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SYMBOL CALCULUS OF SQUARE-INTEGRABLE OPERATOR-VALUED MAPS

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ABSTRACT. We develop an abstract framework for the investigation of quantization and dequantization procedures based on orthogonality relations that do not necessarily involve group representations. To illustrate the usefulness of our abstract method, we show that it behaves well with respect to infinite tensor products. This construction subsumes examples from the study of magnetic Weyl calculus, magnetic pseudo-differential Weyl calculus, metaplectic representation on locally compact abelian groups, irreducible representations associated with finite-dimensional coadjoint orbits of some special infinite-dimensional Lie groups, and square-integrability properties shared by arbitrary irreducible representations of nilpotent Lie groups.

1. Introduction. Square-integrable representations of locally compact groups play a well-known role in Lie theory, representation theory, and their applications to physics. The present paper is devoted to developing a set of techniques applicable to operator-valued maps on measure spaces $\pi\colon (\Sigma,\mu) \to \mathbb{B}(\mathcal{H})$ that satisfy a square integrability property analogous to that of locally compact group representations, see equation (2.4) below, although π may not be a group representation and μ may not be a Haar measure. This investigation was motivated by several situations when Σ is a group that fails to be locally compact so it does not admit any Haar measure (as, for instance, in the study of canonical commutation relations where suitable substitutes of

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the group algebra for inductive limit groups [**12, 13, 14**] are sought), or when Σ is locally compact but π is not a projective group representation (see, for instance, the orthogonality relations for irreducible representations of nilpotent Lie groups [**20, 33, 40**] and the references therein, or magnetic Weyl calculus [**3, 4, 21, 27, 29**]).

From a more technical point of view, this article, as many others, is concerned with symbol calculus (also called *quantization* under certain circumstances) seen as a systematic way of associating operators in some infinite-dimensional vector space with functions almost everywhere defined in a suitable related set Σ endowed with a measure. Our primary focus is on operators acting in a Hilbert space $\mathcal{H}$, although other types of topological vector spaces will also be considered. Moreover, in order to define symbols for arbitrarily bounded linear operators on $\mathcal{H}$, in subsection (3.2), we will also need extensions of the symbol calculus beyond spaces of functions on Σ, in the same way as the classical Weyl calculus on $\mathbb{R}^n$ needs to be extended from functions to tempered distributions on $\mathbb{R}^{2n}$.

Among the different strategies for beginning and motivating symbol calculus, there are two which are dual to each other. The first one, inspired by Weyl's quantization procedure, consists of associating a bounded linear operator $\pi(s)$ in $\mathcal{H}$ to each point s of the space Σ. This mapping $s \mapsto \pi(s)$, while not supposed to be unitarily-valued or to possess group-like properties, would benefit from some regularity requirements. Boundedness and weak continuity are good properties, and yet a square integrability condition with respect to some measure μ on Σ (generalizing the notion of square integrable representation of a group) is the best starting point. Then, operators $\Pi(f)$ are associated to suitable functions f on Σ by integration techniques, cf., equation (3.4), and square integrability plays an important role in identifying Hilbert-Schmidt operators as corresponding by quantization to L^2-functions. Simple examples show that not all of the elements in $L^2(\Sigma, \mu)$ need to be involved.

A dual approach is a priori to give the symbols (functions defined on Σ) of all the rank-1 operators. Then, the symbols of more general operators are obtained by superposition, modeled by integration on Σ, followed eventually by extension techniques. If suitably implemented, the construction is essentially the inverse of that described above. However, this is achieved only after the formalism has been extended.

Many classes of operators form *-algebras under operator multiplication and taking adjoints. Clearly, it is desirable to use quantization to induce isomorphic versions of these *-algebras on classes of symbols. As a matter of fact, due to the square integrability assumption, one obtains compatible scalar products on the *-algebras, making them Hilbert algebras. This makes available extension techniques which permit the treatment of symbols not associated to Hilbert-Schmidt operators. However, most of the known examples strongly suggest the existence of an extra mathematical structure, resulting in Gelfand triples both at the level of vectors and at the level of symbols, suitably interconnected. One could simply recall the role played in pseudodifferential theory by the Schwartz space and its dual, the space of tempered distributions. Another example, leading to Gelfand triples of Banach spaces, is Segal algebra, available on locally compact groups. In our general framework, we will indicate a systematic way to construct Gelfand triples connected to the symbol calculus associated with the family $\{\pi(s) \mid s \in \Sigma\}$ which will have a rich algebraic content.

The objective of Sections 2 and 3 is the construction of the symbol calculus associated to the data $(\Sigma, \mu, \pi, \mathcal{H})$ where Σ is a space endowed with a measure μ. That space serves as a family of indices for a set of bounded operators $\{\pi(s) \mid s \in \Sigma\}$ in the Hilbert space $\mathcal{H}$. We do not assume that $\pi(s)$ is unitary, and we do not require anything about the product $\pi(s)\pi(t)$ for $s, t \in \Sigma$. The map $\pi(\cdot)$ is assumed bounded and weakly continuous. The main requirement is relation (2.4), a condition of square integrability extending a well-known concept from group representation theory [**6**].

In Section 2, we show that the class of square-integrable families of operators is closed under some basic operations as compressions and tensor products. We also show that these families are irreducible, in the sense that their commutant is trivial and they do not have any nontrivial common invariant subspace, which suggests the interesting problem of pointing out the topological groups for which every unitary irreducible representation admits a measure on the group for which the representation is square integrable, see, for instance, Corollaries 2.5 and 2.6 for answers to this question in the case of compact and nilpotent Lie groups, respectively.

In Section 3, the first purpose is to raise the family π, essentially by integration, to a correspondence $f \mapsto \Pi(f)$, sending a closed subspace of $L^2(\Sigma;\mu)$ to the ideal of all Hilbert-Schmidt operators in $\mathcal{H}$. Actually, the fact that Π is an "integrated form" of π (in the spirit of group representation theory) is only seen a posteriori. The initial construction is based only on the "representation coefficient" map Φ. The linear maps Π, Φ and Λ are isomorphisms of H^*-algebras. By transport of structure via these isomorphisms, one then defines classes of trace-class, compact and bounded-type symbols forming Banach *-algebras. Radon measures on Σ can also be incorporated when Σ is a locally compact space.

We also develop the dequantization procedure for the operator calculus, that is, we develop methods for recovering the symbol of a given operator. In order to do that in an effective way, we need to explore new spaces of symbols and their natural composition law that recovers the twisted convolution in the case of group representations and corresponds to the composition of operators in the representation space.

Section 4 deals with the Gelfand triples that occur in our general framework in connection with suitable dense subspaces of the Hilbert space under consideration. This is suggested by the classical example of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions on $\mathbb{R}^n$, which is continuously and densely embedded into the Hilbert space $L^2(\mathbb{R}^n)$ and is closely related to the square-integrable family of unitary operators $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{B}(L^2(\mathbb{R}^n))$ defined by the Weyl system. We study the abstract version of the operators $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ and some related structures, which provide a unifying perspective on different types of applications in Section 7.

In Section 5, we develop some very basic aspects of the Berezin-Toeplitz quantization in our abstract framework, in order to suggest how this important topic fits into our picture.

Section 6 explores infinite tensor products of square-integrable families of operators, a circle of ideas that plays an important role in the representation theory of canonical commutation relations (CCR) with infinitely many degrees of freedom; see, for instance, [**18, 19, 24, 34**]. We prove that these infinite tensor products always have a certain property of approximate square integrability in Theorem 6.6, and then, to a very limited extent, we also discuss the Berezin-Toeplitz quantization which suggests that several interesting problems arise in this area. It is noteworthy that the relationship between CCR and the infinite tensor products was also studied from a different perspective in [**12, 13, 14**].

Finally, in Section 7, we briefly present four topics from earlier literature where one can find special cases of the general ideas developed in the present paper:

(i) the magnetic pseudo-differential Weyl calculus;
(ii) the study of metaplectic representation on locally compact abelian groups;
(iii) irreducible representations associated with finite-dimensional coadjoint orbits of some special infinite-dimensional Lie groups;
(iv) square-integrability properties shared by arbitrary irreducible representations of nilpotent Lie groups.

It would also be quite interesting to understand the relationship between our abstract approach and the Weyl and Berezin calculus on bounded symmetric domains as developed, for instance, in [**1, 37**].

Preliminary conventions and notation. A convenient reference for square-integrable representations of locally compact groups and their role in representation theory is [**32,** Appendix VII]; see also the references therein.

If Σ is a topological space (always Hausdorff), we set $BC(\Sigma)$ for the C^*-algebra of all bounded continuous complex-valued functions on Σ. If Σ is locally compact, we write $C_0(\Sigma)$ for the C^*-algebra of the continuous functions vanishing at infinity. For any measure μ on Σ and $q \in [1, \infty]$, we denote the usual Lebesgue space of order q on (Σ, μ) by $L^q(\Sigma; \mu)$.

For two complex Hausdorff locally convex spaces $\mathcal{E}$ and $\mathcal{F}$, we will write $\mathcal{E} \otimes \mathcal{F}$ for the algebraic tensor product. When endowed with projective topology, that space will be denoted by $\mathcal{E} \otimes_p \mathcal{F}$ and its completion in this topology by $\mathcal{E} \widehat{\otimes}_p \mathcal{F}$. Analogously, $\mathcal{E} \widehat{\otimes}_i \mathcal{F}$ will be the completion of $\mathcal{E} \otimes_i \mathcal{F}$, which is $\mathcal{E} \otimes \mathcal{F}$ endowed with injective topology.

Under the same assumptions on $\mathcal{E}$ and $\mathcal{F}$, we denote by $\mathbb{B}(\mathcal{E}, \mathcal{F})$ the vector space of all linear *continuous* operators from $\mathcal{E}$ onto $\mathcal{F}$ and use the abbreviation $\mathbb{B}(\mathcal{E}) := \mathbb{B}(\mathcal{E}, \mathcal{E})$. Then, $\mathcal{E}' := \mathbb{B}(\mathcal{E}, \mathbb{C})$ is the (topological) dual of $\mathcal{E}$.

We recall [**6**] that a Hilbert algebra is a *-algebra $(\mathscr{A}, \#, ^{\#})$ endowed with a scalar product $\langle\cdot,\cdot\rangle : \mathscr{A}\times\mathscr{A}\to\mathbb{C}$ such that

(i) $\langle g^{\#}, f^{\#}\rangle = \langle f, g\rangle$ for all $f, g \in \mathcal{A}$,
(ii) $\langle f\#g, h\rangle = \langle g, f^{\#}\#h\rangle$ for all $f, g, h \in \mathcal{A}$,
(iii) for every $g \in \mathscr{A}$, the map

$$\mathsf{L}_g : \mathscr{A} \longrightarrow \mathscr{A},$$

$\mathsf{L}_g(f) := g\#f$ is continuous.
(iv) $\mathscr{A}\#\mathscr{A}$ is total in $\mathscr{A}$.

A complete Hilbert algebra is called an H^*-*algebra*.

Clearly, one also has

$$\langle f\#g, h\rangle = \langle f, h\#g^{\#}\rangle \quad \text{for all } f, g, h \in \mathscr{A},$$

and the map $\mathsf{R}_g : \mathscr{A}\to\mathscr{A}$ given by $\mathsf{R}_g(f) := f\#g$ is also continuous; therefore, $\mathscr{A}\times\mathscr{A}\xrightarrow{\#}\mathscr{A}$ is separately continuous.

To give some basic examples, let us fix a complex Hilbert space $\mathcal{H}$. By convention, the scalar product $\langle\cdot,\cdot\rangle$ is anti-linear in the second variable and we denote the conjugate space of $\mathcal{H}$ by $\overline{\mathcal{H}}$. By the Riesz theorem, the dual $\mathcal{H}'$ of $\mathcal{H}$ is canonically antilinearly isomorphic to $\mathcal{H}$, so there is a linear isomorphism permitting the identification of $\overline{\mathcal{H}}$ with $\mathcal{H}'$. Recall that the space $\mathbb{B}_2(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$ forms a *-ideal in $\mathbb{B}(\mathcal{H})$ and a Hilbert space with the scalar product

$$\langle S, T\rangle_{\mathbb{B}_2(\mathcal{H})} := \mathrm{Tr}(ST^*).$$

Actually, $\mathbb{B}_2(\mathcal{H})$ is an H^*-algebra; the subspace $\mathbb{B}_2(\mathcal{H})\mathbb{B}_2(\mathcal{H})$ (coinciding with the *-ideal $\mathbb{B}_1(\mathcal{H})$ of all trace-class operators) is dense in $\mathbb{B}_2(\mathcal{H})$.

Let us denote by Λ the canonical unitary operator

$$\Lambda\colon \mathcal{H}\,\widehat{\otimes}\,\overline{\mathcal{H}} \longrightarrow \mathbb{B}_2(\mathcal{H}), \qquad \Lambda(u\otimes v) := \lambda_{u,v} := \langle\cdot, v\rangle u, \tag{1.1}$$

where $\mathcal{H}\,\widehat{\otimes}\,\overline{\mathcal{H}}$ stands for the Hilbert completion of the algebraic tensor product $\mathcal{H}\otimes\overline{\mathcal{H}}$. On the space $\mathcal{H}\otimes\overline{\mathcal{H}}$, we consider the unique structure of H^*-algebra such that the above operator Λ is an H^*-algebra isomorphism. Its restriction $\Lambda : \mathcal{H}\otimes\overline{\mathcal{H}}\to\mathbb{F}(\mathcal{H})$ is a Hilbert algebra isomorphism between the algebraic tensor product and finite rank operators. We record for further use some relations valid for $u, v, u', v' \in \mathcal{H}$

and $S \in \mathbb{B}(\mathcal{H})$:

$$S\lambda_{u,v} = \lambda_{Su,v}, \quad \lambda_{u,v}S = \lambda_{u,S^*v}, \quad \lambda_{u,v}\lambda_{u',v'} = \langle u', v\rangle\lambda_{u,v'}, \tag{1.2}$$

$$\lambda_{u,v}^* = \lambda_{v,u}, \quad \mathrm{Tr}\,(\lambda_{u,v}) = \langle u, v\rangle. \tag{1.3}$$

Very often, besides the norm topology of a Hilbert algebra $\mathscr{A}$, there is another finer locally convex topology. We recall that a Fréchet *-algebra is a *-algebra $(\mathscr{A}, \#, ^\#)$ with a Fréchet locally convex space topology $\mathscr{T}$ such that the involution $\mathscr{A} \ni f \to f^\# \in \mathscr{A}$ is continuous, and the product

$$\mathscr{A} \times \mathscr{A} \ni (f, g) \longrightarrow f\#g \in \mathscr{A},$$

is separately continuous. Then, a *Fréchet-Hilbert algebra* $(\mathscr{A}, \#, ^\#, \mathscr{T}, \langle\cdot,\cdot\rangle)$ is both a Fréchet *-algebra and a Hilbert algebra, the topology $\mathscr{T}$ being finer than that of the topology associated to the scalar product.

2. Square-integrable operator-valued maps. Let us fix a complex Hilbert space $\mathcal{H}$, a Borel space Σ with a Σ-algebra $\mathcal{M}$, and a positive measure μ on Σ. The set of measurable complex-valued functions on Σ will be denoted by $\mathscr{M}(\Sigma)$. We assume that $\pi : \Sigma \to \mathbb{B}(\mathcal{H})$ is a weakly measurable, almost everywhere defined map. One defines the sesquilinear mapping

$$\phi^\pi \equiv \phi : \mathcal{H} \times \mathcal{H} \longrightarrow \mathscr{M}(\Sigma), \qquad \phi_{u,v}(s) := \langle \pi(s)u, v\rangle. \tag{2.1}$$

This extends concepts such as representation coefficients, wavelet transform and short time Fourier transform.

Notation 2.1. We will use the notation

$$\Phi^\pi \equiv \Phi : \mathcal{H}\,\widehat{\otimes}\,\overline{\mathcal{H}} \longrightarrow L^2(\Sigma;\mu) \equiv L^2(\Sigma), \tag{2.2}$$

if the mapping ϕ^π admits such an *isometric* extension.

Remark 2.2. The map Φ^π from Notation 2.1 exists if and only if

$$\int_\Sigma \langle \pi(s)u_1, v_1\rangle\langle v_2, \pi(s)u_2\rangle\, d\mu(s) = \langle u_1, u_2\rangle\langle v_2, v_1\rangle, \tag{2.3}$$

for all $u_1, u_2, v_1, v_2 \in \mathcal{H}$. To achieve this equality, a renormalization of the measure μ may be used, if necessary. Also note that it is enough to check that equation (2.3) is satisfied for vectors u_1, u_2, v_1 and v_2 merely in some dense subset of $\mathcal{H}$. A simple polarization argument also shows that it suffices to verify equation (2.3) for $u_1 = u_2$ and $v_1 = v_2$, that is,

$$\int_\Sigma \Big|\langle \pi(s)u, v\rangle\Big|^2 d\mu(s) = \|u\|^2\|v\|^2, \quad \text{for all } u, v \in \mathcal{H}. \tag{2.4}$$

The next definition is convenient for our purposes.

Definition 2.3. For any complex Hilbert space $\mathcal{H}$, and any measure space $(\Sigma, \mathcal{M}, \mu)$, we define $\mathrm{SQ}(\mathbb{B}(\mathcal{H}), \mu)$ as the set of all weakly measurable, almost everywhere defined maps $\pi\colon \Sigma \to \mathbb{B}(\mathcal{H})$ satisfying the square-integrability condition (2.3).

Before continuing, we remark that Definition 2.3 was motivated by several important examples of operator-valued maps that satisfy the above square-integrability condition:

- (i) unitary irreducible representations of compact groups (see Corollary 2.5);
- (ii) unitary irreducible representations of connected, simply connected, nilpotent Lie groups (see Corollary 2.6 and Proposition 7.2);
- (iii) the magnetic Weyl systems on $\mathbb{R}^{2n}$ (see subsection 7.1);
- (iv) operator calculi on locally compact abelian groups (see Proposition 7.1);
- (v) localized Weyl calculus for some unitary representations of infinite-dimensional Lie groups (see subsection 7.3).

We will now obtain some simple results which point out that the square-integrability property of operator-valued maps should be viewed as a kind of irreducibility in the sense of representation theory, that is, it implies the absence of nontrivial invariant subspaces. It is well known that the assertions in the next proposition are equivalent ways of describing the irreducibility property if the operator set $\pi(\Sigma)$ is assumed to be closed under operator adjoints. Since we do not assume the self-adjointness hypothesis, we will prove these assertions separately.

