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SOME GENERATING RELATIONS FOR EXTENDED APPELL'S AND LAURICELLA'S HYPERGEOMETRIC FUNCTIONS

HONGMEI LIU

ABSTRACT. In this paper, by the extension of beta functions containing three extra parameters, we generalize Appell's and Lauricella's hypergeometric functions. Some integral representations, transformation formulae, differentiation formulae and recurrence relations are obtained for these new generalizations.

1. Introduction. A generalization of Euler's beta function is defined by **[11]**

$$B_p^{(\alpha,\beta)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) \mathrm{d}t$$

$$(\mathrm{Re}\,(\alpha)>0, \mathrm{Re}\,(\beta)>0, \mathrm{Re}\,(p)>0, \mathrm{Re}\,(x)>0, \mathrm{Re}\,(y)>0). \tag{1.1}$$

When $\alpha=\beta$, it reduces to

$$\begin{aligned} B_p(x,y) &= B_p^{(\alpha,\alpha)}(x,y) \\ &= \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left\{\frac{-p}{t(1-t)}\right\} \mathrm{d}t \quad (\mathrm{Re}\,(p)>0), \end{aligned}$$

which was firstly introduced by Chaudhry et al. **[2]** in 1997 and has a connection with the Macdonald, error and Whittakers functions. Clearly, the classical beta function $B(x,y)$ is given by

$$B(x,y) := B_0(x,y) = B_0^{(\alpha,\beta)}(x,y).$$

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In 2004, making use of $B_p(x, y)$, Chaudhry et al. [**3**] extended the Gauss' hypergeometric function as follows:

$$F_p(a, b; c; z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!},$$

where $p \geq 0$ and $\mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0$. More recently, by appealing to $B_p^{(\alpha,\beta)}(x, y)$, Özergin et al. [**11**] further extended Gauss' hypergeometric function by

$$F_p^{(\alpha,\beta)}(a, b; c; z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}.$$

It can be noticed that

$$F_p(a, b; c; z) = F_p^{(\alpha,\alpha)}(a, b; c; z) \text{ and } {}_2F_1(a, b; c; z) = F_0^{(\alpha,\beta)}(a, b; c; z).$$

In 2010, using $B_p(x, y)$, Özarslan and Özergin [**10**] generalized Appell's hypergeometric functions of two variables, $F_1(a, b, c; d; x, y; p)$ and $F_2(a, b, c; d, e; x, y; p)$, as well as Lauricella's hypergeometric function of three variables, $F_{D,p}^3(a, b, c, d; e; x, y, z)$. Appell's and Lauricella's hypergeometric functions arise frequently in various physical and quantum chemical applications (see [**13, 5, 6, 8, 9, 12, 14**]). Motivated by Özarslan and Özergin's work as well as the other works mentioned above, in this paper, we will generalize Appell's hypergeometric functions and Lauricella's hypergeometric function by the extended beta function $B_p^{(\alpha,\beta)}(x, y)$.

Let us define the new extended Appell's hypergeometric functions

$$F_1^{(\alpha,\beta)}(a, b, c; d; x, y; p) \quad \text{and} \quad F_2^{(\alpha,\beta,\alpha',\beta')}(a, b, c; d, e; x, y; p)$$

and the new extended Lauricella's hypergeometric function

$$F_{D,p}^{(3;\alpha,\beta)}(a, b, c, d; e; x, y, z)$$

by

(1.2)

$$F_1^{(\alpha,\beta)}(a,b,c;d;x,y;p) := \sum_{n,m=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n,d-a)}{B(a,d-a)}$$
$$(b)_n(c)_m \frac{x^n}{n!}\frac{y^m}{m!} \quad (\max\{|x|,|y|\}<1),$$

$$F_2^{(\alpha,\beta,\alpha',\beta')}(a,b,c;d,e;x,y;p)$$

(1.3)

$$:= \sum_{n,m=0}^{\infty} \frac{(a)_{m+n}B_p^{(\alpha,\beta)}(b+n,d-b)B_p^{(\alpha',\beta')}(c+m,e-c)}{B(b,d-b)B(c,e-c)}\frac{x^n}{n!}\frac{y^m}{m!}$$
$$(|x|+|y|<1),$$

and

(1.4)

