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CONTENT FORMULAS FOR POWER SERIES AND KRULL DOMAINS

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ABSTRACT. Let R be an integral domain with quotient field K, and let X be an indeterminate over R. In this paper, we consider content formulae for power series in terms of $*$-operations for PVMDs, Krull domains and Dedekind domains, where $*$ is the star-operation, d, w, t, or v. We prove that R is a Krull domain if and only if $c(f/g)_w = (c(f)c(g)^{-1})_w$ for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal if and only if $c(f/g)_t = (c(f)c(g)^{-1})_t$ for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal, and R is a Dedekind domain if and only if for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal, $c(f/g) = c(f)c(g)^{-1}$.

1. Introduction. Throughout this paper, R denotes an integral domain with quotient field K. Let $R^* = R - \{0\}$, and let $F(R)$ be the set of nonzero fractional ideals of R. For $A \in F(R)$, set $A^{-1} = \{x \in K \mid xA \subseteq R\}$. A star operation $*$ on R is a mapping $I \to I_*$ of $F(R)$ into $F(R)$ such that, for all $0 \neq a \in K$ and all $A, B \in F(R)$,

(i) $(a)_* = (a)$, $(aA)_* = aA_*$,
(ii) $A \subseteq A_*$ and $A \subseteq B$ implies $A_* \subseteq B_*$, and
(iii) $(A_*)_* = A_*$.

An ideal $A \in F(R)$ is called a $*$-ideal if $A_* = A$ and is called $*$-invertible if $(AA^{-1})_* = R$. Examples of star-operations are the d-, t- and v-operations, which are well-known star operations and are defined

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in the following way. The d-operation is $A_d = A$, the t-operation is $A_t = \bigcup B_v$ and the v-operation is $A_v = (A^{-1})^{-1}$, where B ranges over nonzero finitely generated subideals of A. The w-operation on R is defined by

$$A_w = \{x \in K \mid Jx \subseteq A \text{ for some finitely generated ideal } J \text{ with } J^{-1} = R\},$$

and it gives another example of a star operation **[10]**. For $A \in F(R)$, we have $A \subseteq A_w \subseteq A_t \subseteq A_v$.

Let X be an indeterminate over R. For a Laurent power series $f \in K[[X]][X^{-1}]$, the content $c(f)$ is the R-submodule of K generated by the coefficients of f. Note that $c(f)$ is not necessarily a fractional ideal of R. In general, $c(f)$ is a fractional ideal of R if and only if $f \in R[[X]]_{R^*}[X^{-1}]$. It is clear that $f/g \in K[[X]][X^{-1}]$ for all nonzero $f, g \in K[[X]][X^{-1}]$, and so $c(f/g)$ can be defined. Here $c(f/g)$ need not be a fractional ideal. Recall that $0 \neq f \in R[X]$ is called $*$-*Gaussian* if $c(fg)_* = (c(f)c(g))_*$ for all nonzero $g \in R[X]$. If each nonzero $f \in R[X]$ is $*$-Gaussian, we say that R is $*$-Gaussian. It is well known that R is d-Gaussian if and only if R is a Prüfer domain and that R is v-Gaussian (equivalently, t-Gaussian) if and only if R is integrally closed. Recall that an integral domain R is a Prüfer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of R is t-invertible (or equivalently, w-invertible). It was shown **[5]** that R is w-Gaussian if and only if R is a PVMD. In **[2]**, Anderson and Kang considered the power series analogues of those results for d-, t- and v-operations, and gave the conditions equivalent to $c(fg)_v = (c(f)c(g))_v$ for all $f \in R[[X]]^*$ and (linear) $g \in R[X]^*$ and $c(fg)_v = (c(f)c(g))_v$ for all $f, g \in R[[X]]^*$, respectively. They proved that:

(1) R is completely integrally closed $\Leftrightarrow c(fg)_v = (c(f)c(g))_v$ for all $f \in R[[X]]^*$ and (linear) $g \in R[X]^* \Leftrightarrow c(f/g)_v = (c(f)c(g)^{-1})_v$ for all $f \in R[[X]]^*$ and (linear) $g \in R[X]^*$ with $c(f/g)$ a fractional ideal.

(2) $c(fg)_v = (c(f)c(g))_v$ for all $f, g \in R[[X]]^* \Leftrightarrow c(f/g)_v = (c(f)c(g)^{-1})_v$ for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal.