Proposition 2.4. *Let $\pi \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}), \mu)$. Then, the following assertions hold.*

(i) *If a closed linear subspace* $\mathcal{H}_0 \subseteq \mathcal{H}$ *has the property* $\pi(\Sigma)\mathcal{H}_0 \subseteq \mathcal{H}_0$*, then either* $\mathcal{H}_0 = \{0\}$ *or* $\mathcal{H}_0 = \mathcal{H}$*.*

(ii) *If the operator* $T \in \mathbb{B}(\mathcal{H})$ *has the property* $T\pi(s) = \pi(s)T$ *for almost every* $s \in \Sigma$*, then* $T = z1_{\mathcal{H}}$ *for some* $z \in \mathbb{C}$*.*

Proof.

(i) Assume that $\mathcal{H}_0 \subsetneqq \mathcal{H}$. Then, there exists some nonzero vector $v \in \mathcal{H}$ with $v \perp \mathcal{H}_0$. Hence, by using the hypothesis $\pi(\Sigma)\mathcal{H}_0 \subseteq \mathcal{H}_0$, we obtain $v \perp \pi(s)u$ for all $s \in \Sigma$ and $u \in \mathcal{H}_0$. Setting $u_1 = u_2 = u$ and $v_1 = v_2 = v$ in equation (2.3), it follows that, for all $u \in \mathcal{H}_0$, we have $\|u\|^2\|v\|^2 = 0$; hence, necessarily, $u = 0$. Consequently, $\mathcal{H}_0 = 0$, and this concludes the proof.

(ii) First, note that, for every operator $T \in \mathbb{B}(\mathcal{H})$ satisfying the condition $T\pi(s) = \pi(s)T$ for almost every $s \in \Sigma$, we have $\langle \pi(\cdot)Tu_1, v_1\rangle = \langle \pi(\cdot)u_1, T^*v_1\rangle$; hence, by equation (2.3), we obtain

$$\langle Tu_1, u_2\rangle\langle v_2, v_1\rangle = \langle u_1, u_2\rangle\langle v_2, T^*v_1\rangle, \quad \text{for all } u_1, u_2, v_1, v_2 \in \mathcal{H}.$$

Now, for $u_1 = u_2$ and $v_1 = v_2$ we obtain

$$\frac{\langle Tu, u\rangle}{\|u\|^2} = \frac{\langle Tv, v\rangle}{\|v\|^2} =: z \in \mathbb{C} \quad \text{for all } u, v \in \mathcal{H} \setminus \{0\}.$$

Thus, the numerical range of operator T consists of a single point, and T is then a scalar multiple of the identity operator. Specifically, the above equalities imply $\langle (T - z1_{\mathcal{H}})u, u\rangle = 0$ for all $u \in \mathcal{H}$. Then, by polarization, $\langle (T - z1_{\mathcal{H}})u, v\rangle = 0$ for all u, $v \in \mathcal{H}$; hence, eventually $T = z1_{\mathcal{H}}$, which completes the proof. □

Proposition 2.4 implies that direct sums of square-integrable maps may not be square integrable, that is, if

$$\pi_j(\cdot) \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}_j), \mu_j) \quad \text{for } j = 1, 2,$$

then the map

$$\begin{pmatrix} \pi_1(\cdot) & 0 \\ 0 & \pi_2(\cdot) \end{pmatrix}$$

does not belong to

$$\mathrm{SQ}(\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2), \mu_1 \otimes \mu_2),$$

unless we have either $\mathcal{H}_1 = \{0\}$ or $\mathcal{H}_2 = \{0\}$ (see, however, Proposition 2.7).

Now, we derive other consequences of Proposition 2.4.

Corollary 2.5. *If Σ is a compact group with the Haar probability measure μ and $\pi\colon \Sigma \to \mathbb{B}(\mathcal{H})$ is a unitary representation, then $\pi \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}), \mu)$ if and only if π is an irreducible representation.*

Proof. It is well known that irreducible representations of compact groups are square integrable with respect to the Haar measure, and the converse implication follows by Proposition 2.4. □

We also state next corollary here for the sake of completeness of available information on this circle of ideas, although its nontrivial implication depends on some aspects of representation theory of nilpotent Lie groups to be discussed in subsection 7.4.

Corollary 2.6. *Let Σ be any connected, simply connected, nilpotent Lie group, and let $\pi\colon \Sigma \to \mathbb{B}(\mathcal{H})$ be any unitary representation. Then, π is an irreducible representation if and only if there exists a Borel measure μ on Σ for which $\pi \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}), \mu)$.*

Proof. If $\pi \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}), \mu)$ for some measure μ, then the representation π is irreducible by Proposition 2.4. The converse implication, including details on the construction of the measure μ in terms of the representation π, is the subject of Proposition 7.2. □

Proposition 2.4 suggests the problem of determining topological groups for which every unitary irreducible representation admits a measure on the group where the representation is square integrable. As Corollaries 2.5 and 2.6 show, that property is shared by both the compact topological groups and the connected, simply connected nilpotent Lie groups, which looks somehow surprising since these two types of groups have rather few common features. It would be interesting to find other examples of topological groups whose unitary irreducible representations are square-integrable, with respect to suitable measures.

Despite the above irreducibility properties of square-integrable maps, we note that the direct sums of such maps do have a weaker property, as recorded in the following observation, which is necessary for the proof of Theorem 6.6.

Proposition 2.7. *Let $(\Sigma, \mathcal{M}, \mu)$ be any measure space. Let J be any countable index set for every $j \in J$, let $\mathcal{H}_j$ be any complex Hilbert space and let $\pi_j \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}_j), \mu)$. Assume that, for almost every $s \in \Sigma$, we have $\sup_{j\in J} \|\pi_j(s)\| < \infty$. If we set*

$$\mathcal{H} := \bigoplus_{j\in J} \mathcal{H}_j,$$

then the map defined almost everywhere

$$\pi := \bigoplus_{j\in J} \pi_j(\cdot)\colon \Sigma \longrightarrow \mathbb{B}(\mathcal{H}),$$

is weakly measurable and has the property

$$\int_\Sigma |\langle \pi(s)u_1, v_1\rangle\langle v_2, \pi(s)u_2\rangle|\, d\mu(s) \le \|u_1\|\|v_1\|\|v_2\|\|u_2\| \tag{2.5}$$

for all $u_1, u_2, v_1, v_2 \in \mathcal{H}$.

Proof. Since the index set J is countable and the map π_j is weakly measurable for every $j \in J$, it easily follows that the map π is, in turn, weakly measurable. For arbitrary $j \in J$ and $u_1, u_2, v_1, v_2 \in \mathcal{H}$, we will denote their projections on $\mathcal{H}_j$ by $u_{1j}, u_{2j}, v_{1j}, v_{2j} \in \mathcal{H}_j$, respectively. Then, we have

$$\begin{aligned}
&\int_\Sigma |\langle \pi(s)u_1, v_1\rangle\langle v_2, \pi(s)u_2\rangle|\, d\mu(s)\\
&\le \sum_{j,k\in J} \int_\Sigma |\langle \pi_j(s)u_{1j}, v_{1j}\rangle\langle v_{2k}, \pi_k(s)u_{2k}\rangle|\, d\mu(s)\\
&\le \sum_{j,k\in J} \left(\int_\Sigma |\langle \pi_j(s)u_{1j}, v_{1j}\rangle|^2 d\mu(s)\right)^{1/2} \left(\int_\Sigma |\langle v_{2k}, \pi_k(s)u_{2k}\rangle|^2 d\mu(s)\right)^{1/2}\\
&= \sum_{j,k\in J} \|u_{1j}\|\|v_{1j}\|\|v_{2k}\|u_{2k}\|\\
&= \left(\sum_{j\in J} \|u_{1j}\|\|v_{1j}\|\right)\left(\sum_{k\in J} \|\|v_{2k}\|u_{2k}\|\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq\Big(\sum_{j\in J}\|u_{1j}\|^2\Big)^{1/2}\Big(\sum_{j\in J}\|v_{1j}\|^2\Big)^{1/2}\Big(\sum_{k\in J}\|v_{2k}\|^2\Big)^{1/2}\Big(\sum_{k\in J}\|u_{2k}\|^2\Big)^{1/2}\\
&=\|u_1\|\|v_1\|\|v_2\|\|u_2\|,
\end{aligned}
$$

and this concludes the proof. □

Now, we draw a consequence that will be needed in the proof of Theorem 6.6.

Corollary 2.8. *Let* $(\Sigma,\mathcal{M},\mu)$ *be any measure space, and let* $\mathcal{K}$ *be any separable complex Hilbert space. If* $\pi_0\in\mathrm{SQ}(\mathbb{B}(\mathcal{H}_0),\mu)$, *then*

$$\pi(\cdot):=\pi_0(\cdot)\otimes\mathrm{id}_{\mathcal{K}}\colon\Sigma\longrightarrow\mathbb{B}\,(\mathcal{H}_0\,\widehat{\otimes}\,\mathcal{K})$$

is a weakly measurable map that satisfies (2.5) *for all* $u_1,u_2,v_1,v_2\in\mathcal{H}_0\,\widehat{\otimes}\,\mathcal{K}$.

Proof. Let J be the index set of any orthonormal basis of $\mathcal{K}$ such that $\mathcal{K}=\ell^2(J)$. If we set $\mathcal{H}_j:=\mathcal{H}_0$ and $\pi_j:=\pi_0$ for all $j\in J$, then we may apply Proposition 2.7, and the conclusion follows. □

Below, we discuss a few operations on operator maps satisfying the square-integrability condition required in Definition 2.3. In particular, these operations will provide methods of constructing new square-integrable maps out of other maps satisfying the same hypothesis.

2.1. Tensor products of square-integrable maps.

Proposition 2.9. *For* $j=1,2$, *let* $\mathcal{H}_j$ *be any complex Hilbert space, let* $(\Sigma_j,\mathcal{M}_j,\mu_j)$ *be any* σ*-finite measure space, and let* $\pi_j\in\mathrm{SQ}(\mathbb{B}(\mathcal{H}_j),\mu_j)$. *If we define*

$$
\begin{aligned}
\pi_1\otimes\pi_2\colon\Sigma_1\times\Sigma_2&\longrightarrow\mathbb{B}(\mathcal{H}_1\widehat{\otimes}\mathcal{H}_2),\\
(\pi_1\otimes\pi_2)(s_1,s_2)&:=\pi_1(s_1)\otimes\pi_2(s_2),
\end{aligned}
$$

then

$$\pi_1\otimes\pi_2\in\mathrm{SQ}(\mathbb{B}(\mathcal{H}_1\widehat{\otimes}\mathcal{H}_2),\mu_1\otimes\mu_2).$$

Proof. For any $f_j\in L^2(\Sigma_j,\mu_j)$ with $j=1,2$, define

$$f_1\otimes f_2\in L^2(\Sigma_1\times\Sigma_2,\mu_1\otimes\mu_2)$$

by

$$(f_1 \otimes f_2)(x_1, x_2) := f_1(x_1) f_2(x_2),$$

almost everywhere. Recall that the bilinear map

$$L^2(\Sigma_1, \mu_1) \times L^2(\Sigma_2, \mu_2) \longrightarrow L^2(\Sigma_1 \times \Sigma_2, \mu_1 \otimes \mu_2), \quad (f_1, f_2) \longmapsto f_1 \otimes f_2,$$

gives rise to a unitary operator

$$V\colon L^2(\Sigma_1, \mu_1) \,\widehat{\otimes}\, L^2(\Sigma_2, \mu_2) \longrightarrow L^2(\Sigma_1 \times \Sigma_2, \mu_1 \otimes \mu_2).$$

In fact, using the Fubini theorem, the operator V is an isometry, and in order to prove that it is also surjective, we may assume that $\mu_j(\Sigma_j) < \infty$ for $j = 1, 2$. Then, where we have denoted the characteristic function of the set A by χ_A, it suffices to show that the set

$$\mathcal{Q} := \{A \subseteq \Sigma_1 \times \Sigma_2 \mid \chi_A \in \operatorname{Ran} V\}$$

contains the σ-ring $\mathcal{Q}_0$ of all the $\mu_1 \otimes \mu_2$-measurable subsets of $\Sigma_1 \times \Sigma_2$ since $\{\chi_A \mid A \in \mathcal{Q}_0\}$ spans a dense linear subspace of $L^2(\Sigma_1 \times \Sigma_2, \mu_1 \otimes \mu_2)$. Recall that $\mathcal{Q}_0$ is the σ-ring generated by the sets $A_1 \times A_2$ for all measurable sets $A_j \subseteq \Sigma_j$ with $j = 1, 2$. It suffices to note that $\mathcal{Q}$ is a ring, i.e., it is closed under finite unions and differences, $\mathcal{Q}$ is closed under countable unions of increasing sequences by Lebesgue's dominated convergence theorem since we assumed the measures μ_1 and μ_2 to be finite, and moreover, $\mathcal{Q}$ contains all of the above-mentioned sets $A_1 \times A_2$; hence, by the monotone class theorem, $\mathcal{Q}_0 \subseteq \mathcal{Q}$ [**16,** Chapter I, subsection 6, Theorem B].

Then, for $j = 1, 2$, by using the hypothesis $\pi_j \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}_j), \mu_j)$ along with equation (2.2), we obtain the isometry

$$\Phi^{\pi_j} : \mathcal{H}_j \,\widehat{\otimes}\, \overline{\mathcal{H}}_j \longrightarrow L^2(\Sigma_j, \mu_j), \quad \Phi^{\pi_j}(u \otimes v) = \langle \pi_j(\cdot) u, v \rangle.$$

As the Hilbertian tensor product of two isometries is again an isometry, we obtain that the operator

$$\Phi^{\pi_1} \otimes \Phi^{\pi_2} : (\mathcal{H}_1 \,\widehat{\otimes}\, \overline{\mathcal{H}}_1) \,\widehat{\otimes}\, (\mathcal{H}_2 \,\widehat{\otimes}\, \overline{\mathcal{H}}_2) \longrightarrow L^2(\Sigma_1, \mu_1) \,\widehat{\otimes}\, L^2(\Sigma_2, \mu_2)$$

is an isometry. On the other hand, for $j = 1, 2$, all $u_j, v_j \in \mathcal{H}_j$, and almost all $s_j \in \Sigma_j$, we have

$$\begin{aligned}((\Phi^{\pi_1} \otimes \Phi^{\pi_2})((u_1 \otimes v_1) \otimes (u_2 \otimes v_2)))(s_1, s_2) \\ &= \langle \pi(s_1)u_1, v_1\rangle\langle \pi(s_2)u_2, v_2\rangle \\ &= \langle ((\pi_1 \otimes \pi_2)(s_1, s_2))(u_1 \otimes u_2), v_1 \otimes v_2\rangle \\ &= \phi^{\pi_1\otimes\pi_2}_{u_1\otimes u_2, v_1\otimes v_2}(s_1, s_2).\end{aligned}$$

Now, by composing the isometry $\Phi^{\pi_1} \otimes \Phi^{\pi_2}$ with the flip unitary operator,

$$\begin{gathered}(\mathcal{H}_1 \widehat{\otimes} \overline{\mathcal{H}}_1) \widehat{\otimes} (\mathcal{H}_2 \widehat{\otimes} \overline{\mathcal{H}}_2) \longrightarrow (\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2) \widehat{\otimes} \overline{\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2}, \\ u_1 \otimes v_1 \otimes u_2 \otimes v_2 \longmapsto u_1 \otimes u_2 \otimes v_1 \otimes v_2,\end{gathered}$$

and with the above unitary operator V, it follows that the sesquilinear mapping

$$\phi^{\pi_1\otimes\pi_2}\colon (\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2) \times (\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2) \longrightarrow \mathscr{M}(\Sigma_1 \times \Sigma_2),$$

(see equation (2.1)) gives rise to the isometry

$$\Phi^{\pi_1\otimes\pi_2} = \Phi^{\pi_1} \otimes \Phi^{\pi_2}.$$

This implies that

$$\pi_1 \otimes \pi_2 \in \mathrm{SQ}\big(\mathbb{B}(\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2), \mu_1 \otimes \mu_2\big),$$

and the proof is complete. □

2.2. Compressions of square-integrable maps.

Proposition 2.10. *For $j = 1, 2$, let $\mathcal{H}_j$ be any complex Hilbert space, let $(\Sigma_j, \mathcal{M}_j, \mu_j)$ be any measure space, and let $\pi_j\colon \Sigma_j \to \mathbb{B}(\mathcal{H}_j)$. Assume that $p\colon \Sigma_2 \to \Sigma_1$ is a measurable map satisfying the condition $p_*(\mu_2) = \mu_1$ and that $\iota\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a linear isometry for which $\pi_1 \circ p = \iota^*\pi_2(\cdot)\iota$ almost everywhere on Σ_2.*

If $\pi_2 \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}_2), \mu_2)$, then also $\pi_1 \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}_1), \mu_1)$, and we have the commutative diagram

$$\begin{array}{ccc} L^2(\Sigma_1, \mu_1) & \xrightarrow{f \mapsto f \circ p} & L^2(\Sigma_2, \mu_2) \\ \Phi^{\pi_1} \uparrow & & \uparrow \Phi^{\pi_2} \\ \mathcal{H}_1 \widehat{\otimes} \overline{\mathcal{H}}_1 & \xrightarrow{\iota \otimes \iota} & \mathcal{H}_2 \widehat{\otimes} \overline{\mathcal{H}}_2 \end{array}$$

whose arrows are isometries.

Proof. By using the hypothesis for arbitrary u_1, $v_1 \in \mathcal{H}_1$, we obtain

$$\begin{aligned} \int_{\Sigma_1} |\langle \pi_1(s_1) u_1, v_1 \rangle|^2 d\mu_1(s_1) &= \int_{\Sigma_2} |\langle \pi_1(p(s_2)) u_1, v_1 \rangle|^2 d\mu_2(s_2) \\ &= \int_{\Sigma_2} |\langle \iota^* \pi_2(s_2) \iota(u_1), v_1 \rangle|^2 d\mu_2(s_2) \\ &= \int_{\Sigma_2} |\langle \pi_2(s_2) \iota(u_1), \iota(v_1) \rangle|^2 d\mu_2(s_2) \\ &= \|\iota(u_1)\|^2 \|\iota(v_1)\|^2 = \|u_1\|^2 \|v_1\|^2, \end{aligned}$$

which shows that $\pi_1 \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}_1), \mu_1)$. The assertion on the commutative diagram is then clear, and this concludes the proof. $\square$

3. H^*-algebra $\mathscr{B}_2(\Sigma)$ and larger symbol spaces. We place ourselves in the setting of Section 2; in particular, we are given a map $\pi : \Sigma \to \mathbb{B}(\mathcal{H})$ belonging to $SQ(\mathbb{B}(\mathcal{H}), \mu)$. We do not assume the family

$$\{\phi_{u,v} \equiv \Phi(u \otimes v) \mid u, v \in \mathcal{H}\}$$

to be total in $L^2(\Sigma)$. For instance, this property fails if π is any irreducible representation of any compact group $\Sigma \neq \{1\}$, since we then have $\dim \mathcal{H} < \infty$ and the Peter-Weyl decomposition of $L^2(\Sigma)$ see also Corollary 2.5.