$$F_{D,p}^{(3;\alpha,\beta)}(a,b,c,d;e;x,y,z)$$
$$:= \sum_{m,n,r=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(a+m+n+r,e-a)(b)_m(c)_n(d)_r}{B(a,e-a)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^r}{r!}$$
$$(|x|<1,\ |y|<1,\ |z|<1),$$

respectively. It is obvious that

$$F_1^{(\alpha,\alpha)}(a,b,c;d;x,y;p) = F_1(a,b,c;d;x,y;p),$$
$$F_2^{(\alpha,\alpha,\alpha',\alpha')}(a,b,c;d,e;x,y;p) = F_2(a,b,c;d,e;x,y;p),$$
$$F_{D,p}^{(3;\alpha,\alpha)}(a,b,c,d;e;x,y,z) = F_{D,p}^{3}(a,b,c,d;e;x,y,z).$$

The aim of this paper is to obtain some relations for

$$F_1^{(\alpha,\beta)}(a,b,c;d;x,y;p), \qquad F_2^{(\alpha,\beta,\alpha',\beta')}(a,b,c;d,e;x,y;p)$$

and

$$F_{D,p}^{(3;\alpha,\beta)}(a,b,c,d;e;x,y,z),$$

and the paper is organized as follows. In Section 2, we present the integral representations for these three generalized hypergeometric functions. Section 3 gives some results related to Mellin transforms of the above new hypergeometric functions and Section 4 discusses the differentiation formulae of these new functions. Finally, in Section 5, we present series of recurrence relations for these new Appell's and Lauricella's functions.

2. Integral representations.

Theorem 2.1. *For the new extended Appell's function*

$$F_1^{(\alpha,\,\beta)}(a,b,c;d;x,y;p),$$

we have

$$\begin{aligned}
F_1^{(\alpha,\,\beta)}&(a,b,c;d;x,y;p)\\
&= \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\\
&\quad\times {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t.
\end{aligned}\tag{2.1}$$

Proof. By the extended beta function (1.1),

$$\begin{aligned}
F_1^{(\alpha,\,\beta)}&(a,b,c;d;x,y;p)\\
&= \sum_{n,m=0}^{\infty}\frac{B_p^{(\alpha,\,\beta)}(a+m+n,d-a)}{B(a,d-a)}(b)_n(c)_m\frac{x^n}{n!}\frac{y^m}{m!}\\
&= \sum_{n,m=0}^{\infty}\left\{\int_0^1 t^{a+m+n-1}(1-t)^{d-a-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t\right\}\\
&\qquad\qquad\frac{(b)_n(c)_m}{B(a,d-a)}\frac{x^n}{n!}\frac{y^m}{m!}.
\end{aligned}$$

Interchanging the order of summation and integration, we have

$$\begin{aligned}
F_1^{(\alpha,\,\beta)}&(a,b,c;d;x,y;p)\\
&= \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}\\
&\quad\times {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\left(\sum_{n=0}^{\infty}\frac{(b)_n(xt)^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{(c)_m(yt)^m}{m!}\right)\mathrm{d}t
\end{aligned}$$

$$= \frac{1}{B(a, d-a)} \int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}$$
$$\times \, {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) \mathrm{d}t,$$

which gives the result. □

Theorem 2.2. *For the new extended Appell's function*

$$F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a, b, c; d, e; x, y; p),$$

we have

(2.2)

$$F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a, b, c; d, e; x, y; p)$$
$$= \frac{1}{B(b, d-b)B(c, e-c)} \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a}$$
$$\times \, {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) {}_1F_1\left(\alpha'; \beta'; \frac{-p}{s(1-s)}\right) \mathrm{d}t \, \mathrm{d}s.$$

Proof. Using the definition of extended beta function (1.1), we obtain

$$F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a, b, c; d, e; x, y; p)$$
$$= \sum_{n,m=0}^{\infty} \frac{(a)_{m+n} B_p^{(\alpha,\,\beta)}(b+n, d-b) B_p^{(\alpha',\beta')}(c+m, e-c)}{B(b, d-b)B(c, e-c)} \frac{x^n}{n!} \frac{y^m}{m!}$$
$$= \sum_{n,m=0}^{\infty} \frac{(a)_{m+n}}{B(b, d-b)B(c, e-c)}$$
$$\times \left\{ \int_0^1 t^{b+n-1}(1-t)^{d-b-1} \, {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) \mathrm{d}t \right\}$$
$$\times \left\{ \int_0^1 s^{c+m-1}(1-s)^{e-c-1} \, {}_1F_1\left(\alpha'; \beta'; \frac{-p}{s(1-s)}\right) \mathrm{d}s \right\} \frac{x^n}{n!} \frac{y^m}{m!}.$$