Motivated by these results, we consider the $*$-version of each of the above content formulas, where $*$ is the star-operation, d, t, v or w. We give new characterizations of Krull domains and Dedekind domains: R is a Krull domain $\Leftrightarrow c(f/g)_w = (c(f)c(g)^{-1})_w$ for all $f, g \in R[[X]]^*$

with $c(f/g)$ a fractional ideal $\Leftrightarrow c(f/g)_t = (c(f)c(g)^{-1})_t$ for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal; R is a Dedekind domain $\Leftrightarrow$ for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal, $c(f/g) = c(f)c(g)^{-1}$. As a consequence, it follows that the equivalence in (2) does not hold for $* = d$, t, or w. However, it is shown that the second equivalence in (1) holds for $* = d$ or w.

2. Content formulae on star operations. Recall that a star operation $*$ is said to have finite character if, for each $A \in F(R)$,

$$A_* = \bigcup \{B_* \mid 0 \neq B \subseteq A \text{ is finitely generated}\},$$

and that the d-operation, the t-operation and the w-operation all have finite character while the v-operation need not have finite character.

Lemma 2.1. *Let $*$ be a star operation on an integral domain R. If $c(f/g)_* = (c(f)c(g)^{-1})_*$ for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal, then every nonzero countably generated ideal of R is $*$-invertible.*

Proof. Let I be a nonzero countably generated ideal of R. Then there exists a power series $g \in R[[X]]^*$ such that $I = c(g)$. Since $c(g/g) = R$ is a fractional ideal of R, we clearly have $(c(g)c(g)^{-1})_* = R$. Hence, $I = c(g)$ is $*$-invertible. $\square$

Lemma 2.2. *Let $*$ be a finite character star operation on an integral domain R. Then every nonzero ideal of R is $*$-invertible if and only if every nonzero countably generated ideal of R is $*$-invertible.*

Proof. The necessity is clear.

Conversely, suppose that every nonzero countably generated ideal of R is $*$-invertible. It suffices to show that every $*$-ideal of R is $*$-invertible. Suppose that A is a $*$-ideal of R. If $A \neq I_*$ for any finitely generated ideal $I \subseteq A$, then there is an infinite ascending chain

$$(a_1)_* \subset (a_1, a_2)_* \subset (a_1, a_2, a_3)_* \subset \cdots,$$

where each $a_n \in A - (a_1, a_2, \ldots, a_{n-1})_*$. Note that

$$\bigcup_{n=1}^{\infty} (a_1, a_2, \ldots, a_n)_* = \left(\bigcup_{n=1}^{\infty} (a_1, a_2, \ldots, a_n) \right)_*$$

is a $*$-ideal of countable type and so is $*$-invertible. Thus,

$$\bigcup_{n=1}^{\infty}(a_1, a_2, \ldots, a_n)_* = (a_1, a_2, \ldots, a_m)_*$$

for some m, a contradiction. Therefore, we have $A = I_*$ for some finitely generated ideal I of R, and so A is $*$-invertible. $\square$

By Lemmas 2.1 and 2.2, we have the following:

Theorem 2.3. *Let $*$ be a finite character star operation on an integral domain R. If $c(f/g)_* = (c(f)c(g)^{-1})_*$ for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal, then every nonzero ideal of R is $*$-invertible.*

Corollary 2.4. *If $c(f/g) = c(f)c(g)^{-1}$ for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal, then R is a Dedekind domain.*

Corollary 2.5. *If $c(f/g)_w = (c(f)c(g)^{-1})_w$ (or $c(f/g)_t = (c(f)c(g)^{-1})_t$) for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal, then R is a Krull domain.*

It is well known that, if $*$ is a star operation on an integral domain R, then every nonzero finitely generated ideal of R is $*$-invertible if and only if every nonzero two-generated ideal of R is $*$-invertible. Using an argument similar to Lemma 2.1, we can state the following result.