Consequently, we need to introduce the closed subspace

$$\mathscr{B}_2(\Sigma) := \Phi\big(\mathcal{H} \,\widehat{\otimes}\, \overline{\mathcal{H}}\big) \subseteq L^2(\Sigma),$$

which is unitarily equivalent to $\mathcal{H} \,\widehat{\otimes}\, \overline{\mathcal{H}}$, and hence, with $\mathbb{B}_2(\mathcal{H})$. This space $\mathscr{B}_2(\Sigma)$ is the closure in $L^2(\Sigma)$ of the subspace $\Phi(\mathcal{H} \otimes \overline{\mathcal{H}})$. Clearly, there is a unitary operator

$$\Pi := \Lambda \circ \Phi^{-1} : \mathscr{B}_2(\Sigma) \longrightarrow \mathbb{B}_2(\mathcal{H}), \tag{3.1}$$

uniquely determined by

$$\Pi(\phi_{u,v}) = \lambda_{u,v} = \langle \cdot, v\rangle u, \quad \text{for all } u, v \in \mathcal{H}, \tag{3.2}$$

and satisfying

$$\mathrm{Tr}[\Pi(f)\Pi(g)^*] = \langle f, g\rangle_{(\Sigma)} := \int_\Sigma f(s)\overline{g(s)}\, d\mu(s),$$

for all $f, g \in \mathscr{B}_2(\Sigma)$. For the sake of clarity, we also note the commutative diagram

$$\begin{tikzcd} \mathcal{H}\widehat{\otimes}\overline{\mathcal{H}} \arrow[r, "\Phi"] \arrow[d, "\Lambda"'] & \mathscr{B}_2(\Sigma) \arrow[dl, "\Pi"] \\ \mathbb{B}_2(\mathcal{H}) & \end{tikzcd} \tag{3.3}$$

whose arrows are isomorphisms of H^*-algebras, and, in particular, unitary operators.

Remark 3.1. Commutative diagram (3.3) can be connected with previous constructions. For example, in the context of Proposition 2.9, we can use $\pi_j \in SQ(\mathbb{B}(\mathcal{H}_j), \mu_j)$ to construct the H^*-algebra $\mathscr{B}_2(\Sigma_j)$ and the map Π_j, while $\pi_1 \otimes \pi_2$ serves in the same way to construct the H^*-algebra $\mathscr{B}_2(\Sigma_1 \times \Sigma_2)$ and the map Π. As a direct consequence of Proposition 2.9, $\mathscr{B}_2(\Sigma_1 \times \Sigma_2)$ can be identified with $\mathscr{B}_2(\Sigma_1)\widehat{\otimes}\mathscr{B}_2(\Sigma_2)$ and Π with $\Pi_1 \otimes \Pi_2$. In the setting of Proposition 2.10, one has $f \in \mathscr{B}_2(\Sigma_1)$ if and only if

$$f \circ p \in \Phi^{\pi_2}\big(\iota\mathcal{H}_1 \,\widehat{\otimes}\, \overline{\iota\mathcal{H}_1}\big) \subset \mathscr{B}_2(\Sigma_2),$$

and then $\Pi_1(f) = \iota^*\Pi_2(f \circ p)\iota$. If ι is unitary, the two H^*-algebras $\mathscr{B}_2(\Sigma_1)$ and $\mathscr{B}_2(\Sigma_2)$ are isomorphic through the transformation

$$f \longmapsto f \circ p.$$

3.1. Basic properties of Π.

Proposition 3.2. *For any $f \in \mathscr{B}_2(\Sigma)$, in a weak sense one has*

$$\Pi(f) = \int_\Sigma f(s)\pi(s)^* d\mu(s), \qquad \Pi(f)^* = \int_\Sigma \overline{f(s)}\pi(s)\, d\mu(s). \tag{3.4}$$

Proof. If $u, v \in \mathcal{H}$, then one has

$$\begin{aligned}\langle \Pi(f)u, v\rangle &= \mathrm{Tr}[\lambda_{\Pi(f)u,v}] = \mathrm{Tr}[\Pi(f)\lambda_{u,v}] \\ &= \mathrm{Tr}[\Pi(f)\Pi(\phi_{v,u})^*] = \langle f, \phi_{v,u}\rangle_{(\Sigma)} \\ &= \int_\Sigma f(s)\langle \pi(s)^* u, v\rangle\, d\mu(s).\end{aligned}$$

Then, the second formula follows from the first one. $\square$

The next simple corollary is needed in the proof of Proposition 7.1.

Corollary 3.3. *The adjoint of the isometry*

$$\Phi \circ \Lambda^{-1} \colon \mathbb{B}_2(\mathcal{H}) \longrightarrow L^2(\Sigma)$$

is given by the weakly convergent integral

$$(\Phi \circ \Lambda^{-1})^* f = \int_\Sigma f(s)\pi(s)^* d\mu(s),$$

for every $f \in L^2(\Sigma)$.

Proof. One has

$$\Phi \circ \Lambda^{-1} = \Pi^{-1} = \Pi^*,$$

since Π is unitary. The assertion now follows by Proposition 3.2. $\square$

By transport of structure, a composition law and an involution may be defined by

$$\begin{aligned}\star : \mathscr{B}_2(\Sigma) \times \mathscr{B}_2(\Sigma) \longrightarrow \mathscr{B}_2(\Sigma), &\quad f \star g := \Pi^{-1}[\Pi(f)\Pi(g)], \\ {}^\star : \mathscr{B}_2(\Sigma) \longrightarrow \mathscr{B}_2(\Sigma), &\quad f^\star := \Pi^{-1}[\Pi(f)^*].\end{aligned}$$

Therefore, $\mathscr{B}_2(\Sigma)$ is an *H^*-algebra* and $\Pi : \mathscr{B}_2(\Sigma) \to \mathbb{B}_2(\mathcal{H})$ is an *H^*-algebra isomorphism*. Thus, for all $f, g, h \in \mathscr{B}_2(\Sigma)$:

$$\langle f^\star, g^\star \rangle_{(\Sigma)} = \langle g, f \rangle_{(\Sigma)},$$
$$\langle f \star g, h \rangle_{(\Sigma)} = \langle f, h \star g^\star \rangle_{(\Sigma)} = \langle g, f^\star \star h \rangle_{(\Sigma)}.$$

The Hilbert subalgebra $\mathscr{B}_1(\Sigma) := \mathscr{B}_2(\Sigma) \star \mathscr{B}_2(\Sigma)$ is dense in $\mathscr{B}_2(\Sigma)$, and, for every $g \in \mathscr{B}_2(\Sigma)$, the maps

$$\mathscr{B}_2(\Sigma) \ni f \longmapsto g \star f \in \mathscr{B}_2(\Sigma), \qquad \mathscr{B}_2(\Sigma) \ni f \longmapsto f \star g \in \mathscr{B}_2(\Sigma)$$

are continuous.

For further use, note the relations

$$\langle \phi_{u_1,v_1}, \phi_{u_2,v_2} \rangle_{(\Sigma)} = \langle u_1, u_2 \rangle \langle v_2, v_1 \rangle \tag{3.5}$$
$$\phi_{u_1,v_1} \star \phi_{u_2,v_2} = \langle u_2, v_1 \rangle \phi_{u_1,v_2}, \qquad \phi^\star_{u,v} = \phi_{v,u}, \tag{3.6}$$

are valid for every u, u_1, u_2, v, v_1, $v_2 \in \mathcal{H}$, as well as

$$f \star \phi_{u,v} \star g = \phi_{\Pi(f)u, \Pi(g)^* v}, \quad \text{for all } f, g \in \mathscr{B}_2(\Sigma), \quad u, v \in \mathcal{H}.$$

In particular, if $\|u\| = 1$, then $\phi_{u,u}$ is a self-adjoint projection, represented by Π as the rank-one operator $\lambda_{u,u}$. Also, note that $\mathscr{B}_1(\Sigma)$ and $\mathscr{B}_2(\Sigma)$ are Banach *-algebras, where $\mathscr{B}_1(\Sigma)$ is endowed with the norm for which the linear bijection $\Pi \colon \mathscr{B}_1(\Sigma) \to \mathbb{B}_1(\mathcal{H})$ is an isometry.

3.2. Extensions and the explicit form of the composition law. In this subsection, we extend some of the above maps to larger symbol spaces. We define a new norm

$$\| \cdot \|_{\mathscr{B}(\Sigma)} : \mathscr{B}_2(\Sigma) \longrightarrow \mathbb{R}_+, \qquad \|f\|_{\mathscr{B}(\Sigma)} := \|\Pi(f)\|_{\mathbb{B}(\mathcal{H})}.$$

The completion of $\mathscr{B}_2(\Sigma)$ under this norm is a C^*-algebra $\mathscr{B}_\infty(\Sigma)$ containing $\mathscr{B}_2(\Sigma)$ as a dense *-ideal (also endowed with the stronger Hilbert topology). Clearly, Π extends to a C^*-algebraic monomorphism $\Pi : \mathscr{B}_\infty(\Sigma) \to \mathbb{B}(\mathcal{H})$ with range $\Pi[\mathscr{B}_\infty(\Sigma)] = \mathbb{B}_\infty(\mathcal{H})$, the ideal of all compact operators in $\mathcal{H}$. Then, we denote by $\mathscr{B}(\Sigma)$ the multiplier C^*-algebra [**38**] of $\mathscr{B}_\infty(\Sigma)$, which is isomorphic by a canonical extension of Π with $\mathbb{B}(\mathcal{H})$ and, in turn, can be identified with the multiplier C^*-algebra of $\mathbb{B}_\infty(\mathcal{H})$. We keep the same notation $\star$ and * for the composition law and the involution on $\mathscr{B}(\Sigma)$. Based on the constructions above, the elements of $\mathscr{B}_1(\Sigma)$ ($\mathscr{B}_2(\Sigma)$, $\mathscr{B}_\infty(\Sigma)$) will be called *trace-class* (*Hilbert-Schmidt, compact*) *symbols*, respectively. To eliminate any possible confusion, we reserve the term *operator-bounded symbols* for the elements of $\mathscr{B}(\Sigma)$.

The spaces $\mathscr{B}_q(\Sigma),\ q = 1, 2, \infty$ remain *-ideals in $\mathscr{B}(\Sigma)$, and the scalar product $\langle \cdot, \cdot \rangle_{\mathscr{B}_2(\Sigma)}$ can be "extended" to sesquilinear forms

$$\langle \cdot, \cdot \rangle_{(\Sigma)} : \mathscr{B}_1(\Sigma) \times \mathscr{B}(\Sigma) \longrightarrow \mathbb{C},$$
$$\langle \cdot, \cdot \rangle_{(\Sigma)} : \mathscr{B}(\Sigma) \times \mathscr{B}_1(\Sigma) \longrightarrow \mathbb{C}.$$

For this, one simply sets $\langle f, g \rangle_{(\Sigma)} := \mathrm{Tr}[\Pi(f)\Pi(g)^*]$ (definitions by approximation are also available). Note for $f \in \mathscr{B}_1(\Sigma)$ and $g \in \mathscr{B}(\Sigma)$ that the inequality holds:

$$|\langle f, g \rangle_{(\Sigma)}| \leq \|f\|_{\mathscr{B}_1(\Sigma)} \|g\|_{\mathscr{B}(\Sigma)} \equiv \|\Pi(f)\|_{\mathbb{B}_1(\mathcal{H})} \|\Pi(g)\|_{\mathbb{B}(\mathcal{H})}.$$

Due to the cyclicity of the trace, if $f, g, h \in \mathscr{B}(\Sigma)$, and one belongs to $\mathscr{B}_1(\Sigma)$ (or two belong to $\mathscr{B}_2(\Sigma)$), then one has

$$\langle f \star g, h \rangle_{(\Sigma)} = \langle f, h \star g^\star \rangle_{(\Sigma)} = \langle g, f^\star \star h \rangle_{(\Sigma)}.$$

Let us set

$$e_s := \Pi^{-1}[\pi(s)^*] \in \mathscr{B}(\Sigma) \quad \text{for all } s \in \Sigma;$$

hence, $\phi_{u,v}(s) = \langle u, \Pi(e_s)v \rangle$ for all u, v s and

$$\pi(s)^* = \Pi(e_s), \qquad \pi(s) = \Pi(e_s^\star). \tag{3.7}$$

Proposition 3.4. *For every $f \in \mathscr{B}_1(\Sigma)$, one has*

$$\langle f, e_s \rangle_{(\Sigma)} = f(s), \qquad \langle f, e_s^\star \rangle_{(\Sigma)} = \overline{f^\star(s)}, \tag{3.8}$$

μ-almost everywhere $s \in \Sigma$.

Proof. By direct computation using equations (1.2), (1.3) and (3.2) for $u, v \in \mathcal{H}$, $s \in \Sigma$, one gets

$$\langle \phi_{u,v}, e_s \rangle_\Sigma = \mathrm{Tr}[\Pi(\phi_{u,v})\pi(s)] = \mathrm{Tr}[\lambda_{u,v}\pi(s)] = \mathrm{Tr}[\lambda_{u,\pi(s)^*v}] = \phi_{u,v}(s).$$

Thus, the same is true for $\phi_{u,v}$ replaced by any element of $\Phi(\mathcal{H} \otimes \overline{\mathcal{H}})$, which is dense in $\mathscr{B}_1(\Sigma)$.

Now, assume that a sequence

$$\{g_n\}_{n \in \mathbb{N}} \subset \Phi\Big(\mathcal{H} \otimes \overline{\mathcal{H}}\Big)$$

converges to $f \in \mathscr{B}_1(\Sigma)$, with respect to the trace norm. Then, for each $s \in \Sigma$,

$$\begin{aligned}|\langle f, e_s\rangle_{(\Sigma)} - \langle g_n, e_s\rangle_{(\Sigma)}| &= |\mathrm{Tr}\,[\Pi(f - g_n)\pi(s)]| \\ &\le \|\Pi(f - g_n)\|_{\mathbb{B}_1(\mathcal{H})}\|\pi(s)\|_{\mathbb{B}(\mathcal{H})} \\ &= \|f - g_n\|_{\mathscr{B}_1(\Sigma)}\|\pi(s)\|_{\mathbb{B}(\mathcal{H})} \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

recalling that $\Pi\colon \mathscr{B}_1(\Sigma) \to \mathbb{B}_1(\mathcal{H})$ is an isometry by the definition of the norm of $\mathscr{B}_1(\Sigma)$. Since convergence in $\mathscr{B}_1(\Sigma)$ implies convergence in $L^2(\Sigma)$ which, in turn, implies μ-almost everywhere convergence of a subsequence, there is a μ-negligible set $M \subset \Sigma$ and a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ such that, for every $s \in \Sigma \setminus M$,

$$f(s) = \lim_{k\to\infty} g_{n_k}(s) = \lim_{k\to\infty} \langle g_{n_k}, e_s\rangle_{(\Sigma)} = \langle f, e_s\rangle_{(\Sigma)}.$$

Then, for s in the same set $s \in \Sigma \setminus M$, one has

$$\langle f, e_s^\star\rangle_{(\Sigma)} = \langle e_s, f^\star\rangle_{(\Sigma)} = \overline{\langle f^\star, e_s\rangle}_{(\Sigma)} = \overline{f^\star(s)}. \quad \square$$

Corollary 3.5. *For every $f, g \in \mathscr{B}_1(\Sigma)$, one has*

$$\int_\Sigma \langle f, e_s\rangle_{(\Sigma)}\langle e_s, g\rangle_{(\Sigma)}\, d\mu(s) = \langle f, g\rangle_{(\Sigma)}.$$

Proof. Follows immediately from Proposition 3.4. $\square$

Now, we can compute the symbol of a trace-class operator.

Corollary 3.6. *For $T \in \mathbb{B}_1(\mathcal{H}) \subset \mathbb{B}_2(\mathcal{H})$, one has*

$$[\Pi^{-1}(T)](s) = \mathrm{Tr}\,[\,T\pi(s)], \tag{3.9}$$

μ-almost everywhere $s \in \Sigma$.

Proof. For the moment, let us denote the mapping defined in equation (3.9) by $\Pi^{(-1)}$. It is enough to show that $\Pi^{(-1)}[\Pi(f)] = f$ holds μ-almost everywhere, for every f belonging to $\mathscr{B}_1(\Sigma)$.

But, for μ-almost every $s \in \Sigma$, one has, by equations (3.7) and (3.8),

$$(\Pi^{(-1)}[\Pi(f)])(s) = \mathrm{Tr}\,[\,\Pi(f)\pi(s)] = \langle f, e_s\rangle_{(\Sigma)} = f(s). \quad \square$$

Corollary 3.7. *For each* $S \in \mathbb{B}_1(\mathcal{H})$ *and each* $f \in \mathscr{B}_2(\Sigma)$ *one has*

$$\mathrm{Tr}\,[\Pi(f)S] = \int_\Sigma f(s)\mathrm{Tr}\,[\pi(s)^* S]d\mu(s),$$

$$\mathrm{Tr}\,[\Pi(f)^* S] = \int_\Sigma \overline{f(s)}\,\mathrm{Tr}\,[\pi(s)S]d\mu(s). \tag{3.10}$$

Proof. Using the definitions, the fact that Π is unitary and formula (3.9),

$$\begin{aligned}\mathrm{Tr}[\Pi(f)S] =& \langle \Pi(f), \Pi[\Pi^{-1}(S^*)]\rangle_{\mathbb{B}_2(\mathcal{H})} = \langle f, \Pi^{-1}(S^*)\rangle_{(\Sigma)}\\ &= \int_\Sigma f(s)\overline{[\Pi^{-1}(S^*)](s)}d\mu(s) = \int_\Sigma f(s)\overline{\mathrm{Tr}[S^*\pi(s)]}d\mu(s)\\ &= \int_\Sigma f(s)\mathrm{Tr}\,[\pi(s)^* S]\,d\mu(s).\end{aligned}$$

Relation (3.10) follows similarly. □

Corollary 3.7, which can be alternatively derived from equation (3.4), reinforces Proposition 3.2, recovered by taking S to be a rank 1 operator.