Interchanging the order of summation and integration yields

$$F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a, b, c; d, e; x, y; p) = \frac{1}{B(b, d-b)B(c, e-c)}$$

$$\times \int_0^1 \int_0^1 t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}$$
$$\times \, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) {}_1F_1\left(\alpha';\beta';\frac{-p}{s(1-s)}\right)$$
$$\times \left(\sum_{n,m=0}^{\infty}(a)_{m+n}\frac{(xt)^n}{n!}\frac{(ys)^m}{m!}\right)\mathrm{d}t\,\mathrm{d}s.$$

Taking into account the summation formula

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m+n)\frac{x^n}{n!}\frac{y^m}{m!} = \sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!},$$

we have

$$\sum_{n,m=0}^{\infty}(a)_{m+n}\frac{(xt)^n}{n!}\frac{(ys)^m}{m!} = \sum_{N=0}^{\infty}(a)_N\frac{(xt+ys)^N}{N!} = (1-xt-ys)^{-a},$$

which gives the final result. □

Similarly to Theorems 2.1 and 2.2, we can establish the integral representation for the new Lauricella's hypergeometric function $F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)$ as follows.

Theorem 2.3. *For the function* $F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)$, *we have*

$$\begin{aligned}
&F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)\\
&\quad= \frac{1}{B(a,e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}(1-zt)^{-d}\\
&\qquad\times \, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t.
\end{aligned} \tag{2.3}$$

3. Mellin transforms. In this section, we will give the Mellin transforms of $F_1^{(\alpha,\,\beta)}(a,b,c;d;x,y;p)$, $F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b,c;d,e;x,y;p)$ and $F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)$.

Recall that the Mellin transform of function $f(x)$ is given by (see, e.g., [**4, 15**]):

$$\mathcal{M}\{f(x)\}(s) := \varphi(s) = \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x,$$

and its inverse transform is

$$f(x) = \mathcal{M}^{-1}\{\varphi(s)\} := \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} x^{-s}\varphi(s)\,\mathrm{d}s.$$

The Mellin transform is closely connected to the theory of Dirichlet series and is often used in number theory and the theory of asymptotic expansions. Because of its scale invariance property, the Mellin transform is also widely used in computer science. Furthermore, the use of the Mellin transform in the generating function theory is explained by Özergin et al. [**11**].

Theorem 3.1. *For the function* $F_1^{(\alpha,\beta)}(a,b,c;d;x,y;p)$*, the following Mellin transform representation holds:*

$$\begin{aligned}&\mathcal{M}\{F_1^{(\alpha,\beta)}(a,b,c;d;x,y;p)\}(r)\\ &\quad= \frac{\Gamma(r)B(\alpha-r,r)B(a+r,d-a+r)}{B(\beta-r,r)B(a,d-a)} F_1(a+r,b,c;d+2r;x,y).\end{aligned}$$

Proof. By the definition of the Mellin transform and integral representation (2.1), we have

$$\begin{aligned}&\mathcal{M}\{F_1^{(\alpha,\beta)}(a,b,c;d;x,y;p)\}(r)\\ &\quad= \frac{1}{B(a,d-a)}\\ &\quad\times \int_0^\infty p^{r-1}\left(\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\right.\\ &\qquad\left.\times\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t\right)\mathrm{d}p\\ &\quad= \frac{1}{B(a,d-a)}\\ &\quad\times \int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\end{aligned}$$

$$\times \left(\int_0^\infty p^{r-1} \, {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) \mathrm{d}p \right) \mathrm{d}t.$$

Setting $u = p/(t(1-t))$ gives

$$\begin{aligned} \int_0^\infty p^{r-1} \, {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) \mathrm{d}p &= \int_0^\infty u^{r-1} t^r (1-t)^r \, {}_1F_1(\alpha; \beta; -u) \mathrm{d}u \\ &= t^r (1-t)^r \int_0^\infty u^{r-1} \, {}_1F_1(\alpha; \beta; -u) \mathrm{d}u. \end{aligned}$$