Theorem 2.6. *Let $*$ be a star operation on an integral domain R. If $c(f/g)_* = (c(f)c(g)^{-1})_*$ for all (linear) $f, g \in R[X]^*$ with $c(f/g)$ a fractional ideal, then every nonzero finitely generated ideal of R is $*$-invertible.*

Corollary 2.7. *If $c(f/g) = c(f)c(g)^{-1}$ for all (linear) $f, g \in R[X]^*$ with $c(f/g)$ a fractional ideal, then R is a Prüfer domain.*

Corollary 2.8. *If $c(f/g)_w = (c(f)c(g)^{-1})_w$ (or $c(f/g)_t = (c(f)c(g)^{-1})_t$) for all (linear) $f, g \in R[X]^*$ with $c(f/g)$ a fractional ideal, then R is a PVMD.*

3. Krull domains and formally integrally closed domains. It is well known that an integral domain R is integrally closed if and only if, for $f, g \in R[X]^*$ and $a \in R^*$, $c(fg) \subseteq aR$ implies $c(f)c(g) \subseteq aR$ [**4,** Lemma 3.1]. In this section, we are able to extend the result to power series and begin with reviewing the result on the content of power series from [**2**].

Theorem 3.1. *The following conditions are equivalent for an integral domain R:*

(1) *R is completely integrally closed.*
(2) *$c(fg)_v = (c(f)c(g))_v$ for all $f \in R[[X]]^*$ and (linear) $g \in R[X]^*$.*
(3) *$c(f/g)_v = (c(f)c(g)^{-1})_v$ for all (linear) $f, g \in R[X]^*$ with $c(f/g)$ a fractional ideal.*
(4) *$c(f/g)_v = (c(f)c(g)^{-1})_v$ for all $f \in R[[X]]^*$ and (linear) $g \in R[X]^*$ with $c(f/g)$ a fractional ideal.*
(5) *For $f \in R[[X]]^*$, (linear) $g \in R[X]^*$ and $a \in R^*$, $c(fg) \subseteq aR$ implies $c(f)c(g) \subseteq aR$.*

Proof.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). See [**2,** Theorem 2.1 and Remark 2.2].

(2) $\Rightarrow$ (5). Trivial.

(5) $\Rightarrow$ (1). Let $\alpha = a/b$ be almost integral over R where $a, b \in R^*$. Set $g = b - aX = b(1 - \alpha X)$. Now we have $(1 - \alpha X)(1 + \alpha X + \alpha^2 X^2 + \cdots) = 1$, and so $g(1 + \alpha X + \alpha^2 X^2 + \cdots) = b$. Put $f = (1+\alpha X+\alpha^2X^2+\cdots)$. Since α is almost integral over R, there exists a nonzero element $r \in R$ such that $rf \in R[[X]]$. Then $c(rfg) \subseteq rbR$. It follows that $c(rf)c(g) \subseteq rbR$. Hence, $a \in (1, \alpha, \alpha^2 \ldots)(b, a) \subseteq bR$. Therefore, $\alpha = a/b \in R$. $\square$

Theorem 3.2. *The following conditions are equivalent for an integral domain R:*

(1) *$c(fg)_v = (c(f)c(g))_v$ for all $f, g \in R[[X]]^*$.*
(2) *$c(f/g)_v = (c(f)c(g)^{-1})_v$ for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal.*
(3) *For $f, g \in R[[X]]^*$ and $a \in R^*$, $c(fg) \subseteq aR$ implies $c(f)c(g) \subseteq aR$.*

Proof.

(1) $\Leftrightarrow$ (2) is given in [**2,** Theorem 2.3], and

(1) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Let $f, g \in R[[X]]^*$. Obviously, $c(fg)_v \subseteq (c(f)c(g))_v$. Suppose that $c(fg) \subseteq a/bR$ where $a, b \in R^*$. Then, $c(bfg) = bc(fg) \subseteq aR$. Hence, $c(bf)c(g) = bc(f)c(g) \subseteq aR$. Therefore, $c(f)c(g) \subseteq a/bR$. It follows that $(c(f)c(g))_v \subseteq c(fg)_v$. $\square$

Recall that, if R is a PVMD with t-dim$(R) = 1$, then $c(fg)_t = (c(f)c(g))_t$ for all $f, g \in R[[X]]^*$ (see [**3,** Proposition 3.3]). We next consider the question of which integral domains R satisfies $c(fg)_w = (c(f)c(g))_w$ for all $f, g \in R[[X]]^*$.

Theorem 3.3. *Consider the following conditions on an integral domain R.*

(1) *R is a* PVMD *with w*-dim$(R) = 1$.
(2) $c(fg)_w = (c(f)c(g))_w$ *for all* $f, g \in R[[X]]^*$.
(3) $c(fg)_w = (c(f)c(g))_w$ *for all* $f \in R[[X]]^*$ *and* (*linear*) $g \in R[X]^*$.
(4) $c(f/g)_w = (c(f)c(g)^{-1})_w$ *for all* $f \in R[[X]]^*$ *and* (*linear*) $g \in R[X]^*$ *with* $c(f/g)$ *a fractional ideal.*
(5) $c(f/g)_t = (c(f)c(g)^{-1})_t$ *for all* $f \in R[[X]]^*$ *and* (*linear*) $g \in R[X]^*$ *with* $c(f/g)$ *a fractional ideal.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

Proof.

