigma+D\varrho_k}^{|B(R)|} t^{q/n-1}(|\nabla T^L(u_k)|^*(t-\sigma))^q dt \geq 1-3\gamma. \tag{6.20}$$

Therefore, we may use the uniform monotonicity (6.12) to obtain

$$|||\nabla T^L(u_k)|^* \chi_{(0,D\varrho_k)}||_{n,q} < \left(\frac{\zeta}{2}\right)^{1/q}. \tag{6.21}$$

Equation (6.12) was applied to

$$f = \frac{|\nabla u_k|}{||\nabla u_k||_{n,q}} \quad \text{and} \quad g = \frac{|\nabla T^L(u_k)|\chi_G}{||\nabla u_k||_{n,q}} = \frac{|\nabla u_k|\chi_G}{||\nabla u_k||_{n,q}},$$

where the set G was chosen so that the values of $|\nabla T^L(u_k)|$ on G correspond to the values of $|\nabla T^L(u_k)|^*$ on $(0, D\varrho_k)$. With this setting, we have $f-g = |\nabla u_k|\chi_{B(R)\setminus G}/||\nabla u_k||_{n,q}$ and (6.20) implies $\||\nabla u_k|\chi_{B(R)\setminus G}\|_{n,q}^q \geq 1-3\gamma$. The normalization by $1/||\nabla u_k||_{n,q}$ is harmless since $1-3\gamma \leq ||\nabla u_k||_{n,q}^q \leq 1$.

By the Hardy-Littlewood inequality, we also have

$$\begin{aligned} 1 \geq ||\nabla u_k||_{n,q}^q \geq &\int_0^{\varrho_k} t^{(q/n)-1}(|\nabla T_L(u_k)|^*(t))^q dt \\ &+ \int_{\varrho_k}^{(D+1)\varrho_k} t^{(q/n)-1}(|\nabla T^L(u_k)|^*(t-\varrho_k))^q dt \\ &+ \int_{(D+1)\varrho_k}^{\varrho_k+|\operatorname{supp}\nabla T^L(u_k)|} t^{(q/n)-1}(|\nabla T^L(u_k)|^*(t-\varrho_k))^q dt \\ &+ \int_{\varrho_k+|\operatorname{supp}\nabla T^L(u_k)|}^{|B(R)|} t^{(q/n)-1}(|\nabla T_L(u_k)|^*(t-|\operatorname{supp}\nabla T^L(u_k)|))^q dt. \end{aligned}$$

Thus, by (6.15), for the third summand on the right hand side, we obtain

$$\int_{(D+1)\varrho_k}^{\varrho_k+|\operatorname{supp}\nabla T^L(u_k)|} t^{(q/n)-1}(|\nabla T^L(u_k)|^*(t-\varrho_k))^q dt \leq \theta+2\tau+\gamma. \tag{6.22}$$

Finally, from (6.13), (6.17), (6.21), (6.22), $(q/n)-1<0$ and $\tau<\gamma/6$, we infer

$$\begin{aligned}
&||\nabla T^L(u_k)||_{n,q}^q\\
&\quad=\int_0^{D\varrho_k} t^{q/n-1}(|\nabla T^L(u_k)|^*(t))^q dt+\int_{D\varrho_k}^{|B(R)|} t^{q/n-1}(|\nabla T^L(u_k)|^*(t))^q dt\\
&\quad\le \frac{\zeta}{2}+\int_{(D+1)\varrho_k}^{|B(R)|+\varrho_k}(t-\varrho_k)^{q/n-1}(|\nabla T^L(u_k)|^*(t-\varrho_k))^q dt\\
&\quad\le \frac{\zeta}{2}+\left(\frac{D}{D+1}\right)^{q/n-1}\int_{(D+1)\varrho_k}^{|B(R)|+\varrho_k} t^{q/n-1}(|\nabla T^L(u_k)|^*(t-\varrho_k))^q dt\\
&\quad\le \frac{\zeta}{2}+(1+\tau)(\theta+2\tau+\gamma)\le\frac{\zeta}{2}+\theta+2\tau+\gamma+\tau(1+2+1)\\
&\quad\le \theta+\frac{\zeta}{2}+2\gamma\le\theta+\zeta.
\end{aligned}$$

This is (6.7), which completes the proof of Theorem 1.5. $\square$

REFERENCES

1. Tintarev K. Adimurthi, *On compactness in the Trudinger-Moser inequality*, Ann. Scuola Norm. **13** (2014), 399–416.

2. A. Alvino, V. Ferone and Q. Trombetti, *Moser-type inequalities in Lorentz spaces*, Potent. Anal. **5** (1996), 273–299.

3. R. Černý, *Concentration-compactness principle for generalized Moser-Trudinger inequalities*: *Characterization of the non-compactness in the radial case*, Math. Inequal. Appl. **17** (2014), 1245–1280.

4. ______, *Concentration-compactness principle for Moser-type inequalities in Lorentz-Sobolev spaces*, Potent. Anal. **43** (2015), 97–126.

5. R. Černý, A. Cianchi and S. Hencl, *Concentration-compactness principle for Moser-Trudinger inequalities*: *New results and proofs*, Ann. Mat. Pura Appl. **192** (2013), 225–243.

6. J.A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

7. G.B. Folland, *Real analysis*: *Modern techniques and their applications*, John Wiley & Sons, Inc., New York, 1999.

8. I. Halperin, *Uniform convexity in function spaces*, Duke Math. J. **21** (1954), 195–204.

9. P.L. Lions, *The concentration-compactness principle in the calculus of variations, The limit case*, I, Rev. Mat. Iber. **1** (1985), 145–201.

10. V. Maz'ya, *Sobolev spaces*, Springer-Verlag, Berlin, 1985.

11. J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

12. G. Talenti, *An inequality between* u^* *and* ∇u, Int. Numer. Math. **103**, Birkhäuser, Basel, 1992.

13. N.S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484.

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