According to **[7]**,

$$\int_0^\infty u^{r-1} \, {}_1F_1(\alpha; \beta; -u) \, \mathrm{d}u = \frac{\Gamma(\beta)\Gamma(\alpha - r)\Gamma(r)}{\Gamma(\alpha)\Gamma(\beta - r)},$$

we obtain

$$\begin{aligned} \mathcal{M}\{F_1^{(\alpha,\beta)}(a,b,c;d;x,y;p)\}(r) &= \frac{\Gamma(\beta)\Gamma(\alpha - r)\Gamma(r)}{B(a,d-a)\Gamma(\alpha)\Gamma(\beta - r)} \\ &\quad \times \int_0^1 t^{a+r-1}(1-t)^{d-a+r-1}(1-xt)^{-b}(1-yt)^{-c} \, \mathrm{d}t \\ &= \frac{\Gamma(r)B(\alpha - r, r)B(a+r, d-a+r)}{B(\beta - r, r)B(a, d-a)} F_1(a+r, b, c; d+2r; x, y), \end{aligned}$$

which is the desired result. □

Corollary 3.2. *By the Mellin inverse formula, we have the following complex integral representation for the function* $F_1^{(\alpha,\beta)}(a,b,c;d;x,y;p)$:

$$\begin{aligned} F_1^{(\alpha,\beta)}&(a,b,c;d;x,y;p) \\ &= \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\Gamma(r)B(\alpha - r, r)B(a+r, d-a+r)}{B(\beta - r, r)B(a, d-a)} \\ &\qquad \times F_1(a+r, b, c; d+2r; x, y) p^{-r} \, \mathrm{d}r. \end{aligned}$$

Theorem 3.3. *For the function* $F_2^{(\alpha,\beta,\alpha',\beta')}(a,b,c;d,e;x,y;p)$, *the* *following Mellin transform representation holds*:

$$\begin{aligned}
&\mathcal{M}\{F_2^{(\alpha,\,\beta,\,\gamma,\,\gamma)}(a,b,c;d,e;x,y;p)\}(r)\\
&\quad=\Gamma(r)\sum_{n=0}^{\infty}\frac{(\alpha)_n(r)_nB(b-n,d-b-n)B(c+r+n,e-c+r+n)}{n!(-1)^n(\beta)_nB(b,d-b)B(c,e-c)}\\
&\qquad\times F_2(a,b-n,c+r+n;d-2n,e+2r+2n;x,y).
\end{aligned}$$

Proof. By the integral representation (2.2), we obtain

$$\begin{aligned}
&\mathcal{M}\{F_2^{(\alpha,\,\beta,\,\gamma,\,\gamma)}(a,b,c;d,e;x,y;p)\}(r)\\
&\quad=\int_0^\infty p^{r-1}F_2^{(\alpha,\,\beta,\,\gamma,\,\gamma)}(a,b,c;d,e;x,y;p)\,\mathrm{d}p\\
&\quad=\int_0^1\int_0^1\frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^aB(b,d-b)B(c,e-c)}\\
&\quad\times\left(\int_0^\infty p^{r-1}\mathrm{e}^{-p/(s(1-s))}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}p\right)\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

According to [**7**], there exists the following integral formula for confluent function ${}_1F_1$:

$$\int_0^\infty t^{r-1}\mathrm{e}^{-ct}\,{}_1F_1(a;b;-t)\,\mathrm{d}t=c^{-r}\Gamma(r)\,{}_2F_1\left(a,r;b;-\frac{1}{c}\right).$$

Then, setting $u=p/(t(1-t))$ and using the above formula, we have

$$\begin{aligned}
&\int_0^\infty p^{r-1}\mathrm{e}^{-p/(s(1-s))}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}p\\
&\quad=t^r(1-t)^r\int_0^\infty u^{r-1}\mathrm{e}^{-t(1-t)u/(s(1-s))}\,{}_1F_1(\alpha;\beta;-u)\,\mathrm{d}u\\
&\quad=s^r(1-s)^r\Gamma(r)\,{}_2F_1\left\{\alpha,r;\beta;-\frac{s(1-s)}{t(1-t)}\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathcal{M}\{F_2^{(\alpha,\,\beta,\,\gamma,\,\gamma)}(a,b,c;d,e;x,y;p)\}(r)\\
&\quad=\Gamma(r)\int_0^1\int_0^1\frac{t^{b-1}(1-t)^{d-b-1}s^{c+r-1}(1-s)^{e-c+r-1}}{(1-xt-ys)^aB(b,d-b)B(c,e-c)}\\
&\qquad\times\,{}_2F_1\left(\alpha,r;\beta;-\frac{s(1-s)}{t(1-t)}\right)\mathrm{d}t\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(r)}{B(b,d-b)B(c,e-c)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(r)_n}{n!(-1)^n(\beta)_n} \\
&\times \int_0^1\int_0^1 t^{b-n-1}(1-t)^{d-b-n-1}s^{c+r+n-1} \\
&\quad \times (1-s)^{e-c+r+n-1}(1-xt-ys)^{-a}\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