(1) $\Rightarrow$ (2). It follows from [**3,** Proposition 3.3] since every w-ideal is a t-ideal in a PVMD.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). Suppose that $f \in R[[X]]^*$ and $g \in R[X]^*$ with $c(f/g)$ a fractional ideal. Then there exists an element $r \in R^*$ and an $n \geqslant 0$ such that $rX^n f/g \in R[[X]]$. Thus, we have

$$\begin{aligned} c(f)_w &= (1/(rX^n))(c(rX^n fg/g))_w \\ &= (1/(rX^n))(c(rX^n f/g)c(g))_w \\ &= (c(f/g)c(g))_w. \end{aligned}$$

Since $c(g)$ is w-invertible by [**5,** Corollary 1.6], $(c(f)c(g)^{-1})_w = (c(f/g)c(g)c(g)^{-1})_w = c(f/g)_w$.

(4) $\Rightarrow$ (3). Suppose $f \in R[[X]]^*$ and $g \in R[X]^*$. Now $c(fg/g) = c(f)$ is a fractional ideal of R. It follows that $c(f)_w = c(fg/g)_w = (c(fg)c(g)^{-1})_w$. Since R is a PVMD by Corollary 2.8, we have $(c(f)c(g))_w = (c(fg)c(g)^{-1}c(g))_w = c(fg)_w$.

(4) $\Leftrightarrow$ (5). It follows from Corollary 2.8. $\square$

Corollary 3.4. *Let R be a Krull domain. Then $c(fg)_w = (c(f)c(g))_w$ for all $f, g \in R[[X]]^*$.*

Corollary 3.5. *Let R be a* PVMD *with w-*dim$(R) = 1$. *Then R is completely integrally closed.*

Example 3.6. Take a ring of entire function R which is a completely integrally closed Bezout domain. Here R does not satisfy $c(fg)_w = (c(f)c(g))_w$ for all $f, g \in R[[X]]^*$ since every ideal is a w-ideal in a Bezout domain, but R satisfies $c(fg)_v = (c(f)c(g))_v$ for all $f, g \in R[[X]]^*$ (consult [**2,** Example 2.10]).

Theorem 3.2 stated that $c(fg)_v = (c(f)c(g))_v$ for all $f, g \in R[[X]]^*$ if and only if $c(f/g)_v = (c(f)c(g)^{-1})_v$ for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal. Next we consider the case of t- and w-operations and give conditions equivalent to a Krull domain.

Theorem 3.7. *The following conditions are equivalent for an integral domain R:*

(1) *R is a Krull domain.*
(2) *For all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal, $c(f/g)_w = (c(f)c(g)^{-1})_w$.*
(3) *For all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal, $c(f/g)_t = (c(f)c(g)^{-1})_t$.*

Proof.

(1) $\Rightarrow$ (2). Note that a Krull domain is a completely integrally closed domain in which every w-ideal is a v-ideal [**10,** Theorem 5.4]. Then the result follows from Theorem 3.2 and Corollary 3.4.

(2) $\Rightarrow$ (3) is clear, while

(3) $\Rightarrow$ (1) is given in Corollary 2.5. $\square$

Remark 3.8. By Theorems 3.3 and 3.7, we have that the analogues of (1) $\Leftrightarrow$ (2) of Theorem 3.2 for t- and w-operations are false, since a PVMD with w-$\dim(R) = 1$ need not be Krull. Recall that an integral domain R is formally integrally closed if $c(fg)_t = (c(f)c(g))_t$ for all $f, g \in R[[X]]^*$. We can also define a formally w-Gaussian domain. An integral domain R is called a formally w-Gaussian domain if $c(fg)_w = (c(f)c(g))_w$ for all $f, g \in R[[X]]^*$. By Corollary 3.4, we have the implications: Krull domain $\Rightarrow$ formally w-Gaussian domain $\Rightarrow$ formally integrally closed domain.

Next we show some necessary and sufficient conditions for R to be a Krull domain in terms of these two classes of domains, but first we give the following result.