Remark 3.8. One can also justify, for every $f \in \mathscr{B}_2(\Sigma)$, the relations

$$f = \int_\Sigma f(s)e_s d\mu(s), \qquad f^\star = \int_\Sigma \overline{f(s)} e_s^\star d\mu(s);$$

for example, if $g \in \mathscr{B}_1(\Sigma)$, then

$$\int_\Sigma f(s)\langle e_s, g\rangle_{(\Sigma)}\,d\mu(s) = \int_\Sigma f(s)\overline{g(s)}d\mu(s) = \langle f, g\rangle_{(\Sigma)}.$$

In general, $\mathscr{B}_2(\Sigma)$ is not a reproducing kernel Hilbert space as the symbols e_s rarely belong to $\mathscr{B}_2(\Sigma)$.

Now, we give explicit formulae for the algebraic structure. In the following statement, we use the fact that, for any complex Hilbert space $\mathcal{H}$, the Banach algebra of trace-class operators $\mathbb{B}_1(\mathcal{H})$ has right approximate units. For instance, the family of orthogonal projections onto finite-dimensional subspaces of $\mathcal{H}$ is a two-sided approximate unit of $\mathbb{B}_1(\mathcal{H})$. This follows by **[10,** Chapter III, Theorem 6.3] on separable Hilbert spaces, and then it extends to arbitrary Hilbert spaces, every operator in $\mathbb{B}_1(\mathcal{H})$ written as a linear combination of self-adjoint operators, and using the fact that the closure of the range of any compact self-adjoint operator is separable.

Theorem 3.9. *Let $\{S_j \mid j \in J\}$ be any right approximate unit in $\mathbb{B}_1(\mathcal{H})$.*

(i) *If $f \in \mathscr{B}_1(\Sigma)$, for μ-almost every $r \in \Sigma$, one has*

$$f^\star(r) = \lim_j \int_\Sigma \mathrm{Tr}\,[\pi(r)\pi(s)S_j]\overline{f(s)}\,d\mu(s).$$

(ii) *If $f, g \in \mathscr{B}_2(\Sigma)$, for μ-almost every $r \in \Sigma$, one has*

$$(f \star g)(r) = \lim_j \int_\Sigma \int_\Sigma \mathrm{Tr}\,[\pi(s)^*\pi(t)^*\pi(r)S_j]f(s)g(t)\,d\mu(s)\,d\mu(t).$$

Proof. Both computations rely on Corollaries 3.6 and 3.7. One has for μ-almost every $r \in \Sigma$,

$$\begin{aligned} f^\star(r) &= \mathrm{Tr}\,[\Pi(f)^*\pi(r)] = \lim_j \mathrm{Tr}\,[\Pi(f)^*\pi(r)S_j] \\ &= \lim_j \int_\Sigma \mathrm{Tr}\,[\pi(s)\pi(r)S_j]\overline{f(s)}\,d\mu(s), \end{aligned}$$

and

$$\begin{aligned} (f \star g)(r) &= \mathrm{Tr}\,[\Pi(f)\Pi(g)\pi(r)] = \lim_j \mathrm{Tr}\,[\Pi(f)\Pi(g)\pi(r)S_j] \\ &= \lim_j \int_\Sigma f(s)\mathrm{Tr}\,[\pi(s)^*\Pi(g)\pi(r)S_j]\,d\mu(s) \\ &= \lim_j \int_\Sigma f(s)\,d\mu(s) \int_\Sigma g(t)\mathrm{Tr}\,[\pi(s)^*\pi(t)^*\pi(r)S_j]\,d\mu(t), \end{aligned}$$

and this concludes the proof. □

Remark 3.10. Let us assume that Σ is a locally compact space and we have $\|\pi(s)\|_{\mathbb{B}(\mathcal{H})} \le C < \infty$ for all $s \in \Sigma$. Let $\mathscr{R}(\Sigma)$ be the Banach space of all Radon bounded complex measures on Σ, seen alternatively both as functions on the Borel sets of Σ and as elements of the topological anti-dual of $C_0(\Sigma)$. Using the Hahn-Banach theorem, one can easily find a norm-preserving extension of every $\rho \in \mathscr{R}(\Sigma)$ to an anti-linear continuous functional $\rho : BC(\Sigma) \to \mathbb{C}$, where $BC(\Sigma)$ denotes the

Banach space of all bounded continuous functions on Σ. We will use the notation $\langle\langle \rho, f\rangle\rangle = \int_\Sigma \overline{f}\, d\rho$ for this "duality" (linear in ρ and anti-linear in f). On $\mathscr{R}(\Sigma)$, the usual norm of an anti-dual coincides with the measure norm expressed as the total variation applied to the entire space Σ. Since, for every $u, v \in \mathcal{H}$, one has $\phi_{v,u} \in \mathscr{B}_2(\Sigma) \cap BC(\Sigma)$, one can define $\Pi(\rho) \in \mathbb{B}(\mathcal{H})$ in a weak sense by

$$\langle \Pi(\rho)u, v\rangle := \int_\Sigma \langle \pi(s)^* u, v\rangle\, d\rho(s) = \langle\langle \rho, \overline{\langle \pi(\cdot)^* u, v\rangle}\rangle\rangle = \langle\langle \rho, \phi_{v,u}\rangle\rangle.$$

Now, it is also obvious that e_t coincides with the Dirac measure concentrated in t and that, if ρ has a density g with respect to the initial measure μ, then $\Pi(\rho) = \Pi(g)$. The estimate,

$$\|\Pi(\rho)\|_{\mathbb{B}(\mathcal{H})} \le \|\rho\|_{\mathscr{R}(\Sigma)} \sup_{s\in\Sigma} \|\pi(s)\|_{\mathbb{B}(\mathcal{H})},$$

is easy and certifies that $L^1(\Sigma, \mu) \subset \mathscr{R}(\Sigma) \subset \mathscr{B}(\Sigma)$.

4. Fréchet-Hilbert algebras and their Gelfand triples. In most applications, there is some supplementary structure which can be used to enrich and enlarge the formalism. Let $\mathcal{G}$ be a Fréchet space continuously and densely embedded in $\mathcal{H}$, and set $\alpha : \mathcal{G} \to \mathcal{H}$ for the embedding. We will show that this extra data generates many new, useful objects, even if we do not require $\pi(s)\mathcal{G} \subset \mathcal{G}$ for $s \in \Sigma$. Assuming that $\mathcal{G}$ is nuclear would simplify the overall picture, but we also want to cover the case of Banach spaces.

Lemma 4.1. *The projective tensor product $\mathcal{G} \widehat{\otimes}_p \overline{\mathcal{G}}$ is a Fréchet space continuously and densely embedded in the Hilbert space $\mathcal{H} \widehat{\otimes} \overline{\mathcal{H}}$.*

Proof. Clearly, $\mathcal{G} \widehat{\otimes}_p \overline{\mathcal{G}}$ is a Fréchet space. It is enough to embed it continuously and densely into $\mathcal{H}\widehat{\otimes}_p\overline{\mathcal{H}}$. The canonical mapping,

$$\alpha \otimes \alpha : \mathcal{G} \otimes \overline{\mathcal{G}} \longrightarrow \mathcal{H} \otimes \overline{\mathcal{H}},$$

is linear and continuous when we put the corresponding projective topologies [**36,** Proposition 43.6] on both tensor products, so it extends to a linear continuous map

$$\alpha \widehat{\otimes}_p \alpha : \mathcal{G} \widehat{\otimes}_p \overline{\mathcal{G}} \longrightarrow \mathcal{H}\widehat{\otimes}_p\overline{\mathcal{H}}.$$

This map obviously has a dense range, and it is injective [**36,** Example 43.2]. By composing with the canonical injection $\mathcal{H}\widehat{\otimes}_p\overline{\mathcal{H}} \hookrightarrow \mathcal{H}\mathbin{\widehat{\otimes}}\overline{\mathcal{H}}$, we get the linear continuous map $\mathfrak{a} : \mathcal{G}\mathbin{\widehat{\otimes}_p}\overline{\mathcal{G}} \to \mathcal{H}\mathbin{\widehat{\otimes}}\overline{\mathcal{H}}$ (injective and with dense image). $\square$

By slight abuse of notation, we will treat $\mathcal{G}$ as a dense subspace of $\mathcal{H}$ and $\mathcal{G}\mathbin{\widehat{\otimes}_p}\overline{\mathcal{G}}$ as a dense subspace of $\mathcal{H}\mathbin{\widehat{\otimes}}\overline{\mathcal{H}}$. Let us set

$$\mathscr{G}(\Sigma) := \Phi\Big[\mathcal{G}\mathbin{\widehat{\otimes}_p}\overline{\mathcal{G}}\Big] \subset \mathscr{B}_2(\Sigma) \subset L^2(\Sigma).$$

Theorem 4.2. *With the algebraic and topological structure induced from $\mathcal{G}\mathbin{\widehat{\otimes}_p}\overline{\mathcal{G}}$, along with the scalar product $\langle\cdot,\cdot\rangle_{(\Sigma)}$, the space $\mathscr{G}(\Sigma)$ becomes a Fréchet-Hilbert algebra (as defined at the end of the introduction) composed of trace-class symbols, continuously and densely embedded into $\mathscr{B}_2(\Sigma)$. Its dual $\mathscr{G}'(\Sigma)$ contains all of the operator-bounded symbols. The restriction $\Phi : \mathcal{G}\mathbin{\widehat{\otimes}_p}\overline{\mathcal{G}} \to \mathscr{G}(\Sigma)$ is an isomorphism of Fréchet-Hilbert algebras.*

Proof. Clearly, $\mathscr{G}(\Sigma)$ is turned into a Fréchet space by transport of structure. It is densely contained in $\mathscr{B}_2(\Sigma)$ because $\mathcal{G}\mathbin{\widehat{\otimes}_p}\overline{\mathcal{G}}$ is densely contained in $\mathcal{H}\mathbin{\widehat{\otimes}}\overline{\mathcal{H}}$.

The basic complete tensor products in the case of Hilbert spaces are described in [**36,** Section 48]. For instance, by [**36,** Theorem 48.3], $\mathcal{H}\widehat{\otimes}_p\overline{\mathcal{H}}$ can be identified with $\mathbb{B}_1(\mathcal{H})$ while $\mathcal{H}\widehat{\otimes}_i\overline{\mathcal{H}}$ is canonically isomorphic to $\mathbb{B}_\infty(\mathcal{H})$. We recall that the Hilbert space tensor product $\mathcal{H}\mathbin{\widehat{\otimes}}\overline{\mathcal{H}}$ is isomorphic to $\mathbb{B}_2(\mathcal{H})$. Taking into account the continuous embedding of $\mathcal{G}\mathbin{\widehat{\otimes}_p}\overline{\mathcal{G}}$ into $\mathcal{H}\widehat{\otimes}_p\overline{\mathcal{H}}$, the assertions $\mathscr{G}(\Sigma) \subset \mathscr{B}_1(\Sigma)$ follow, and then $\mathscr{B}_1(\Sigma)' \subset \mathscr{G}'(\Sigma)$ becomes obvious. However, the dual of $\mathbb{B}_1(\mathcal{H})$ is $\mathbb{B}(\mathcal{H})$, which permits the identification of $\mathscr{B}_1(\Sigma)'$ with $\mathscr{B}(\Sigma) \subset \mathscr{G}'(\Sigma)$.

By the very definition of the structure of $\mathscr{G}(\Sigma)$ by transport of structure from $\mathcal{G}\widehat{\otimes}_p\overline{\mathcal{G}}$ via Φ, it is enough to check that $\mathcal{G}\widehat{\otimes}_p\overline{\mathcal{G}}$ is a Fréchet-Hilbert algebra. Recall from the definition at the end of the introduction that a Hilbert algebra need not be complete with respect to the topology defined by its scalar product. On $\mathcal{G}\otimes\overline{\mathcal{G}}$ the algebraic structure is uniquely defined by $(u\otimes v)^* := v\otimes u$ and

$$(u\otimes v)\cdot(u'\otimes v') = \langle u', v\rangle(u\otimes v') \in \mathcal{G}\otimes\overline{\mathcal{G}},$$

valid for $u, v, u', v' \in \mathcal{G}$. Then, one can conclude by density, if it is checked, that the involution and the multiplication are continuous, with respect to the projective topology. For the involution, this is very easy; we will treat the multiplication.

Let $\{p_\lambda \mid \lambda \in \Lambda\}$ be a directed family of seminorms defining the topology of $\mathcal{G}$. Since $\mathcal{G}$ is continuously contained in $\mathcal{H}$, by [**36,** Proposition 7.7] there exist $\lambda_0 \in \Lambda$ and $C > 0$ such that $\|u\| \leq C p_{\lambda_0}(u)$ for all $u \in \mathcal{G}$. Also, by [**36,** Proposition 43.1], the projective topology on $\mathcal{G} \otimes \overline{\mathcal{G}}$ is defined by the family of seminorms $\{p_{\lambda,\mu} \mid (\lambda, \mu) \in \Lambda \times \Lambda\}$, given by

$$p_{\lambda,\mu}(\mathfrak{w}) := \inf\left\{ \sum_l p_\lambda(w_l) p_\mu(w'_l) \;\middle|\; \mathfrak{w} = \sum_l w_l \otimes w'_l \right\};$$

all the sums should be finite. For any $\lambda, \mu \in \Lambda$ and any $\mathfrak{u}, \mathfrak{v} \in \mathcal{G} \otimes \overline{\mathcal{G}}$, one has

$$\begin{aligned}
p_{\lambda,\mu}(\mathfrak{u}\cdot\mathfrak{v}) &= \inf\left\{ \sum_l p_\lambda(w_l) p_\mu(w'_l) \;\middle|\; \mathfrak{u}\cdot\mathfrak{v} = \sum_l w_l \otimes w'_l \right\} \\
&\leq \inf\Bigg\{ \sum_{j,k} p_\lambda(\langle v_k, u'_j\rangle u_j) p_\mu(v'_k) \\
&\qquad\qquad \Bigg| \; \mathfrak{u} = \sum_j u_j \otimes u'_j, \mathfrak{v} = \sum_k v_k \otimes v'_k \Bigg\} \\
&\leq \inf\Bigg\{ \sum_{j,k} \|u'_j\| \|v_k\| p_\lambda(u_j) p_\mu(v'_k) \\
&\qquad\qquad \Bigg| \; \mathfrak{u} = \sum_j u_j \otimes u'_j, \mathfrak{v} = \sum_k v_k \otimes v'_k \Bigg\} \\
&\leq C^2 \inf\Bigg\{ \sum_j p_\lambda(u_j) p_{\lambda_0}(u'_j) \sum_k p_{\lambda_0}(v_k) p_\mu(v'_k) \\
&\qquad\qquad \Bigg| \; \mathfrak{u} = \sum_j u_j \otimes u'_j, \mathfrak{v} = \sum_k v_k \otimes v'_k \Bigg\} \\
&= C^2 \inf\left\{ \sum_j p_\lambda(u_j) p_{\lambda_0}(u'_j) \;\middle|\; \mathfrak{u} = \sum_j u_j \otimes u'_j \right\}
\end{aligned}$$

$$\times \inf \left\{ \sum_k p_{\lambda_0}(v_k) p_\mu(v'_k) \mid \mathfrak{v} = \sum_k v_k \otimes v'_k \right\}$$
$$= C^2 p_{\lambda,\lambda_0}(\mathfrak{u}) p_{\lambda_0,\mu}(\mathfrak{v}),$$

which justifies the continuity of the product. □

For further reference, we indicate here the continuous embeddings

$$\mathscr{G}(\Sigma) \hookrightarrow \mathscr{B}_1(\Sigma) \hookrightarrow \mathscr{B}_2(\Sigma) \hookrightarrow \mathscr{B}_\infty(\Sigma) \hookrightarrow \mathscr{B}(\Sigma) \hookrightarrow \mathscr{G}'(\Sigma). \tag{4.1}$$

The first and last embeddings become isomorphisms if and only if $\mathcal{G} = \mathcal{H}$. In most of the cases, the operators $\pi(s) = \Pi(e_s)^*$ are unitary; in such cases, e_s is not a compact symbol.

Remark 4.3. The above direct construction of the Fréchet-Hilbert algebra $\mathscr{G}(\Sigma)$ by transport of structure is convenient because it is universal, but the output is rather implicit (although, clearly,

$$\{\phi_{u,v} \mid u, v \in \mathcal{G}\}$$

is a total family in $\mathscr{G}(\Sigma)$; see [**36,** Theorem 45.1] for a stronger result). Fortunately, in most of the interesting examples, the space $\mathscr{G}(\Sigma)$ has some independent definition as a space of functions (or distributions) on Σ. The scale of spaces given in equation (4.1) can be used to extend the composition law by duality techniques and to define optimal "Moyal-type" *-algebras, as in [**30**].

Now, we need some notions concerning duality of Fréchet spaces [**36,** Section 19]. When, on the (topological) dual $\mathcal{G}'$, we consider the weak*-topology, we write $\mathcal{G}'_\sigma$. Recall that, in this topology, the convergence is merely the pointwise convergence of functionals and that a base of neighborhoods of $0 \in \mathcal{G}'_\Sigma$ is composed of the polars of all the finite subsets of $\mathcal{G}$. However, we are also going to use the stronger topology γ of uniform convergence on convex compact subsets of $\mathcal{G}$, and then $\mathcal{G}'$ will be denoted by $\mathcal{G}'_\gamma$. One can take the polars of the convex compact subsets of $\mathcal{G}$ as a base of $0 \in \mathcal{G}'_\gamma$.

Using the transpose $\alpha' : \mathcal{H}' \to \mathcal{G}'$, cf., [**36,** Section 23], and the Riesz antilinear identification of $\mathcal{H}$ with its strong dual $\mathcal{H}'$, one gets an injective continuous antilinear embedding of $\mathcal{H}$ into the dual $\mathcal{G}'$ (or, equivalently, a linear embedding of $\overline{\mathcal{H}}$ in $\mathcal{G}'$). Thus, we identify $\mathcal{H}$

with a subspace of $\mathcal{G}'$, which is dense if, on $\mathcal{G}'$, one considers either of the topologies σ or γ. Hence, we have a Gelfand triple $(\mathcal{G}, \mathcal{H}, \mathcal{G}'_\nu)$ for $\nu = \sigma, \gamma$. Since the duality between $\mathcal{G}$ and $\mathcal{G}'$ is compatible with the scalar product, we can use notation for this duality, such as $\langle u, w\rangle := w(u)$, antilinear in $w \in \mathcal{G}'$ and linear in $u \in \mathcal{G}$.