By integral representation of Appell's function F_2, we obtain the result. □

Considering that the parameters b and d are symmetric to c and e in $F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b,c;d,e;x,y;p)$, we have

Theorem 3.4. *For the function $F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b,c;d,e;x,y;p)$, the following Mellin transform representation holds*:

$$
\begin{aligned}
&\mathcal{M}\{F_2^{(\gamma,\,\gamma,\,\alpha,\,\beta)}(a,b,c;d,e;x,y;p)\}(r) \\
&\quad = \Gamma(r)\sum_{n=0}^{\infty}\frac{(\alpha)_n(r)_nB(c-n,e-c-n)B(b+r+n,d-b+r+n)}{n!(-1)^n(\beta)_nB(b,d-b)B(c,e-c)} \\
&\qquad \times F_2(a,c-n,b+r+n;e-2n,d+2r+2n;x,y).
\end{aligned}
$$

Finally, we have

Theorem 3.5. *For the function $F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)$, the following Mellin transform representation holds:*

$$
\begin{aligned}
&\mathcal{M}\{F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)\}(r) \\
&\quad = \frac{\Gamma(r)B(\alpha-r,r)B(a+r,e-a+r)}{B(\beta-r,r)B(a,e-a)}F_D^{(3)}(a+r,b,c,d;e+2r;x,y,z).
\end{aligned}
$$

Corollary 3.6. *By the Mellin inverse formula, we have the following complex integral representation*:

$$
\begin{aligned}
&F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z) \\
&\quad = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\Gamma(r)B(\alpha-r,r)B(a+r,e-a+r)}{B(\beta-r,r)B(a,e-a)} \\
&\qquad \times F_D^{(3)}(a+r,b,c,d;e+2r;x,y,z)p^{-r}\,\mathrm{d}r.
\end{aligned}
$$

4. Differentiation formulae.

Theorem 4.1. *For $F_1^{(\alpha,\,\beta)}(a,b,c;d;x,y;p)$, the following differentiation formula holds*:

$$\frac{\mathrm{d}^{m+n}}{\mathrm{d}x^m\mathrm{d}y^n}\{F_1^{(\alpha,\,\beta)}(a,b,c;d;x,y;p)\} = \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}}F_1^{(\alpha,\,\beta)}(a+m+n,b+m,c+n;d+m+n;x,y;p).$$

Proof. Taking the derivatives of $F_1^{(\alpha,\,\beta)}(a,b,c;d;x,y;p)$ with respect to x and y gives

$$\frac{\mathrm{d}^2}{\mathrm{d}x\mathrm{d}y}\{F_1^{(\alpha,\,\beta)}(a,b,c;d;x,y;p)\} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{B_p^{(\alpha,\,\beta)}(a+m+n,d-a)}{B(a,d-a)}(b)_n(c)_m\frac{x^{n-1}}{(n-1)!}\frac{y^{m-1}}{(m-1)!}.$$

Then, using the formula $B(b,c-b)=(c/b)B(b+1,c-b)$, and setting $n\to n+1$, $m\to m+1$, yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x\mathrm{d}y}\{F_1^{(\alpha,\,\beta)}(a,b,c;d;x,y;p)\} = \frac{a(a+1)bc}{d(d+1)}F_1^{(\alpha,\,\beta)}(a+2,b+1,c+1;d+2;x,y;p).$$

By induction, the result can be derived. □

Similarly to Theorem 4.1, we can establish the differentiation formulae for $F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b,c;d,e;x,y;p)$ and $F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)$.