Lemma 3.9. *The following conditions are equivalent for an integral domain R:*

(1) *R is a Krull domain.*
(2) *For any two countably generated ideals A, B with $A_w \subseteq B_w$, there exists a countably generated ideal C such that $A_w = (BC)_w$.*
(3) *For any two countably generated ideals A, B with $A_t \subseteq B_t$, there exists a countably generated ideal C such that $A_t = (BC)_t$.*

Proof.

(1) $\Rightarrow$ (2). If R is a Krull domain and A, B are two countably generated ideals with $A_w \subseteq B_w$, then $(B(B^{-1}A))_w = A_w$. Note that $B^{-1}A \subseteq R$ and B^{-1} is a w-ideal of finite type. Set $B^{-1} = H_w$ for some finitely generated fractional ideal H of R. Set $C = HA$, as required.

(2) $\Rightarrow$ (1). Let I be a nonzero countably generated ideal of R. Pick $0 \neq a \in I$. Then $(aR)_w \subseteq I_w$, and thus, $aR = (II')_w$ for some ideal I' of R. Therefore, I is w-invertible. By Lemma 2.2, it follows that R is a Krull domain.

(1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) using a similar proof as that above. $\square$

Theorem 3.10. *The following conditions are equivalent for an integral domain R:*

(1) *R is a Krull domain.*
(2) *R is formally w-Gaussian and, for any $f \in R[[X]]^*$, there exists $g \in R[[X]]_{R^*}$ such that $c(fg)_w = R$.*
(3) *R is formally integrally closed and, for any $f \in R[[X]]^*$, there exists $g \in R[[X]]_{R^*}$ such that $c(fg)_t = R$.*
(4) *R is formally w-Gaussian and, for any $f \in R[[X]]^*$,*

$$fR[[X]]_{R^*} \bigcap R[[X]]$$

contains an element g such that $c(g)_w = R$.
(5) *R is formally integrally closed and, for any $f \in R[[X]]^*$,*

$$fR[[X]]_{R^*} \bigcap R[[X]]$$

contains an element g such that $c(g)_t = R$.
(6) *R is formally w-Gaussian and, for any $f, g \in R[[X]]^*$ with $c(f)_w \subseteq c(g)_w$, there exist $h, k \in R[[X]]^*$ such that $fh = gk$ with $c(h)_w = R$.*
(7) *R is formally integrally closed and, for any $f, g \in R[[X]]^*$ with $c(f)_t \subseteq c(g)_t$, there exist $h, k \in R[[X]]^*$ such that $fh = gk$ with $c(h)_t = R$.*

Proof.

(1) $\Rightarrow$ (2). For $f \in R[[X]]^*$, $c(f)$ is w-invertible. Hence, there exists $g \in R[[X]]_{R^*}$ such that $c(g)_w = c(f)^{-1}$, and thus, $c(fg)_w = (c(f)c(g))_w = R$.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). Let I be a nonzero countably generated ideal of R. Then there exists $f \in R[[X]]^*$ such that $I = c(f)$. Thus, we have $c(fg)_t = (c(f)c(g))_t = R$ for some $g \in R[[X]]_{R^*}$. Therefore, $I = c(f)$ is t-invertible. It follows from Lemma 2.2 that R is a Krull domain.

(1) $\Rightarrow$ (4). For $f \in R[[X]]^*$, we have $fR[[X]]_{R^*} \bigcap R[[X]] = fc(f)^{-1}[[X]]$ by [**2,** Theorem 2.3] and $c(f)^{-1} = (a_0, a_1, \ldots, a_n)_w$, where each $a_i \in K$. Set $g = f(a_0 + a_1X + \cdots + a_nX^n)$. Then we have

$$c(g)_w = (c(f)(a_0, a_1, \ldots, a_n))_w = R.$$

(4) $\Rightarrow$ (5). Trivial.

(5) $\Rightarrow$ (3). If $f \in R[[X]]^*$, then there exists $g \in fR[[X]]_{R^*} \bigcap R[[X]]$ such that $c(g)_t = R$. Let $g = fh/b$ where $h \in R[[X]]$ and $b \in R^*$. Then $h/b \in R[[X]]_{R^*}$ and $c(f(h/b))_t = R$.