Note that $\mathcal{G}$ can be seen both as the dual of its weak*-dual $\mathcal{G}'_\sigma$ and as the dual of $\mathcal{G}'_\gamma$, cf., [**36**]. In general, it does not coincide with the dual of $\mathcal{G}'_\beta$, involving the strong topology β of uniform convergence on the bounded subsets of $\mathcal{G}$. By a simple duality argument, it follows that $\mathcal{H}$ (hence, $\mathcal{G}$ also) will be dense in $\mathcal{G}'_\nu$ for $\nu = \sigma, \gamma$. This also happens for $\nu = \beta$ if $\mathcal{G}$ is assumed reflexive. If $\mathcal{G}$ is a Banach space, we have a Banach-Gelfand triple. In such a case, the strong topology β on $\mathcal{G}'$ coincides with the norm topology given by

$$\|w\|_{\mathcal{G}'} := \sup\{|\langle u, w\rangle| \mid u \in \mathcal{G}, \|u\|_{\mathcal{G}} \leq 1\}.$$

The same construction can be applied to the continuous and dense embedding $\mathcal{G} \,\widehat{\otimes}_p\, \overline{\mathcal{G}} \hookrightarrow \mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$, obtaining an ampler Gelfand triple

$$(\mathcal{G} \,\widehat{\otimes}_p\, \overline{\mathcal{G}}, \mathcal{H} \,\widehat{\otimes}\, \overline{\mathcal{H}}, (\mathcal{G} \,\widehat{\otimes}_p\, \overline{\mathcal{G}})'_\nu);$$

one uses the transpose $\mathcal{H} \,\widehat{\otimes}\, \overline{\mathcal{H}} \overset{\mathfrak{a}'}{\hookrightarrow} (\mathcal{G} \,\widehat{\otimes}_p\, \overline{\mathcal{G}})'_\nu$ of the map $\mathfrak{a}$ constructed in the proof of Lemma 4.1. Here, once again, we may use any of the topologies $\nu = \sigma, \gamma, \beta$.

Now, we recall that we have an isomorphism of H^*-algebras $\Phi : \mathcal{H} \,\widehat{\otimes}\, \overline{\mathcal{H}} \to \mathscr{B}_2(\Sigma)$ (in particular, a unitary map) which restricts to an isomorphism of Fréchet-Hilbert algebras $\Phi : \mathcal{G} \,\widehat{\otimes}_p\, \overline{\mathcal{G}} \to \mathscr{G}(\Sigma)$. The inverse of the transpose will be a continuous extension (denoted by abuse of the same letter)

$$\Phi : (\mathcal{G} \,\widehat{\otimes}_p\, \overline{\mathcal{G}})'_\nu \longrightarrow \mathscr{G}'(\Sigma)_\nu; \tag{4.2}$$

the topological dual of $\mathscr{G}(\Sigma)$ has been denoted by $\mathscr{G}'(\Sigma)$.

We summarize the discussion above as a corollary. We denote an isomorphism of Gelfand triples (unitary at the level of Hilbert spaces) for which the "small" spaces are Fréchet-Hilbert algebras by *isomorphism of Hilbert algebra Gelfand triples.* The Hilbert spaces are H^*-algebras, and the isomorphism respects the *-algebra structures, whenever this makes sense.

Corollary 4.4. *Assume that the Fréchet space $\mathcal{G}$ is continuously and densely embedded in $\mathcal{H}$. There is a canonical isomorphism of Hilbert algebra Gelfand triples*

$$\Phi^\pi \equiv \Phi : (\mathcal{G} \widehat{\otimes}_p \overline{\mathcal{G}}, \mathcal{H} \widehat{\otimes} \overline{\mathcal{H}}, (\mathcal{G} \widehat{\otimes}_p \overline{\mathcal{G}})'_\nu) \longrightarrow (\mathscr{G}(\Sigma), \mathscr{B}_2(\Sigma), \mathscr{G}'(\Sigma)_\nu). \tag{4.3}$$

Notice that for $u, v \in \mathcal{G}'$ one has a well-defined element

$$\phi_{u,v} := \Phi(u \otimes v) \in \mathscr{G}'(\Sigma).$$

Remark 4.5. A canonical isomorphism $(\mathcal{G} \widehat{\otimes}_p \overline{\mathcal{G}})' \sim \mathbb{B}(\mathcal{G}, \mathcal{G}'_\sigma)$ exists which is purely algebraical and involves the space of all linear operators $A : \mathcal{G} \to \mathcal{G}'$ which are continuous when we put the weak*-topology on $\mathcal{G}'$. See [**36,** page 465] for more details. Using this, it is easy to deduce from the above considerations that

$$\Pi = \Lambda \circ \Phi^{-1} : \mathscr{B}_2(\Sigma) \longrightarrow \mathbb{B}_2(\mathcal{H})$$

extends to a linear isomorphism

$$\Pi : \mathscr{G}'(\Sigma) \longrightarrow \mathbb{B}(\mathcal{G}, \mathcal{G}'_\sigma).$$

Thus, the elements of $\mathscr{G}'(\Sigma)$ can be seen as symbols of linear operators $T : \mathcal{G} \to \mathcal{G}'$ that are continuous with respect to the weak*-topology on the dual. The relation

$$\langle \Pi(g)u, v\rangle = \int_\Sigma g(t)\langle \pi(t)^* u, v\rangle \, d\mu(t) = \langle g, \phi_{v,u}\rangle_{(\Sigma)}, \tag{4.4}$$

valid a priori for $g \in \mathscr{B}_2(\Sigma)$ and $u, v \in \mathcal{H}$, also stands true for $g \in \mathscr{G}'(\Sigma)$ and $u, v \in \mathcal{G}$ with the obvious reinterpretation of the duality $\langle \cdot, \cdot \rangle_{(\Sigma)}$.

Remark 4.6. Some extra structure is present if, in addition, $\mathcal{G}$ satisfies the approximation property. That property is shared by many specific examples of Fréchet spaces; see, for instance, [**22,** Section 18]. Here, it serves to identify the injective tensor product $\mathcal{G}' \widehat{\otimes}_i \overline{\mathcal{G}}'$ with another topological tensor product $\mathcal{G}' \in \overline{\mathcal{G}}'$, and thus, to simplify the picture. Under this extra assumption, one has isomorphisms

$$\mathbb{B}(\mathcal{G}, \mathcal{G}'_\sigma) \sim (\mathcal{G} \widehat{\otimes}_p \overline{\mathcal{G}})'_\gamma \cong \mathcal{G}'_\gamma \widehat{\otimes}_i \overline{\mathcal{G}}'_\gamma.$$

The second [**22,** page 346] is an isomorphism of locally convex spaces, and it involves the topology of uniform convergence on compact convex sets on various dual spaces. By general principles, it justifies the Hilbert algebra Gelfand triple

$$(\mathcal{G}\,\widehat{\otimes}_p\,\overline{\mathcal{G}},\ \mathcal{H}\widehat{\otimes}\,\overline{\mathcal{H}}, \mathcal{G}'_\gamma\,\widehat{\otimes}_i\,\overline{\mathcal{G}}'_\gamma).$$

Remark 4.7. Let us show how suitable subspaces of $\mathscr{B}_2(\Sigma)$ can be used to define the Fréchet spaces $\mathcal{G}$ and $\mathscr{G}(\Sigma)$ which are the topic of this section.

Let $\mathfrak{G}(\Sigma)$ be a continuously and densely embedded Fréchet space in $\mathscr{B}_2(\Sigma)$. We fix a unit vector $w \in \mathcal{H}$, and we set $w(s) := \pi(s)^*w$ for every $s \in \Sigma$ and $\phi_w(u) := \phi_{u,w}$ for every $u \in \mathcal{H}$. Then, we define

$$\mathcal{G}_w^{\mathfrak{G}(\Sigma)} \equiv \mathcal{G} := \phi_w^{-1}[\mathfrak{G}(\Sigma)] \subset \mathcal{H}.$$

It can be shown that $\mathcal{G}$ is a continuously and densely embedded Fréchet space in $\mathcal{H}$.

We will now give a concrete realization for $\mathscr{G}(\Sigma) := \Phi(\mathcal{G}\widehat{\otimes}_p\overline{\mathcal{G}})$. Let us also define $\Upsilon_w : \mathscr{B}_2(\Sigma) \to L^2(\Sigma \times \Sigma)$ by

$$\begin{aligned}[\Upsilon_w(g)](s,t) :&= \langle g, \phi_{w(t),w(s)}\rangle_{(\Sigma)}\\ &= \int_\Sigma g(r)\langle \pi(r)\pi(t)w, \pi(s)w\rangle\, d\mu(r).\end{aligned}$$

It turns out that Υ_w is an isometry with its range contained in $\mathscr{B}_2(\Sigma)\,\widehat{\otimes}\,\overline{\mathscr{B}_2(\Sigma)}$. Then, it is rather easy to show that

$$\mathscr{G}(\Sigma) = \Upsilon_w^{-1}\Big[\mathfrak{G}(\Sigma)\,\widehat{\otimes}_p\,\overline{\mathfrak{G}(\Sigma)}\Big].$$

The proof is based on the identity

$$\Upsilon_w \circ \Phi = \phi_w \otimes \phi_w,$$

which is an immediate consequence of equation (3.5).

5. The Berezin-Toeplitz quantization. In this section, we show that some very basic aspects of the Berezin-Toeplitz quantization can be naturally recovered in our abstract framework. More details on this circle of ideas can be found in [**7**].

Let us fix a unit vector $w \in \mathcal{H}$ and consider the family

$$W := \{w(s) := \pi(s)^*w \mid s \in \Sigma\}.$$

As a consequence of the existence of isometry (2.2), we have, in a weak sense,

$$1 = \int_\Sigma \lambda_{w(s),w(s)}\, d\mu(s) \tag{5.1}$$

by using the notation introduced in equation (1.1). For later use, we note that equation (5.1) implies

$$\operatorname{Tr} T = \int_\Sigma \langle Tw(s), w(s)\rangle\, d\mu(s) \tag{5.2}$$

for every operator $T \in \mathbb{B}_1(\mathcal{H})$. In fact, this follows by writing T as a linear combination of four nonnegative trace-class operators and then using the diagonalization of these four operators along with equation (2.3). We set

$$\phi_w : \mathcal{H} \longrightarrow \mathscr{B}_2(\Sigma) \subset L^2(\Sigma), \qquad [\phi_w(u)](s) := \langle u, w(s)\rangle$$

whose adjoint operator $\phi_w^\dagger : L^2(\Sigma) \to \mathcal{H}$ is given by

$$\phi_w^\dagger(f) = \int_\Sigma f(s)w(s)\, d\mu(s) = \Pi(f)w.$$

The associated kernel is the function $p_w : \Sigma \times \Sigma \to \mathbb{C}$, given by

$$p_w(s,t) := \langle w(t), w(s)\rangle = [\phi_w(w(t))](s) = \overline{[\phi_w(w(s))](t)},$$

defining a self-adjoint integral operator $P_w = \mathfrak{Int}(p_w)$ in $L^2(\Sigma)$. One can easily check that $P_w = \phi_w\phi_w^\dagger$ is the final projection of the isometry ϕ_w, so $P_w[L^2(\Sigma)]$ is a closed subspace of $\mathscr{B}_2(\Sigma)$. Since $\phi_w^\dagger\phi_w = 1$, we obtain the inversion formula

$$u = \int_\Sigma [\phi_w(u)](t)w(t)\, d\mu(t), \tag{5.3}$$

leading to the reproducing formula $\phi_w(u) = P_w[\phi_w(u)]$, i.e.,

$$[\phi_w(u)](s) = \int_\Sigma \langle w(t), w(s)\rangle [\phi_w(u)](t)\, d\mu(t).$$

Thus, $\mathscr{P}_w(\Sigma) := P_w[L^2(\Sigma)]$ is a reproducing Hilbert space with reproducing kernel p_w.

If, in addition, $\sup \|\pi(\cdot)\|\langle\infty$, and Σ is endowed with a topology for which π is weakly continuous, then the reproducing kernel Hilbert space $\mathscr{P}_w(\Sigma)$ consists of bounded continuous functions on Σ.

Let us define, for $f \in L^\infty(\Sigma)$,

$$\Omega_w^\pi(f) \equiv \Omega_w(f) := \int_\Sigma f(s)\lambda_{w(s),w(s)}\, d\mu(s)$$

and call it the *Berezin operator associated to the frame W*. This should be taken in a weak sense, i.e., for any $u, v \in \mathcal{H}$, set

$$\begin{aligned} \langle \Omega_w(f)u, v\rangle :&= \int_\Sigma f(s)\langle \lambda_{w(s),w(s)}u, v\rangle\, d\mu(s) \\ &= \int_\Sigma f(s)[\phi_w(u)](s)\overline{[\phi_w(v)](s)}\, d\mu(s). \end{aligned} \tag{5.4}$$

We gather the basic properties of Ω_w in the following statement; see **[2, 7]** for other results of this type.

Proposition 5.1.

(i) *The estimate $\|\Omega_w(f)\|_{\mathbb{B}(\mathcal{H})} \le \|f\|_{L^\infty(\Sigma)}$ holds.*

(ii) *The map Ω_w sends μ-almost everywhere positive functions in positive operators.*

(iii) *If $f \in L^1(\Sigma, \|w(\cdot)\|^2\mu)$, then $\Omega_w(f)$ is a trace-class operator and*

$$\begin{aligned} \mathrm{Tr}\,[\Omega_w(f)] &= \int_\Sigma \langle \Omega_w(f)w(s), w(s)\rangle\, d\mu(s) \\ &= \int_\Sigma f(s)\|w(s)\|^2 d\mu(s). \end{aligned}$$

(iv) *Assume that Σ is a locally compact space and that μ is a Radon measure with full support. If $f \in C_0(\Sigma)$, then $\Omega_w(f)$ is a compact operator.*

Proof.

(i) We estimate

$$\begin{aligned} |\langle \Omega_w(f)u, v\rangle| &\le \|f\|_{L^\infty} \int_\Sigma |[\phi_w(u)](s)||[\phi_w(v)](s)|\, d\mu(s) \\ &\le \|f\|_{L^\infty}\|\phi_w(u)\|_{L^2}\|\phi_w(v)\|_{L^2} \\ &= \|f\|_{L^\infty}\|u\|\|v\|, \end{aligned}$$

and this gives the result.

(ii) This follows from the fact that

$$\langle \Omega_w(f)u, u\rangle := \int_\Sigma f(s)|[\phi_w(u)](s)|^2 d\mu(s).$$

(iii) We may assume that $f \geq 0$. On one hand, if $\{v_k\}_k$ is any orthonormal basis in $\mathcal{H}$, one has, by the Parseval equality,

$$\begin{aligned}\mathrm{Tr}\,[\Omega_w(f)] &= \sum_k \langle \Omega_w(f)v_k, v_k\rangle \\ &= \int_\Sigma f(s) \sum_k \Big|\langle w(s), v_k\rangle\Big|^2 d\mu(s) \\ &= \int_\Sigma f(s)\|w(s)\|^2 d\mu(s);\end{aligned}$$

hence, the assumption $f \in L^1(\Sigma, \|w(\cdot)\|^2\mu)$ implies $\Omega_w(f) \in \mathbb{B}_1(\mathcal{H})$. The asserted formula of the trace can then be obtained either by equation (5.2) or directly, by equation (5.1),

$$\begin{aligned}\int_\Sigma \langle \Omega_w(f)w(s), w(s)\rangle\, d\mu(s) &= \int_\Sigma \int_\Sigma f(t)|\langle\, w(t), w(s)\rangle|^2 d\mu(s)\, d\mu(t) \\ &= \int_\Sigma f(t)\, d\mu(t) \int_\Sigma |\langle\, w(t), w(s)\rangle\,|^2 d\mu(s) \\ &= \int_\Sigma f(t)\|w(t)\|^2 d\mu(t).\end{aligned}$$

(iv) Since μ was selected to be a Radon measure, it is finite on compact subsets of Σ, and thus,

$$C_{\mathrm{c}}(\Sigma) \subset L^1(\Sigma) \cap L^\infty(\Sigma).$$

If f is continuous and has compact support, then $\Omega_w(f)$ is a compact operator by (iii). Then, the assertion follows by density from (i). $\square$

Remark 5.2. Formula (5.4), which can also be written

$$\langle \Omega_w(f)u, v\rangle = \langle f, \overline{\phi_w(u)}\phi_w(v)\rangle_{(\Sigma)},$$

opens the way to various extensions by duality. Returning to the setting of Section 4, let us also assume that $\mathscr{G}(\Sigma)$ is stable under the pointwise product. Then, for $w, u, v \in \mathcal{G}$, one has $\overline{\phi_w(u)} \cdot \phi_w(v) \in \mathscr{G}(\Sigma) \cdot \mathscr{G}(\Sigma) \subset$

$\mathscr{G}(\Sigma)$. This gives meaning to $\Omega^{\pi}_{w}(f)$ as a linear continuous operator $\mathcal{G} \to \mathcal{G}'_{\sigma}$ for $f \in \mathscr{G}'(\Sigma)$.

We now give a Toeplitz-like form of the operator

$$\Delta_w(f) := \phi_w \circ \Omega_w(f) \circ \phi_w^{\dagger}.$$

Proposition 5.3. *For every $f \in L^{\infty}(\Sigma)$, one has*

$$\Delta_w(f) = P_w \circ M_f \circ P_w,$$

where $M_f \colon L^2(\Sigma) \to L^2(\Sigma)$ is the multiplication by f operator.

Proof. We have now obtained

$$\begin{aligned}
[\Delta_w(f)h](s) &= \langle \Omega_w(f)\big[\phi_w^{\dagger}(h)\big], w(s)\rangle \\
&= \int_{\Sigma} f(t)\big[\phi_w\big(\phi_w^{\dagger}(h)\big)\big](t)\overline{[\phi_w(w(s))](t)}\, d\mu(t) \\
&= \int_{\Sigma} f(t)\, d\mu(t) \int_{\Sigma} h(t')[\phi_w(w(t'))](t)\overline{[\phi_w(w(s))](t)}\, d\mu(t') \\
&= \int_{\Sigma}\int_{\Sigma} f(t)h(t')p_w(t,t')\overline{p_w(t,s)}\, d\mu(t)\, d\mu(t') \\
&= \int_{\Sigma}\bigg[\int_{\Sigma} p_w(s,t)f(t)p_w(t,t')\, d\mu(t)\bigg] h(t')\, d\mu(t') \\
&= [(P_w \circ M_f \circ P_w)h](s).
\end{aligned}$$

□

We define the *covariant symbol of the operator* $A \in \mathbb{B}[L^2(\Sigma)]$ to be the complex function on Σ given by

$$[\sigma_w(A)](s) := \langle A\phi_w[w(s)], \phi_w[w(s)]\rangle_{(\Sigma)} = \langle \phi_w^{\dagger} A \phi_w[w(s)], w(s)\rangle.$$

We also introduce the *covariant symbol of the operator* $S \in \mathbb{B}(\mathcal{H})$ as

$$[\tau_w(S)](s) := \langle S w(s), w(s)\rangle.$$

Note that, for $S = \Pi(f)$, we obtain

$$[\tau_w(S)](s) = \langle \Pi(e_s^{\star} \star f \star e_s)w, w\rangle$$

for any $s \in \Sigma$. On the other hand, the covariant symbols of Berezin-Toeplitz operators are also easy to compute in terms of the reproducing kernel p_w. Specifically,

$$[\sigma_w(\Delta_w(g))](s) = [\tau_w(\Omega_w(g))](s) = \int_\Sigma g(t)|\langle w(s), w(t)\rangle|^2 d\mu(t).$$

Now, we investigate the connection between $\Omega_w \equiv \Omega_w^\pi$ and Π.