Theorem 4.2. *For $F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b,c;d,e;x,y;p)$, the following differentiation formula holds*:

$$\frac{\mathrm{d}^{m+n}}{\mathrm{d}x^m\mathrm{d}y^n}\{F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b,c;d,e;x,y;p)\} = \frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n} \times F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a+m+n,b+m,c+n;d+m,e+n;x,y;p).$$

Theorem 4.3. *For $F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)$, the following differentiation formula holds*:

$$\frac{\mathrm{d}^{m+n+r}}{\mathrm{d}x^m\mathrm{d}y^n\mathrm{d}z^r}\{F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)\} = \frac{(a)_{m+n+r}(b)_m(c)_n(d)_r}{(e)_{m+n+r}} \times F_{D,p}^{(3;\,\alpha,\,\beta)}(a+m+n+r,b+m,c+n,d+r;e+m+n+r;x,y,z).$$

Now, let us consider the differentiation formulae of the new Appell's and Lauricella's functions with respect to the parameter α. To compute them, we need the following equation for confluent function ${}_1F_1$:

$$\frac{\mathrm{d}\{{}_1F_1(a+1;a;z)\}}{\mathrm{d}a} = -\frac{ze^z}{a^2}. \tag{4.1}$$

Theorem 4.4. *For $F_1^{(\alpha,\,\beta)}(a,b,c;d;x,y;p)$, the following differentiation formula holds*:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\{F_1^{(\alpha+1,\,\alpha)}(a,b,c;d;x,y;p)\} = \frac{pB(a-1,d-a-1)}{\alpha^2B(a,d-a)}F_1(a-1,b,c;d-2;x,y;p).$$

Proof. By the integral representation (2.1), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\alpha}\{F_1^{(\alpha+1,\,\alpha)}(a,b,c;d;x,y;p)\}\\&= \frac{\mathrm{d}}{\mathrm{d}\alpha}\left\{\frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\right.\\&\qquad\left.{}_1F_1\left(\alpha+1;\alpha;\frac{-p}{t(1-t)}\right)\mathrm{d}t\right\}\\&= \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\frac{\mathrm{d}}{\mathrm{d}\alpha}\\&\qquad\left\{{}_1F_1\left(\alpha+1;\alpha;\frac{-p}{t(1-t)}\right)\right\}\mathrm{d}t,\end{aligned}$$

which, combined with (4.1), gives

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\{F_1^{(\alpha+1,\,\alpha)}(a,b,c;d;x,y;p)\} = \frac{p}{\alpha^2B(a,d-a)}$$

$$\times \int_0^1 t^{a-2}(1-t)^{d-a-2}(1-xt)^{-b}(1-yt)^{-c}\mathrm{e}^{-p/(t(1-t))}\,\mathrm{d}t.$$

Then, by appealing to the integral representation of function $F_1(a,b,c;d,x,y;p)$, we obtain the result. $\square$

Similarly, the next two theorems for $F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b,c;d,e;x,y;p)$ and $F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)$, can be established.

Theorem 4.5. *For $F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b,c;d,e;x,y;p)$, the following differentiation formulae hold*:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\{F_2^{(\alpha+1,\,\alpha,\,\beta,\,\beta)}(a,b,c;d,e;x,y;p)\} = \frac{pB(b-1,d-b-1)}{\alpha^2 B(b,d-b)}F_2(a,b-1,c;d-2,e;x,y;p),$$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\{F_2^{(\beta,\,\beta,\,\alpha+1,\,\alpha,)}(a,b,c;d,e;x,y;p)\} = \frac{pB(c-1,e-c-1)}{\alpha^2 B(c,e-c)}F_2(a,b,c-1;d,e-2;x,y;p).$$

Theorem 4.6. *For $F_{D,p}^{(3;\,\alpha,\,\beta)}(a,b,c,d;e;x,y,z)$, the following differentiation formula holds*:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\{F_{D,p}^{(3;\,\alpha+1,\,\alpha)}(a,b,c,d;e;x,y,z)\} = \frac{pB(a-1,e-a-1)}{\alpha^2 B(a,e-a)}F_{D,p}^3(a-1,b,c,d;e-2;x,y,z).$$

5. Recurrence relations. In this section, we will establish several recurrence relations for the new functions by using recurrences of the confluent function ${}_1F_1$. Firstly, let us present the recurrence formulae of confluent function ${}_1F_1$ (see **[1, 7]**).