(2) $\Rightarrow$ (6). Let $f, g \in R[[X]]^*$ with $c(f)_w \subseteq c(g)_w$. Then $c(gg')_w = R$ for some $g' \in R[[X]]_{R^*}$, and thus, we have

$$c(fg')_w = (c(f)c(g'))_w \subseteq (c(g)c(g'))_w = c(gg')_w = R.$$

Therefore, $fg' \in R[[X]]$. Set $h = gg'$ and $k = fg'$, as required.

(6) $\Rightarrow$ (1). Let A, B be two countably generated ideals of R where $A_w \subseteq B_w$. Then there exist $f, g \in R[[X]]^*$ such that $c(f) = A$ and $c(g) = B$. Therefore, there also exist $h, k \in R[[X]]^*$ such that $fh = gk$ with $c(h)_w = R$. Put $C = c(k)$. Then we have

$$A_w = (c(f)c(h))_w = c(fh)_w = (c(g)c(k))_w = (BC)_w.$$

Thus, by Lemma 3.9, R is a Krull domain.

(2) $\Rightarrow$ (7) $\Rightarrow$ (1). The proof is similar to the implications (2) $\Rightarrow$ (6) $\Rightarrow$ (1) above. $\square$

4. Dedekind domains. Using arguments similar to those in Section 3, we obtain companion results for some of the results in Section 4 for the d-operation. Firstly, we give the d-analogue for (2) $\Leftrightarrow$ (4) of Theorem 3.1.

Proposition 4.1. *$c(fg) = c(f)c(g)$ for all $f \in R[[X]]^*$ and (linear) $g \in R[X]^*$ if and only if $c(f/g) = c(f)c(g)^{-1}$ for all $f \in R[[X]]^*$ and (linear) $g \in R[X]^*$ with $c(f/g)$ a fractional ideal.*

Proof. Suppose that $c(fg) = c(f)c(g)$ for all $f \in R[[X]]^*$ and (linear) $g \in R[X]^*$. If $c(f/g)$ is a fractional ideal of R, then there exists an element $r \in R^*$ and an $n \geqslant 0$ such that $rX^n f/g \in R[[X]]$. Thus,

$$c(f) = (1/(rX^n))c(rX^n fg/g) = (1/(rX^n))c(rX^n f/g)c(g) = c(f/g)c(g).$$

Since R is a Prüfer domain, we have

$$c(f)c(g)^{-1} = c(f/g)c(g)c(g)^{-1} = c(f/g).$$

Conversely, assume that $f \in R[[X]]^*$ and $g \in R[X]^*$. Note that $c(fg/g) = c(f)$ is a fractional ideal of R. It follows that $c(f) =$

$c(fg)c(g)^{-1}$. Since R is a Prüfer domain by Corollary 2.7, we have

$$c(f)c(g) = c(fg)c(g)^{-1}c(g) = c(fg). \qquad \square$$

Theorem 4.2. *R is a Dedekind domain if and only if, for all $f, g \in R[[X]]^*$ with $c(f/g)$ a fractional ideal, $c(f/g) = c(f)c(g)^{-1}$.*

Proof. It follows from Corollary 2.4 and Theorem 3.7. $\square$

We call R a formally Gaussian domain if $c(fg) = c(f)c(g)$ for all $f, g \in R[[X]]^*$. Recall that a one-dimensional Prüfer domain is formally Gaussian [**2,** Corollary 2.6]. Note that the d-analogue of Theorem 3.2 (1) $\Leftrightarrow$ (2) is not true since a Prüfer domain with $\dim(R) = 1$ need not be a Dedekind domain.

Next, we show when a formally Gaussian domain is a Dedekind domain. Using a similar proof as that of Lemma 3.9 and Theorem 3.10, respectively, we have the following.

Theorem 4.3. *R is a Dedekind domain if and only if, for any two countably generated ideals A, B with $A \subseteq B$, there exists a countably generated ideal C such that $A = BC$.*

Theorem 4.4. *Let R be a formally Gaussian domain. Then the following conditions are equivalent*:

(1) *R is a Dedekind domain.*
(2) *For any $f \in R[[X]]^*$, there exists $g \in R[[X]]_{R^*}$ such that $c(fg) = R$.*
(3) *For any $f \in R[[X]]^*$, $fR[[X]]_{R^*} \bigcap R[[X]]$ contains an element g such that $c(g) = R$.*
(4) *For any $f, g \in R[[X]]^*$ with $c(f) \subseteq c(g)$, there exist $h, k \in R[[X]]^*$ such that $fh = gk$ with $c(h) = R$.*

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