Proposition 5.4. *For any* $f \in L^1(\Sigma, \|w(\cdot)\|^2\mu)$*, one has* $\Omega_w^\pi(f) = \Pi[\mathfrak{S}_w^\pi(f)]$*, where, for* μ*-almost every* $s \in \Sigma$*,*

$$[\mathfrak{S}_w^\pi(f)](s) := \int_\Sigma f(t)\langle \pi(s)^* w(t), w(t)\rangle \, d\mu(t) = \langle f, \tau_w^\pi[\pi(s)]\rangle_{(\Sigma)}.$$

Proof. Using equations (3.9), (5.1), (5.2) and (5.4), one has

$$\begin{aligned}
&\Pi^{-1}[\Omega_w^\pi(f)](s) = \mathrm{Tr}[\pi(s)\Omega_w^\pi(f)] \\
&= \int_\Sigma \langle \Omega_w^\pi(f)w(t), \pi(s)^* w(t)\rangle \, d\mu(t) \\
&= \int_\Sigma \int_\Sigma f(r)\overline{\phi_{w(t),w}(r)}\phi_{\pi(s)^*w(t),w}(r) \, d\mu(t) \, d\mu(r) \\
&= \int_\Sigma \int_\Sigma f(r)\langle \pi(r)^* w, w(t)\rangle \langle w(t), \pi(s)\pi(r)^* w\rangle \, d\mu(r) \, d\mu(t) \\
&= \int_\Sigma f(r)\langle \pi(r)^* w, \pi(s)\pi(r)^* w\rangle \, d\mu(r) \\
&= \int_\Sigma f(t)\langle \pi(s)^* w(t), w(t)\rangle \, d\mu(t) = [\mathfrak{S}_w^\pi(f)](s),
\end{aligned}$$

and this concludes the proof. □

6. Infinite tensor products of square-integrable maps. We need the next definition in order to describe square-integrability properties of infinite tensor products of operator-valued maps.

Definition 6.1. Let $\mathcal{H}$ be any complex Hilbert space, and let $(\Sigma, \mathcal{M})$ be any measurable space as above. Also, let $\boldsymbol{\mu} = \{\mu_\alpha\}_{\alpha \in A}$ be any family of positive measures defined on the σ-algebra $\mathcal{M}$, where the index set A is directed with respect to some partial ordering. A weakly measurable map $\pi : \Sigma \to \mathbb{B}(\mathcal{H})$ is said to be *square integrable* with respect to the family of measures $\boldsymbol{\mu}$ if it satisfies the condition

$$\lim_{\alpha \in A} \int_{\Sigma} \langle \pi(s) u_1, v_1 \rangle \langle v_2, \pi(s) u_2 \rangle \, d\mu_\alpha(s) = \langle u_1, u_2 \rangle \langle v_2, v_1 \rangle \tag{6.1}$$

for all $u_1, u_2, v_1, v_2 \in \mathcal{H}$.

Remark 6.2. As in Remark 2.2, a polarization argument also shows that it suffices to verify equation (6.1) for $u_1 = u_2$ and $v_1 = v_2$, that is,

$$\lim_{\alpha \in A} \int_{\Sigma} \Big| \langle \pi(s) u, v \rangle \Big|^2 d\mu_\alpha(s) = \|u\|^2 \|v\|^2 \tag{6.2}$$

for all $u, v \in \mathcal{H}$. Also note that, if there exists a measure μ_∞ such that, for some $\alpha_0 \in A$, we have

$$L^1(\Sigma, \mu_\infty) = \bigcap_{\alpha \geq \alpha_0} L^1(\Sigma, \mu_\alpha),$$

and, for all $\phi \in L^1(\Sigma, \mu_\infty)$, we have

$$\int_{\Sigma} \phi \, d\mu_\infty = \lim_{\alpha \in A} \int_{\Sigma} \phi \, d\mu_\alpha,$$

then the operator-valued map π is square-integrable with respect to the family of measures $\boldsymbol{\mu} = \{\mu_\alpha\}_{\alpha \in A}$ if and only if $\pi \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}), \mu_\infty)$.

By using infinite tensor products of Hilbert spaces, we introduce the following notion of infinite tensor product of square integrable maps. We emphasize that this notion does not involve measure spaces, but rather measurable spaces, that is, merely pairs $(\Sigma, \mathcal{M})$ where $\mathcal{M}$ is a σ-algebra of subsets of a set Σ. In particular, the functions on $\mathcal{M}$ must be everywhere defined. However, we use the above notation from square integrable operator-valued maps to facilitate the application of this infinite tensor product construction in that setting in Theorem 6.6.

Definition 6.3. For $j \geq 1$, let $\mathcal{H}_j$ be any complex Hilbert space, let $(\Sigma_j, \mathcal{M}_j)$ be any measurable space, and let $\pi_j \colon \Sigma_j \to \mathbb{B}(\mathcal{H}_j)$. Assume that we have

(i) a distinguished vector $w_j \in \mathcal{H}_j$ with $\|w_j\| = 1$;
(ii) a distinguished point $t_j \in \Sigma_j$ with $\pi_j(t_j) = 1_{\mathcal{H}_j}$.

Denote

$$\mathbf{w} := \{w_j\}_{j\geq 1} \in \prod_{j\geq 1} \mathcal{H}_j,$$

and define the complex Hilbert space

$$\mathcal{H} := \widehat{\bigotimes_{j\geq 1}}^{\mathbf{w}} \mathcal{H}_j,$$

as the inductive limit of the sequence of Hilbert spaces

$$\mathcal{H}^{(N)} := \widehat{\bigotimes_{1\leq j\leq N}} \mathcal{H}_j,$$

with respect to the isometric embeddings

$$\mathcal{H}^{(N)} \longrightarrow \mathcal{H}^{(N+1)}, \qquad x \longmapsto x \otimes w_{N+1}.$$

Then, define

$$\Sigma := \Big\{ s = \{s_j\}_{j\geq 1} \in \prod_{j\geq 1} \Sigma_j \,\Big|\, \text{there exists } j(s) \geq 1 \text{ for all } j \geq j(s),\ s_j = t_j \Big\}$$

and

$$\pi\colon \Sigma \longrightarrow \mathbb{B}(\mathcal{H}), \qquad \pi(\{s_j\}_{j\geq 1}) = \bigotimes_{j\geq 1} \pi_j(s_j).$$

As in [**15,** Appendix D], we say that Σ is the *restricted Cartesian product* of $\{\Sigma_j\}_{j\geq 1}$ along the sequence of distinguished points $\{t_j\}_{j\geq 1}$, and we endow it with the restricted product of the σ-algebras $\{\mathcal{M}_j\}_{j\geq 1}$, see [**15,** Definition D.2]. Moreover, π is the *restricted tensor product* of maps $\{\pi_j\}_{j\geq 1}$ along the sequences of distinguished points $\{t_j\}_{j\geq 1}$ and distinguished unit vectors $\mathbf{w} = \{w_j\}_{j\geq 1}$.

Remark 6.4. The definition of an infinite tensor product of operator-valued functions raises several issues which may seem mutually exclusive, namely, in general, the map π in Definition 6.3 cannot be defined on the whole Cartesian product of $\{\Sigma_j\}_{j\geq 1}$ because of restrictions required by the definition of an infinite tensor product of operators. More precisely, if $T_j \in \mathbb{B}(\mathcal{H}_j)$ for every $j \geq 1$, then, in order to define $\otimes_{j\geq 1} T_j$ on the inductive limit $\widehat{\otimes_{j\geq 1}}^{\mathbf{w}} \mathcal{H}_j$, we need to have $T_j w_j = w_j$ whenever $j \geq 1$ is large enough. On the other hand, the problem with the restricted Cartesian product Σ is that, if some measure μ_j is given on the measurable space $(\Sigma_j, \mathcal{M}_j)$ for every $j \geq 1$, then it is not clear how to use the sequence of measures $\{\mu_j\}_{j\geq 1}$ in order to define a natural measure on Σ. From this point of view, it might seem preferable to work with the full Cartesian product of $\{\Sigma_j\}_{j\geq 1}$ and to assume that each μ_j is a probability measure. In a forthcoming paper, we will see how these problems can be dealt with in some special cases when every $(\Sigma_j, \mathcal{M}_j, \mu_j)$ is given by some Gaussian measures on an Euclidean space, and every π_j is a projective representation of the additive group underlying that Euclidean space.

6.1. Square integrability for infinite tensor products. In order to avoid the difficulties mentioned in Remark 6.4, we will use an alternative approach which leads to Theorem 6.6 and was inspired by some methods used in the representation theory of canonical commutation relations; see, for instance, [**18, 19**], [**24,** Section B] and [**34,** Lemma 4.2]. The key of this approach is the approximate orthogonality property introduced in Definition 6.1. The next elementary lemma will be needed in the proof of Theorem 6.6.

Lemma 6.5. *Let $\{a_{MN}\}_{M,N\geq 1}$ be any double sequence of complex numbers satisfying the following conditions.*

(i) *There exists $a \in \mathbb{C}$ for which $\lim\limits_{M\to\infty} \lim\limits_{N\to\infty} a_{MN} = a$.*
(ii) *We have* $\sup\{|a_{MN}| \mid M, N \geq 1\} < \infty$.
(iii) *There exists $\lim\limits_{M\to\infty} a_{MN} =: b_N$ uniformly for $N \geq 1$.*

Then, $\lim\limits_{N\to\infty} \lim\limits_{M\to\infty} a_{MN} = a$.

Proof. The sequence $\{b_N\}_{N\geq 1}$ is clearly bounded. We need to prove that it is convergent to a, and, to this end, we will prove that every convergent subsequence has the limit a.

If $\{b_{N_j}\}_{j\geq 1}$ is any convergent subsequence, then the uniform convergence hypothesis ensures that the double sequence $\{a_{MN_j}\}_{M,j\geq 1}$ has the property

$$a = \lim_{M\to\infty} \lim_{j\to\infty} a_{MN_j} = \lim_{j\to\infty} \lim_{M\to\infty} a_{MN_j};$$

hence, $\lim\limits_{j\to\infty} b_{N_j} = a$ for any convergent subsequence $\{b_{N_j}\}_{j\ge 1}$, and the assertion follows. $\square$

Theorem 6.6. *Assume the setting of Definition* 6.3*. Moreover, assume that there is a σ-finite measure μ_j on the measurable space $(\Sigma_j, \mathcal{M}_j)$ with $\pi_j \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}_j), \mu_j)$ for all $j \ge 1$. For every $N \ge 1$, define the map*

$$\theta^{(N)} \colon \Sigma_1 \times \cdots \times \Sigma_N \longrightarrow \Sigma$$

by

$$\theta^{(N)}(s_1, \dots, s_N) = (s_1, \dots, s_N, t_{N+1}, t_{N+2}, \dots),$$

and consider the measure

$$\mu^{(N)} := (\theta^{(N)})_*(\mu_1 \otimes \cdots \otimes \mu_N)$$

on Σ. Then, the restricted tensor product $\pi \colon \Sigma \to \mathbb{B}(\mathcal{H})$ of the maps $\{\pi_j\}_{j\ge1}$ along the sequences of distinguished points $\{t_j\}_{j\ge1}$ and distinguished unit vectors $\mathbf{w} = \{w_j\}_{j\ge1}$ is weakly measurable and square integrable with respect to the sequence of measures $\{\mu^{(N)}\}_{N\ge1}$.

Proof. The proof has two parts since we will first record some preliminary facts. It is clear from Definition 6.3 that the map π is weakly measurable. To prove that this map is square integrable with respect to the sequence of measures $\{\mu^{(N)}\}_{N\ge1}$, we must prove that, for arbitrary $u, v \in \mathcal{H}$, we have

$$\lim_{N\to\infty} \int_\Sigma |\langle \pi(\cdot)u, v\rangle|^2 d\mu^{(N)} = \|u\|^2 \|v\|^2, \tag{6.3}$$

which is equivalent to

$$\lim_{N\to\infty} \int_{\Sigma^{(N)}} |\langle \pi(\theta^{(N)}(s))u, v\rangle|^2 d(\mu_1 \otimes \cdots \otimes \mu_N)(s) = \|u\|^2 \|v\|^2,$$

where $\Sigma^{(N)} := \Sigma_1 \times \cdots \times \Sigma_N$ for every $N \ge 1$.

(i) For every $N \ge 1$, denote by $\mathcal{K}^{(N)}$ the infinite tensor product of the sequence of Hilbert spaces $\{\mathcal{H}_j\}_{j\ge N+1}$ along the sequence of unit vectors $\{w_j\}_{j\ge N+1}$. The associativity of infinite tensor products shows that there exists a natural unitary operator

$$W^{(N)} \colon (\mathcal{H}_1 \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{H}_N) \widehat{\otimes} \mathcal{K}^{(N)} \longrightarrow \mathcal{H},$$

which also allows us to define

$$\mathcal{H}^{(N)} := W((\mathcal{H}_1 \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{H}_N) \otimes w_{N+1} \otimes w_{N+2} \otimes \cdots) \subseteq \mathcal{H},$$

with the orthogonal projection $P^{(N)} : \mathcal{H} \to \mathcal{H}^{(N)}$. Note that,

$$M \geq N \geq 1 \Longrightarrow P^{(M)} \pi(\theta^{(N)}(\cdot)) = \pi(\theta^{(N)}(\cdot)) P^{(M)}, \tag{6.4}$$

since, for every $N \geq 1$, one has the commutative diagram

$$\begin{tikzcd} \Sigma_1 \times \cdots \times \Sigma_N \arrow[r, "\theta^{(N)}"] \arrow[d, "\pi_1(\cdot)\otimes\cdots\otimes\pi_N(\cdot)\otimes 1_{\mathcal{K}^{(N)}}"'] & \Sigma \arrow[d, "\pi"] \\ \mathbb{B}((\mathcal{H}_1 \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{H}_N) \widehat{\otimes} \mathcal{K}^{(N)}) \arrow[r] & \mathbb{B}(\mathcal{H}) \end{tikzcd} \tag{6.5}$$

whose existence follows by the definition of π (see Definition 6.3) and whose bottom arrow is the spatial isomorphism of von Neumann algebras given by $T \mapsto W^{(N)} T (W^{(N)})^*$. For the same reasons, and by also taking into account Propositions 2.7 and 2.9, we obtain for all $u_1, v_1, u_2, v_2 \in \mathcal{H}$ and $M \geq N \geq 1$,

$$\begin{aligned} &\int_\Sigma |\langle \pi(\cdot) u_1, P^{(M)} v_1 \rangle \langle P^{(M)} v_2, \pi(\cdot) u_2 \rangle| \, d\mu^{(N)} \\ &\quad \leq \|P^{(M)} u_1\| \|P^{(M)} v_1\| \|P^{(M)} u_2\| \|P^{(M)} v_2\| \\ &\quad \leq \|u_1\| \|P^{(M)} v_1\| \|u_2\| \|P^{(M)} v_2\|. \end{aligned} \tag{6.6}$$

The above inequality holds under a stronger form if $N \geq M \geq 1$, namely,

$$\begin{aligned} N \geq M \Longrightarrow &\int_\Sigma \langle \pi(s) u_1, P^{(M)} v_1 \rangle \langle P^{(M)} v_2, \pi(s) u_2 \rangle \, d\mu^{(N)}(s) \\ &= \langle P^{(N)} u_1, P^{(N)} u_2 \rangle \langle P^{(M)} v_2, P^{(M)} v_1 \rangle \end{aligned} \tag{6.7}$$

since, in this case, we have $P^{(M)} P^{(N)} = P^{(N)} P^{(M)} = P^{(M)}$, and then the above equality follows by Proposition 2.9 along with equations (6.4) and (6.5).

(ii) Now, we return to the proof of equation (6.3). By using equation (6.7), we obtain

$$\lim_{M\to\infty} \lim_{N\to\infty} a_{MN}(u_1, v_1, u_2, v_2) = \langle u_1, u_2 \rangle \langle v_2, v_1 \rangle, \tag{6.8}$$

where

$$a_{MN}(u_1,v_1,u_2,v_2) := \int_\Sigma \langle \pi(s)u_1, P^{(M)}v_1\rangle\langle P^{(M)}v_2, \pi(s)u_2\rangle\, d\mu^{(N)}(s),$$

for all $M,N \geq 1$ and $u_1,v_1,u_2,v_2 \in \mathcal{H}$.

We will prove that the limits in equation (6.8) can be interchanged by using Lemma 6.5. To this end, it suffices to consider the case $u_1 = u_2 =: u$ (by a polarization argument). Since, by equation (6.6),

$$|a_{MN}(u,v_1,u,v_2)| \leq \|u\|^2\|v_1\|\|v_2\|, \tag{6.9}$$

for all $M,N \geq 1$, we still need to show that there exists a sequence $\{b_N(u,v_1,u,v_2)\}_{N\geq 1}$ for which the conditions

$$\lim_{M\to\infty} a_{MN}(u,v_1,u,v_2) = b_N(u_1,v_1,u_2,v_2), \tag{6.10}$$

are uniformly satisfied for $N \geq 1$. In order to check the above condition, first note that, for arbitrary $u,v \in \mathcal{H}$ and $M_1,M_2 \geq 1$, we have

$$\begin{aligned}\int_\Sigma \Big|\langle \pi(\cdot)u, P^{(M_1)}v\rangle - \langle \pi(\cdot)u, P^{(M_2)}v\rangle\Big|^2 d\mu^{(N)} \\ \leq \|u\|^2\|P^{(M_1)}v - P^{(M_2)}v\|^2.\end{aligned} \tag{6.11}$$

In fact, we may assume that $M_2 < M_1$. If $N < M_1$, and we set $v_1 = v_2 := v - P^{(M_2)}v$, then

$$P^{(M_1)}v_j = P^{(M_1)}v - P^{(M_2)}v,$$

for $j = 1,2$; hence, by using equation (6.6) for $M := M_1$ and $u_1 = u_2 := u$, we obtain equation (6.11). On the other hand, if $N \geq M_1$, then equation (6.11) follows at once by equation (6.7).