(5.1)
$$(a-b)\,{}_1F_1(a-1;b;z)+a\,{}_1F_1(a+1;b;z)+(b-2a-z)\,{}_1F_1(a;b;z)=0,$$
(5.2)
$$(1-b)b\,{}_1F_1(a;b-1;z)+(a-b)z\,{}_1F_1(a;b+1;z)+b(b+z-1)\,{}_1F_1(a;b;z)=0,$$
(5.3)
$$(a+z-1)\,{}_1F_1(a;b;z)+(b-a)\,{}_1F_1(a-1;b;z)+(1-b)\,{}_1F_1(a;b-1;z)=0,$$
(5.4)
$$b\,{}_1F_1(a;b;z)-b\,{}_1F_1(a-1;b;z)-z\,{}_1F_1(a;b+1;z)=0,$$
(5.5)
$$(a-b+1)\,{}_1F_1(a;b;z)-a\,{}_1F_1(a+1;b;z)-(1-b)\,{}_1F_1(a;b-1;z)=0,$$
(5.6)
$$b(a+z)\,{}_1F_1(a;b;z)-ab\,{}_1F_1(a+1;b;z)-(b-a)z\,{}_1F_1(a;b+1;z)=0,$$

$$\begin{aligned}(a-b+1)(b-a)\,{}_1F_1(a-1;b;z)&+a(a+z-1)\,{}_1F_1(a+1;b;z)\\&+(b-1)(b-2a-z)\,{}_1F_1(a;b-1;z)=0,\end{aligned}\tag{5.7}$$
$$\begin{aligned}ab(b+z-1)\,{}_1F_1(a+1;b;z)&+(1-b)b(a+z)\,{}_1F_1(a;b-1;z)\\&+(a-b)(a-b+1)z\,{}_1F_1(a;b+1;z)=0.\end{aligned}\tag{5.8}$$

Now we take (5.1) as an example. Based on (5.1), we have

$$\begin{aligned}&\frac{\alpha-\beta}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\\&\qquad{}_1F_1\left(\alpha-1;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t\\&+\frac{\alpha}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\\&\qquad{}_1F_1\left(\alpha+1;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t\\&+\frac{\beta-2\alpha}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\end{aligned}$$

$$
\begin{aligned}
&{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t\\
&+\frac{p}{B(a,d-a)}\int_0^1 t^{a-2}(1-t)^{d-a-2}(1-xt)^{-b}(1-yt)^{-c}\\
&{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t=0.
\end{aligned}
$$

From the integral representation of $F_1^{(\alpha,\beta)}(a,b,c;d;x,y;p)$, the following recurrence relation holds:

$$
\begin{aligned}
&(\alpha-\beta)F_1^{(\alpha-1,\beta)}+\alpha F_1^{(\alpha+1,\beta)}+(\beta-2\alpha)F_1^{(\alpha,\beta)}\\
&\quad+\frac{pB(a-1,d-a-1)}{B(a,d-a)}F_1^{(\alpha,\beta)}(a-1,b,c;d-2;x,y;p)=0,
\end{aligned}
$$

where, for the sake of making the contiguous expressions more transparent, we omit the parameters $(a,b,c;d;x,y;p)$.

Similarly to the above procedure, for

$$F_2^{(\alpha,\beta,\alpha',\beta')}(a,b,c;d,e;x,y;p)\quad\text{and}\quad F_{D,p}^{(3;\alpha,\beta)}(a,b,c,d;e;x,y,z),$$

the following

$$
\begin{aligned}
&(\alpha-\beta)F_2^{(\alpha-1,\beta,\alpha',\beta')}+\alpha F_2^{(\alpha+1,\beta,\alpha',\beta')}+(\beta-2\alpha)F_2^{(\alpha,\beta,\alpha',\beta')}\\
&\quad+\frac{pB(b-1,d-b-1)}{B(b,d-b)}F_2^{(\alpha,\beta,\alpha',\beta')}(a,b-1,c;d-2,e;x,y;p)=0,\\
&(\alpha'-\beta')F_2^{(\alpha,\beta,\alpha'-1,\beta')}+\alpha' F_2^{(\alpha,\beta,\alpha'+1,\beta')}+(\beta'-2\alpha')F_2^{(\alpha,\beta,\alpha',\beta')}\\
&\quad+\frac{pB(c-1,e-c-1)}{B(c,e-c)}F_2^{(\alpha,\beta,\alpha',\beta')}(a,b,c-1;d,e-2;x,y;p)=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\alpha-\beta)F_{D,p}^{(3;\alpha-1,\beta)}+\alpha F_{D,p}^{(3;\alpha+1,\beta)}+(\beta-2\alpha)F_{D,p}^{(3;\alpha,\beta)}\\
&\quad+\frac{pB(a-1,e-a-1)}{B(a,e-a)}F_{D,p}^{(3;\alpha,\beta)}(a-1,b,c,d;e-2;x,y,z)=0
\end{aligned}
$$

hold.