Now, we can check equation (6.10) by proving that

$$\{a_{MN}(u,v_1,u,v_2)\}_{M\geq 1}$$

is a Cauchy sequence, uniformly for $N \geq 1$. More precisely, for all $M_1,M_2,N \geq 1$, we have

$$\begin{aligned}
&|a_{M_1N}(u,v_1,u,v_2) - a_{M_2N}(u,v_1,u,v_2)| \\
&\leq \int_\Sigma |\langle \pi(\cdot)u, P^{(M_1)}v_1\rangle(\langle P^{(M_1)}v_2, \pi(\cdot)u\rangle - \langle P^{(M_2)}v_2, \pi(\cdot)u\rangle)|\, d\mu^{(N)} \\
&\quad + \int_\Sigma |(\langle \pi(\cdot)u, P^{(M_1)}v_1\rangle - \langle \pi(\cdot)u, P^{(M_2)}v_1\rangle\langle P^{(M_2)}v_2, \pi(\cdot)u\rangle)|\, d\mu^{(N)} \\
&\leq \|u\|^2(\|P^{(M_1)}v_1\|\|P^{(M_1)}v_2 - P^{(M_2)}v_2\| \\
&\quad + \|P^{(M_2)}v_2\|\|P^{(M_1)}v_1 - P^{(M_2)}v_1\|),
\end{aligned}$$

where the last inequality follows by the Schwartz inequality, along with estimates (6.6)–(6.11). Thus, equation (6.10) follows.

Now, equations (6.9) and (6.10) ensure that Lemma 6.5 applies; hence, the limits in equation (6.8) can be interchanged. In turn, this implies that

$$\lim_{N\to\infty}\lim_{M\to\infty} a_{MN}(u_1,v_1,u_2,v_2) = \langle u_1,u_2\rangle\langle v_2,v_1\rangle \tag{6.12}$$

for all $u_1,v_1,u_2,v_2 \in \mathcal{H}$. On the other hand, by taking into account the definition of $a_{MN}(u_1,v_1,u_2,v_2)$, it follows that, for all $N \geq 1$ and $u,v \in \mathcal{H}$, we have

$$\begin{aligned}
\lim_{M\to\infty} a_{MN}(u,v,u,v) &= \lim_{M\to\infty}\int_\Sigma |\langle \pi(\cdot)u, P^{(M)}v\rangle|^2 d\mu^{(N)} \\
&= \int_\Sigma |\langle \pi(\cdot)u, v\rangle|^2 d\mu^{(N)},
\end{aligned}$$

where the last equality is a direct consequence of equation (6.11). Thus, equation (6.12) implies that equality (6.3) holds true, and this completes the proof. □

6.2. Symbol calculus for infinite tensor products. One can inquire as to what happens at the level of quantizations when operations (as tensor products) are performed with square integrable families. This question is particularly interesting in the setting of Section 5 since our input there was a pointed Hilbert space $(\mathcal{H}, w)$, which is the basic subject when performing infinite tensor products. In the rest of this section, we will take a few steps in this direction, but many interesting problems still remain unsolved; in particular, the symbols that give rise to operators in various Schatten ideals. See, for instance, Proposition 5.1 (iii).

Theorem 6.7. *Assume the setting of Theorem* 6.6, *and denote the space of all complex-valued, bounded measurable functions on* Σ *by* $\widetilde{L}^\infty(\Sigma)$. *Then, the following assertions hold*:

(i) *for every* $N \ge 1$, $f \in \widetilde{L}^\infty(\Sigma)$ *and* $u, v \in \mathcal{H}$, *the integral*

$$\Omega_u^{(N)}(f)v := \int_\Sigma f(\cdot)\langle v, \pi(\cdot)u\rangle \pi(\cdot)u \, d\mu^{(N)}$$

is weakly convergent and defines an operator $\Omega_u^{(N)}(f) \in \mathbb{B}(\mathcal{H})$ *satisfying* $\|\Omega_u^{(N)}(f)\| \le \|u\|^2 \sup_\Sigma |f|$.

(ii) *For all* $u \in \mathcal{H} \setminus \{0\}$ *and* $f \in \widetilde{L}^\infty(\Sigma)$, *there exists* $\Omega_u(f) \in \mathbb{B}(\mathcal{H})$ *satisfying* $\Omega_u(f) = \lim_{N\to\infty} \Omega_u^{(N)}(f)$ *in the weak operator topology, and moreover,* $\|\Omega_u(f)\| \le \|u\|^2 \sup_\Sigma |f|$.

(iii) *If* $u \in \mathcal{H}$ *and* $0 \le f \in \widetilde{L}^\infty(\Sigma)$, *then* $0 \le \Omega_u(f) \in \mathbb{B}(\mathcal{H})$.

(iv) *For every* $u \in \mathcal{H}$ *with* $\|u\| = 1$, *we have* $\Omega_u(1) = 1$.

Proof.

(i) Use again the notation $\Sigma^{(N)} = \Sigma_1 \times \cdots \times \Sigma_N$ to write

$$\begin{aligned}\langle \Omega_u^{(N)}(f)v_1, v_2\rangle &= \int_\Sigma f(\cdot)\langle v_1, \pi(\cdot)u\rangle\langle \pi(\cdot)u, v_2\rangle \, d\mu^{(N)} \\ &= \int_{\Sigma^{(N)}} f(\theta^{(N)}(\cdot))\langle v_1, \pi(\theta^{(N)}(\cdot))u\rangle \\ &\qquad \langle \pi(\theta^{(N)}(\cdot))u, v_2\rangle d(\mu_1 \otimes \cdots \otimes \mu_N);\end{aligned}$$

hence, by taking into account commutative diagram (6.5), the convergence of the above integral and the required estimate for the norm of $\Omega_u^{(N)}(f)$ follow by the estimate provided in Proposition 2.7.

For (ii), we only need to prove the asserted convergence in the weak operator topology, since the norm estimate will then follow by the norm estimates established above.

In order to prove that the sequence $\{\Omega_u^{(N)}(f)\}_{N\ge 1}$ is convergent in the weak operator topology, we will adapt the method of proof of

Theorem 6.6, and we will use the notation from that proof, except that for fixed $f \in \widetilde{L}^\infty(\Sigma)$, we set

$$a_{MN}(u_1, v_1, u_2, v_2) := \int_\Sigma f(\cdot)\langle \pi(\cdot)u_1, P^{(M)}v_1\rangle\langle P^{(M)}v_2, \pi(\cdot)u_2\rangle \, d\mu^{(N)}$$

for all $M, N \geq 1$ and $u_1, v_1, u_2, v_2 \in \mathcal{H}$. Then, we have

$$\lim_{M\to\infty} a_{MN}(u, v_1, u, v_2) = \langle \Omega_u^{(N)}(f)v_1, v_2\rangle$$

uniformly for $N \geq 1$, using reasoning similar to that used for proving equation (6.10). Moreover, we have the following version of equation (6.9)

$$|a_{MN}(u, v_1, u, v_2)| \leq \|u\|^2\|v_1\|\|v_2\| \sup_\Sigma |f|$$

for all $M, N \geq 1$; hence, we can use Lemma 6.5 to prove that $\{\langle \Omega_u^{(N)}(f)v_1, v_2\rangle\}_{N\geq 1}$ is a convergent sequence for all $u, v_1, v_2 \in \mathcal{H}$, that is, the operator sequence $\{\Omega_u^{(N)}(f)\}_{N\geq 1}$ is convergent in the weak operator topology in $\mathbb{B}(\mathcal{H})$ for all $u \in \mathcal{H}$.

Assertion (iii) is clear.

Finally, assertion (iv) follows directly from Theorem 6.6 (see equation 6.3), and this completes the proof. $\square$

Remark 6.8. Theorem 6.7 (iv) should be regarded as a version of equation (5.1). In the present situation of infinite tensor products, we do not have any natural version of the space $L^2(\Sigma)$, and therefore, we need to work inside the space $\mathbb{C}^\Sigma$ of all complex-valued functions on Σ in order to construct the reproducing kernel Hilbert space as above. For instance, if we pick any unit vector $w \in \mathcal{H}$, then we can define

$$w(\cdot) = \pi(\cdot)^*w : \Sigma \longrightarrow \mathcal{H} \quad \text{and} \quad \phi_w : \mathcal{H} \longrightarrow \mathbb{C}^\Sigma,$$

$$\phi_w(u) = \langle u, w(\cdot)\rangle = \langle \pi(\cdot)u, w\rangle.$$

Then, the linear map ϕ_w is injective by equation (6.3); hence, we may define $\mathscr{P}_w(\Sigma) := \operatorname{Ran}\phi_w$, and make this function space into a Hilbert space such that $\phi_w\colon \mathcal{H} \to \mathscr{P}_w(\Sigma)$ is a unitary operator. Note that $\mathscr{P}_w(\Sigma)$ is a reproducing kernel Hilbert space since, for every $s \in \Sigma$ and the point evaluation $\mathrm{ev}_s\colon \mathscr{P}_w(\Sigma) \to \mathbb{C}$, $\mathrm{ev}_s(f) = f(s)$ is continuous.

The corresponding reproducing kernel is

$$p_w \colon \Sigma \times \Sigma \longrightarrow \mathbb{C}, \qquad p_w(s,t) = \langle w(t), w(s) \rangle,$$

as above, see [**32,** Theorem I.1.6].

The version of inversion formula (5.3) in the present setting is

$$u = \lim_{N\to\infty} \Omega_w^{(N)}(1)u = \lim_{N\to\infty} \int_\Sigma \langle u, \pi(\cdot)^* w\rangle \pi(\cdot)^* w \, d\mu^{(N)},$$

where the limit and the integral are taken in the weak sense. This is obtained by using Theorem 6.7 (ii) and (iv) for the mappings $\pi_j(\cdot)^* \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}_j), \mu_j)$ and provides a generalization of a result [**24,** Theorem 3.2] from the representation theory of canonical commutation relations.

7. Examples. It is quite clear that the formalism of Section 6 is meant to cover at least two situations: square integrable irreducible unitary group representations with their associated twisted convolution algebras and the Weyl pseudodifferential calculus. Now, we will briefly indicate other examples in order to show the generality of our setting. They are developed only to the extent where the identification of the relevant objects becomes obvious, although we plan to give more specific applications in forthcoming papers. The references cited in this section contain much more than we are able to review here.

7.1. The magnetic Weyl calculus. Take $\Sigma := \mathcal{X} \times \mathcal{X}^*$, where $\mathcal{X}$ is an n-dimensional real vector space and $\mathcal{X}^*$ is its dual (so the "phase-space" Σ is non-canonically isomorphic to $\mathbb{R}^{2n}$). Below, setting $B = 0$ and $A = 0$, one may recover the standard Weyl calculus [**8**].

The magnetic Weyl calculus [**21, 23, 27, 28, 31**] has, as background, the problem of quantization of a physical system consisting of a spinless particle moving in the Euclidean space $\mathcal{X} \cong \mathbb{R}^n$ under the influence of a magnetic field, i.e., a closed 2-form B on $\mathcal{X}$ ($dB = 0$), given in a base by matrix-component functions

$$B_{jk} = -B_{kj} : \mathcal{X} \longrightarrow \mathbb{R}, \qquad j,k = 1,\dots,n.$$

For simplicity, and in order to have full formalism, we are going to assume that the components B_{jk} belong to $C^\infty_{\mathrm{pol}}(\mathcal{X})$, the class of smooth functions on $\mathcal{X}$ with polynomial bounds on all the derivatives. The magnetic field can be written in many ways as the differential $B = dA$ of some 1-form A on $\mathcal{X}$ called a *vector potential*. One has $B = dA = dA'$ if and only if $A' = A + d\varphi$ for some 0-form φ (then they are called equivalent). It is easy to see that vector potential can also be chosen for class $C^{\infty}_{\mathrm{pol}}(\mathcal{X})$.

One would like to develop a symbol calculus taking the magnetic field into account. The basic requirements are:

(i) it should reduce to the standard Weyl calculus for $A = 0$, and
(ii) the operators $\Pi^{A}(f)$ and $\Pi^{A'}(f)$ should be unitarily equivalent (independently on the symbol f) if A and A' are equivalent; this is called *gauge covariance* and has a fundamental physical meaning.

To justify the formulae, one could think of the emerging symbol calculus as a functional calculus for the family of non commuting self-adjoint operators $(Q_1, \ldots, Q_n; P_1^A, \ldots, P_n^A)$ in $\mathcal{H} := L^2(\mathcal{X})$. Here, Q_j is one of the components of the Poisson operator, but the momentum $P_j := -i\partial_j$ is replaced by the *magnetic momentum* $P_j^A := P_j - A_j(Q)$, where $A_j(Q)$ indicates the operator of multiplication with the function $A_j \in C^{\infty}_{\mathrm{pol}}(\mathcal{X})$. Notice the commutation relations

$$i[Q_j, Q_k] = 0, \qquad i[P_j^A, Q_k] = \delta_{j,k}, \qquad i\left[P_j^A, P_k^A\right] = B_{jk}(Q).$$

Now, one computes the magnetic Weyl system

$$\pi^A : \Sigma \longrightarrow \mathbb{B}(\mathcal{H}), \qquad \pi^A(x,\xi) := \exp[i(x \cdot P^A - Q \cdot \xi)],$$

and explicitly obtains

$$[\pi^A(x,\xi)u](y) = e^{-i(y+x/2)\cdot\xi} \exp\left[(-i) \int_{[y,y+x]} A\right] u(y+x).$$

The extra phase factor involves the circulation of the 1-form A through the segment

$$[x,y] := \{(1-t)x + ty \mid t \in [0,1]\}.$$

These operators depend strongly continuously on (x,ξ) and satisfy $\pi^A(0,0) = 1$ and

$$\pi^A(x,\xi)^* = \pi(x,\xi)^{-1} = \pi^A(-x,-\xi),$$

thus being unitary. However, they do not form a projective representation of $\Sigma = \mathcal{X} \times \mathcal{X}^*$. Actually, they satisfy

$$\pi^A(x,\xi)\pi^A(y,\eta) = \Omega^B[(x,\xi),(y,\eta);Q]\pi^A(x+y,\xi+\eta),$$

where $\Omega^B[(x,\xi),(y,\eta);Q]$ only depends on the 2-form B and denotes the operator of multiplication in $L^2(\mathcal{X})$ by the function

$$\begin{aligned}\mathcal{X} \ni z \longrightarrow\, &\Omega^B[(x,\xi),(y,\eta);z] \\ &:= \exp\left[\frac{i}{2}(y\cdot\xi - x\cdot\eta)\right]\exp\left[(-i)\int_{\langle z,z+x,z+x+y\rangle} B\right].\end{aligned}$$

Here, the distinguished factor is constructed with the flux (invariant integration) of the magnetic field through the triangle defined by the corners z, $z+x$ and $z+x+y$.

Straightforward computation leads to

$$\begin{aligned}[\Phi^A(u\otimes v)](x,\xi) : = \langle\pi^A(x,\xi)u,v\rangle = &\int_{\mathcal{X}} e^{-iy\cdot\xi}\exp\Big[(-i)\int_{[y-x/2,y+x/2]} A\Big] \\ &\times u(y+x/2)\overline{v(y-x/2)}\,dy.\end{aligned}$$

It can be decomposed into the product of multiplication by a function with values in the unit circle, a change of variables with the Jacobian identically equal to 1, and a partial Fourier transform. All are isomorphisms between the corresponding spaces, so the orthogonality relation holds with $\mathscr{B}_2(\Sigma) = L^2(\Sigma)$.

Thus, one can apply all of the prescriptions and get the correspondence $f \mapsto \Pi^A(f)$ and the composition law $(f,g) \to f \star^B g$ (depending only on the magnetic field). In fact, many are interested in the (symplectic) Fourier transformed version

$$a(Q,P^A) \equiv \mathfrak{Op}^A(a) := \Pi^A[\mathfrak{F}^{-1}(f)]$$

and in the multiplication $\#^B$ obtained by transport of the structure and therefore satisfying $\mathfrak{Op}^A(a)\mathfrak{Op}^A(b) = \mathfrak{Op}^A(a\#^B b)$. The resulting involution is just complex conjugation; thus, $\mathfrak{Op}^A(a)^* = \mathfrak{Op}^A(\overline{a})$. For the convenience of the reader, we indicate the explicit formulae in which we set $\Gamma^A([x,y]) := \int_{[x,y]} A$ and $\Gamma^B(\langle x,y,z\rangle) := \int_{\langle x,y,z\rangle} B$. The *magnetic Moyal product* is

$$(a\#^B b)(X) = \pi^{-2n} \int_\Sigma \int_\Sigma \exp\{-2i[(x-z)\cdot(\xi-\eta)-(x-y)\cdot(\xi-\zeta)]\} \\ \times \exp\Big[-i\Gamma^B(\langle x-y+z, y-z+x, z-x+y\rangle)\Big] \\ \times f(Y)\, g(Z)\, dZ\, dY$$

and the magnetic Weyl calculus is given by

(7.1)

$$[\mathfrak{Op}^A(a)u](x) = (2\pi)^{-n} \int_{\mathcal{X}} \int_{\mathcal{X}'} \exp[i(x-y)\cdot\xi] \exp[-i\Gamma^A([x,y])] \\ \times a\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi.$$

An important property of equation (7.1) is gauge covariance, as hinted above. If $A' = A + d\rho$ defines the same magnetic field as A, then $\mathfrak{Op}^{A'}(f) = e^{i\rho}\mathfrak{Op}^A(f)e^{-i\rho}$. By destroying the magnetic phase factors in all the above formulae, one obtains the defining relations of the usual Weyl calculus.

In this case, a convenient choice of the auxiliary space is the Schwartz space $\mathcal{G} = \mathcal{S}(\mathcal{X})$, which is a nuclear Fréchet space continuously and densely embedded in $L^2(\Sigma)$; thus, $\mathcal{G}'$ will be the space of tempered distributions. By simple examination of the map Φ^A, this leads to $\mathscr{G}(\Sigma) = \mathcal{S}(\mathcal{X}\times\mathcal{X}^*)$. It can easily be shown that (by suitable restriction or extensions)

$$\mathfrak{Op}^A[\mathcal{S}(\mathcal{X}\times\mathcal{X}^*)] = \mathbb{B}[\mathcal{S}'(\mathcal{X}), \mathcal{S}(\mathcal{X})]$$

and

$$\mathfrak{Op}^A[\mathcal{S}'(\mathcal{X}\times\mathcal{X}^*)] = \mathbb{B}[\mathcal{S}(\mathcal{X}), \mathcal{S}'(\mathcal{X})].$$

The symbol algebras for the magnetic Weyl calculus were studied in detail **[27]** while, in **[21]**, the full pseudodifferential theory was developed.

7.2. Operator calculi on locally compact abelian groups. In this subsection, we will present some square integrable operator-valued maps related to the metaplectic representation in the framework of locally compact abelian groups, the representation of which was studied in [**25, 35, 39**].

The framework is provided by any locally compact abelian group $(G,+)$ with its dual group $\widehat{G}$, which is the set of all continuous homomorphisms from G into the circle group $\mathbb{T}$. Recall that $\widehat{G}$ is in turn a locally compact abelian group with the pointwise operations and with the topology given by uniform convergence on compact sets. The natural duality pairing between $\widehat{G}$ and G is denoted by $\langle\cdot,\cdot\rangle\colon \widehat{G}\times G\to\mathbb{T}$. For every Haar measure ν on G, there exists a unique Haar measure ν^* on $\widehat{G}$ for which the Fourier transform

$$\mathcal{F}\colon L^1(G,\nu)\longrightarrow BC(\widehat{G}),\qquad (\mathcal{F}f)(\xi)=\int_G\langle\xi,-x\rangle f(x)\,d\nu(x)$$

gives rise to a unitary operator $L^2(G,\nu)\to L^2(\widehat{G},\nu^*)$.

Proposition 7.1. *Let G be any locally compact abelian group with a Haar measure ν and denote $\mathcal{H}=L^2(G,\nu)$. For $k=1,2$, define*

$$\pi_k\colon G\times\widehat{G}\longrightarrow\mathbb{B}(\mathcal{H}),\qquad (\pi_k(x,\xi)u)(z)=\langle\xi,kz+(k-1)x\rangle u(z+x).$$

Then, the following assertions hold:

(i) *we have $\pi_1\in\mathrm{SQ}(\mathbb{B}(\mathcal{H}),\nu\times\nu^*)$, and*

$$\Phi^{\pi_1}\colon\mathcal{H}\,\widehat{\otimes}\,\overline{\mathcal{H}}\longrightarrow L^2(G\times\widehat{G},\nu\times\nu^*)$$

is a unitary operator.

(ii) *If the map $x\mapsto 2x$ is an automorphism of G, then there exists a constant $c>0$ for which $\pi_2\in\mathrm{SQ}(\mathbb{B}(\mathcal{H}),\nu\times c\nu^*)$, and the corresponding operator*

$$\Phi^{\pi_2}\colon\mathcal{H}\,\widehat{\otimes}\,\overline{\mathcal{H}}\longrightarrow L^2(G\times\widehat{G},\nu\times c\nu^*)$$

is unitary.

Proof. For $k=1,2$ and every

$$u\in L^1(G\times\widehat{G},\nu\times\nu^*)\cap L^2(G\times\widehat{G},\nu\times\nu^*),$$

define

$$T^{\pi_k} u = \iint_{G\times\widehat{G}} u(x,\xi)\pi_k(x,\xi)\,d\nu(x)\,d\nu^*(\xi).$$

Now, we prove the assertions separately.

(i) It follows by **[39]** that T^{π_1} extends to a unitary operator $L^2(G\times\widehat{G},\nu\times\nu^*)\to\mathbb{B}_2(\mathcal{H})$. By taking operator adjoints and complex-conjugates of functions, we then obtain a unitary operator

$$L^2(G\times\widehat{G},\nu\times\nu^*)\longrightarrow\mathbb{B}_2(\mathcal{H}),$$

given by

$$u\longmapsto\iint_{G\times\widehat{G}} u(x,\xi)\pi_1(x,\xi)^*d\nu(x)\,d\nu^*(\xi),$$

for every

$$u\in L^1(G\times\widehat{G},\nu\times\nu^*)\cap L^2(G\times\widehat{G},\nu\times\nu^*),$$

and this is just the adjoint of

$$\Phi^{\pi_1}\circ\Lambda^{-1}\colon \mathcal{H}\,\widehat{\otimes}\,\overline{\mathcal{H}}\longrightarrow L^2(G\times\widehat{G},\nu\times\nu^*),$$

in Corollary 3.3. Since Λ is a unitary operator, it follows that Φ^{π_1} is, in turn, unitary as asserted.

(ii) If the map $x\mapsto 2x$ is an automorphism of G, then it follows **[35,** Theorem 1] that there exists a constant $c>0$ for which T^{π_2} extends to a unitary operator $L^2(G\times\widehat{G},\nu\times c\nu^*)\to\mathbb{B}_2(\mathcal{H})$. Now, the assertion can be proved, just as above. □

The hypothesis that the map $x\mapsto 2x$ is an automorphism of G from Proposition 7.1 (ii) is satisfied by many important examples of groups, as, for instance, the linear spaces $G=\mathbb{R}^d$, or the additive groups of local fields, like the p-adic fields $\mathbb{Q}_p$ which are interesting for quantization and pseudodifferential theory with applications to number theory, as shown in **[17]**.

We also note that the aforementioned hypothesis is quite natural inasmuch as it is satisfied if and only if an appropriate version of the Stone-von Neumann theorem holds. More precisely, according to **[25,** Theorem 1], if we define the cocycle

$$\sigma\colon (G\times\widehat{G})\times(G\times\widehat{G})\longrightarrow\mathbb{T},\qquad \sigma((x,\xi),(y,\eta))=\langle\xi,y\rangle\overline{\langle\eta,x\rangle},$$

then the locally compact abelian group $G \times \widehat{G}$ has just one equivalence class of unitary irreducible projective representations with the cocycle σ if and only if the map $x \mapsto 2x$ is an automorphism of G. One projective representation of that type is the map π_2 from our Proposition 7.1 (ii). See also [**32,** Appendix VIII] for a discussion of this circle of ideas.

Much extra structure is present due to the existence of a Fourier transform and the structure theorem for locally compact abelian groups. In particular, besides choosing the Bruhat-Schwartz space for the ingredient $\mathcal{G}$, there also exists a better choice which relies on writing G as $\mathbb{R}^m \times G_0$ with G_0 containing an open compact subgroup. We do not review the theory; instead, we refer to [**11**], which also contains many constructions and results involving a certain class of coorbit spaces, discretization techniques and Gabor frames. If $G = \mathbb{R}^m$, i.e., G_0 is trivial, the emerging formalism essentially boils down to the Kohn-Nirenberg pseudodifferential calculus [**8**].

7.3. Unitary representations of some infinite-dimensional Lie groups. For a full presentation, we refer to [**3, 4, 5, 33**].

The starting point is a unitary strongly continuous representation $\varpi : M \to \mathbb{U}(\mathcal{H})$, where M is a locally convex Lie group with the Lie algebra $\mathfrak{m}$ and a smooth exponential $\exp_M : \mathfrak{m} \to M$. On the dual $\mathfrak{m}'$ of $\mathfrak{m}$ we consider the weak* topology.

We also fix a real finite-dimensional vector space Σ with dual Σ' and a linear map $\theta : \Sigma \to \mathfrak{m}$. The basic idea is to use $\pi := \varpi \circ \exp_M \circ \theta : \Sigma \to \mathbb{B}(\mathcal{H})$ as well as the Fourier transform $\widehat{\cdot} : L^2(\Sigma) \to L^2(\Sigma')$ in order to build a more general form of the Weyl calculus. So, one should set

$$\mathfrak{Op}^{\varpi,\theta}(a) \equiv \Pi(\widehat{a}) := \int_\Sigma \widehat{a}(s)\, \varpi\left[\exp_M(\theta(s))\right] ds. \tag{7.2}$$

In [**4**], the outcome was called the *localized Weyl calculus associated to the representation ϖ along the linear map θ.* The single requirement needed to develop the basic part of the theory is orthogonality, i.e.,

$$\int_\Sigma |\langle \varpi[\exp_M(\theta(s))]u, v\rangle|^2 ds = \|u\|^2 \|v\|^2,$$

for all $u, v \in \mathcal{H}$. Under this requirement, the general theory gives a definite sense to equation (7.2), at least for $\widehat{a} \in \mathscr{B}_2(\Sigma)$, and the constructions and results of Section 2 and subsection 3.2 are valid. A good choice for the auxiliary space $\mathcal{G}$ is the space of smooth vectors of the representation ϖ,

$$\mathcal{G} \equiv \mathcal{H}_\infty := \{u \in \mathcal{H} \mid M \ni m \longmapsto \varpi(m)u \in \mathcal{H} \text{ is } C^\infty\}.$$

It carries a natural Fréchet topology [**4,** Remark 2.1] in terms of "differential operators" indexed by the universal associative enveloping algebra $U(\mathfrak{m}_\mathbb{C})$ of the complexified Lie algebra $\mathfrak{m}_\mathfrak{C}$. In the infinite-dimensional case, the denseness of $\mathcal{H}_\infty$ in $\mathcal{H}$ is not always verified, so we must require it. But, as soon as this is achieved, all the present results hold. In [**4**], extra regularity assumptions are imposed in order to obtain good control upon the spaces involved. One of the aims is to identify $\mathscr{G}(\Sigma)$ with the Schwartz space $\mathscr{S}(\Sigma)$ and

$$\mathfrak{Op}^{\varpi,\theta}[\widehat{\mathscr{G}(\Sigma)}] = \Pi[\mathscr{G}(\Sigma)],$$

with the Fréchet space $\mathbb{B}(\mathcal{H})_\infty$ of all the smooth vectors (operators) under the continuous unitary representation

$$\varpi^{(2)} : M \times M \longrightarrow \mathbb{B}[\mathbb{B}_2(\mathcal{H})], \qquad [\varpi^{(2)}(m,n)]\,T := \varpi(m)T\varpi(n)^{-1}.$$

Another aim is to determine when $\mathscr{B}_2(\Sigma) = L^2(\Sigma)$ holds.

The above general setting was studied in some detail in two specific situations: firstly, when M is a finite-dimensionally and simply connected nilpotent Lie group. Such a situation was pointed out in [**33**] and will be discussed in subsection 7.4. Secondly, the case when M is an infinite-dimensional Lie group that can be written as the semidirect product $\mathcal{F} \ltimes G$ between a (finite-dimensional) connected nilpotent Lie group G, with Lie algebra $\mathfrak{g}$, and a suitable (typically infinite-dimensional) locally convex space $\mathcal{F}$ of smooth functions on G. This was studied in [**3, 4**]. The connection with the square-integrable families of operators is established by [**4,** Corollary 4.7 (3)].

An important particular case comes from the presence of a smooth magnetic field (i.e., a closed differential 2-form) on G, inasmuch as the aforementioned function space $\mathcal{F}$ should be invariant under left translations on G and should contain the coefficients of the magnetic field as well as their derivatives of arbitrarily high order. This shows that, if the magnetic field fails to have polynomial coefficients, then $\mathcal{F}$ is infinite-dimensional. In this situation, an irreducible representation is given by

$$\varpi\colon M = \mathcal{F} \rtimes G \longrightarrow \mathbb{B}(L^2(G)), \qquad (\varpi(\phi, x)f)(y) = e^{i\phi(y)} f(x^{-1}y),$$

and, in **[3]**, it was proved that, although M is an infinite-dimensional Lie group, its irreducible representation ϖ can be obtained by the geometric quantization from a certain *finite-dimensional* coadjoint orbit $\mathcal{O}$ of M. Moreover, the coadjoint orbit $\mathcal{O}$ is symplectomorphic to the cotangent bundle T^*G. The appearance of the finite-dimensional vector space Σ from the above general framework is connected to that coadjoint orbit $\mathcal{O}$, and the linear mapping θ is canonically assigned to a vector potential generating the magnetic field. Specifically, one may use $\Sigma = \mathfrak{g} \times \mathfrak{g}'$. In this setting, the outcome of the operator calculus is an extension to nilpotent Lie groups of the magnetic Weyl calculus briefly presented in subsection 7.1, which can be recovered for the abelian group $G = (\mathbb{R}^n, +)$.

7.4. Operator calculus for representations of nilpotent Lie groups. We will briefly describe some square integrable operator-valued maps related to unitary irreducible representations of nilpotent Lie groups. This method was previously used in the proof of Corollary 2.6. The details of this construction can essentially be found in **[5, 33]**.

Let G be any connected, simply connected, nilpotent Lie group with the Lie algebra $\mathfrak{g}$. Then, the exponential map $\exp_G\colon \mathfrak{g} \to G$ is a diffeomorphism with its inverse denoted by $\log_G\colon G \to \mathfrak{g}$. The adjoint action of G is

$$\mathrm{Ad}_G\colon G \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \mathrm{Ad}_G(g)x := \frac{d}{dt}\Big|_{t=0} \big(g \exp_G(tx) g^{-1}\big).$$

We denote by $\mathfrak{g}^*$ the linear dual space of $\mathfrak{g}$ and by $\langle \cdot, \cdot \rangle\colon \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the natural duality pairing. The coadjoint action of G is

$$\mathrm{Ad}_G^*\colon G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \qquad (g, \xi) \longmapsto \mathrm{Ad}_G^*(g)\xi = \xi \circ \mathrm{Ad}_G(g^{-1}).$$

Pick any $\xi_0 \in \mathfrak{g}^*$ with its corresponding coadjoint orbit

$$\mathcal{O} := \mathrm{Ad}_G^*(G)\xi_0 \subseteq \mathfrak{g}^*.$$

The isotropy group at ξ_0 is

$$G_{\xi_0} := \{g \in G \mid \mathrm{Ad}_G^*(g)\xi_0 = \xi_0\},$$

with the corresponding isotropy Lie algebra

$$\mathfrak{g}_{\xi_0} = \{X \in \mathfrak{g} \mid \xi_0 \circ \mathrm{ad}_{\mathfrak{g}} X = 0\}.$$

Let $n := \dim \mathfrak{g}$ and fix any sequence of ideals in $\mathfrak{g}$,

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

such that

$$\dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1$$

and

$$[\mathfrak{g}, \mathfrak{g}_j] \subseteq \mathfrak{g}_{j-1} \quad \text{for } j = 1, \dots, n.$$

Pick any $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$ for $j = 1, \dots, n$, so that the set $\{X_1, \dots, X_n\}$ will be a Jordan-Hölder basis in $\mathfrak{g}$.

The set of *jump indices* of the coadjoint orbit $\mathcal{O}$, with respect to the above Jordan-Hölder basis is

$$e := \{j \in \{1, \dots, n\} \mid \mathfrak{g}_j \not\subseteq \mathfrak{g}_{j-1} + \mathfrak{g}_{\xi_0}\},$$

and does not depend on the choice of $\xi_0 \in \mathcal{O}$. The corresponding *predual of the coadjoint orbit* $\mathcal{O}$ is

$$\mathfrak{g}_e := \mathrm{span}\{X_j \mid j \in e\} \subseteq \mathfrak{g}$$

and it turns out that the map $\mathcal{O} \to \mathfrak{g}_e^*$, $\xi \mapsto \xi|_{\mathfrak{g}_e}$ is a diffeomorphism.

Proposition 7.2. *Assume that the above setting holds, and let* $\pi \colon G \to \mathbb{B}(\mathcal{H})$ *be a fixed, unitary irreducible representation associated with the coadjoint orbit* $\mathcal{O}$*. Then, there exists a Lebesgue measure* μ_e *on* $\mathfrak{g}_e$ *for which, if we denote by* μ *the measure on* G *obtained as the pushforward of* μ_e *by the map* $\exp_G|_{\mathfrak{g}_e} \colon \mathfrak{g}_e \to G$*, then* $\pi \in \mathrm{SQ}(\mathbb{B}(\mathcal{H}), \mu)$*, and its corresponding operator* $\Phi^\pi \colon \mathcal{H} \widehat{\otimes} \overline{\mathcal{H}} \to L^2(G, \mu)$ *is unitary.*

Proof. It easily follows by [**33,** Theorems 2.2.6–2.2.7] that there exists a Lebesgue measure μ_e on $\mathfrak{g}_e$ for which, if we define the measure μ as in the above statement, then the operator

$$T^\pi \colon L^1(G, \mu) \cap L^2(G, \mu) \longrightarrow \mathbb{B}(\mathcal{H}),$$

given by

$$T^\pi f = \int_{\mathfrak{g}_e} f(\exp_G x)\pi(\exp_G x)\, d\mu_e(x) = \int_G f(s)\pi(s)\, d\mu(s),$$

extends to a unitary operator $L^2(G,\mu) \to \mathbb{B}_2(\mathcal{H})$. The assertion is obtained by the same method as in the proof of Proposition 7.1. $\square$

The space of smooth vectors

$$\mathcal{H}_\infty := \{v \in \mathcal{H} \mid \pi(\cdot)v \in \mathcal{C}^\infty(G,\mathcal{H})\}$$

is a Fréchet space in a natural way and is a dense linear subspace of $\mathcal{H}$ which is invariant under the unitary operator $\pi(g)$ for every $g \in G$. We denote the space of all continuous antilinear functionals on $\mathcal{H}_\infty$ by $\mathcal{H}_{-\infty}$, and then, we have the natural inclusions

$$\mathcal{H}_\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-\infty}.$$

Now, consider the unitary representation

$$\pi \otimes \overline{\pi} \colon G \times G \to \mathbb{B}(\mathbb{B}_2(\mathcal{H})),$$

defined by

$$(\pi \otimes \pi)(g_1, g_2)T = \pi(g_1)T\pi(g_2)^{-1} \text{ for all } g_1, g_2 \in G, \text{ for all } T \in \mathbb{B}_2(\mathcal{H})).$$

It is well known that $\pi \otimes \overline{\pi}$ is strongly continuous. The corresponding space of smooth vectors is denoted by $\mathbb{B}(\mathcal{H})_\infty$ and is called the space of smooth operators for the representation π. One can prove that $\mathbb{B}(\mathcal{H})_\infty \subseteq \mathbb{B}_1(\mathcal{H})$.

Since

$$\{\langle \cdot, f_1\rangle f_2 \mid f_1, f_2 \in \mathcal{H}_\infty\} \subseteq \mathbb{B}(\mathcal{H})_\infty \subseteq \mathbb{B}_1(\mathcal{H}),$$

and $\mathcal{H}_\infty$ is dense in $\mathcal{H}$, we obtain continuous inclusion maps

$$\mathbb{B}(\mathcal{H})_\infty \hookrightarrow \mathbb{B}_1(\mathcal{H}) \hookrightarrow \mathbb{B}(\mathcal{H}) \hookrightarrow \mathbb{B}(\mathcal{H})^*_\infty,$$

where the latter mapping is constructed by using the well-known isomorphism $(\mathbb{B}_1(\mathcal{H}))^* \simeq \mathbb{B}(\mathcal{H})$ given by the usual semifinite trace on $\mathbb{B}(\mathcal{H})$.

We conclude by noting that the version in the present setting of the above dequantization formula from Corollary 3.6 corresponds to [**33,** Theorem 2.2.6].

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