By the same method, using formulae (5.2)–(5.8), we can present a series of other recurrences for the three new functions. Because the parameters α, β, b, d are symmetric to the parameters α', β', c, e in $F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b,c;d,e;x,y;p)$, we only exhibit one relation for it.

Using (5.2), we obtain the recurrences

$$\begin{aligned}
&(1-\beta)\beta F_1^{(\alpha,\,\beta-1)} + \beta(\beta-1)F_1^{(\alpha,\,\beta)}\\
&\quad - \frac{p(\alpha-\beta)B(a-1,d-a-1)}{B(a,d-a)}F_1^{(\alpha,\,\beta+1)}(a-1,b,c;d-2;x,y;p)\\
&\quad - \frac{p\beta B(a-1,d-a-1)}{B(a,d-a)}F_1^{(\alpha,\,\beta)}(a-1,b,c;d-2;x,y;p)=0,\\
&(1-\beta)\beta F_2^{(\alpha,\,\beta-1,\,\alpha',\,\beta')} + \beta(\beta-1)F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}\\
&\quad - \frac{p(\alpha-\beta)B(b-1,d-b-1)}{B(b,d-b)}F_2^{(\alpha,\,\beta+1,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p)\\
&\quad - \frac{p\beta B(b-1,d-b-1)}{B(b,d-b)}F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p)=0,
\end{aligned}$$

and

$$\begin{aligned}
&(1-\beta)\beta F_{D,p}^{(3;\,\alpha,\,\beta-1)} + \beta(\beta-1)F_{D,p}^{(3;\,\alpha,\,\beta)}\\
&\quad - \frac{p(\alpha-\beta)B(a-1,e-a-1)}{B(a,e-a)}F_{D,p}^{(3;\,\alpha,\,\beta+1)}(a-1,b,c,d;e-2;x,y,z)\\
&\quad - \frac{p\beta B(a-1,e-a-1)}{B(a,e-a)}F_{D,p}^{(3;\,\alpha,\,\beta)}(a-1,b,c,d;e-2;x,y,z)=0.
\end{aligned}$$

From (5.3), we have the recurrences

$$\begin{aligned}
&(\alpha-1)F_1^{(\alpha,\,\beta)} + (\beta-\alpha)F_1^{(\alpha-1,\,\beta)} + (1-\beta)F_1^{(\alpha,\,\beta-1)}\\
&\quad - \frac{pB(a-1,d-a-1)}{B(a,d-a)}F_1^{(\alpha,\,\beta)}(a-1,b,c;d-2;x,y;p)=0,\\
&(\alpha-1)F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')} + (\beta-\alpha)F_2^{(\alpha-1,\,\beta,\,\alpha',\,\beta')} + (1-\beta)F_2^{(\alpha,\,\beta-1,\,\alpha',\,\beta')}\\
&\quad - \frac{pB(b-1,d-b-1)}{B(b,d-b)}F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p)=0,
\end{aligned}$$

and

$$(\alpha-1)F_{D,p}^{(3;\,\alpha,\,\beta)} + (\beta-\alpha)F_{D,p}^{(3;\,\alpha-1,\,\beta)} + (1-\beta)F_{D,p}^{(3;\,\alpha,\,\beta-1)}$$

$$- \frac{pB(a-1,e-a-1)}{B(a,e-a)} F_{D,p}^{(3;\,\alpha,\,\beta)}(a-1,b,c,d;e-2;x,y,z) = 0.$$

By (5.4), the following recurrences can be established:

$$\beta F_1^{(\alpha,\,\beta)} - \beta F_1^{(\alpha-1,\,\beta)} + \frac{pB(a-1,d-a-1)}{B(a,d-a)}$$
$$F_1^{(\alpha,\,\beta+1)}(a-1,b,c;d-2;x,y;p) = 0,$$
$$\beta F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')} - \beta F_2^{(\alpha-1,\,\beta,\,\alpha',\,\beta')}$$
$$+ \frac{pB(b-1,d-b-1)}{B(b,d-b)} F_2^{(\alpha,\,\beta+1,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p) = 0,$$

and

$$\beta F_{D,p}^{(3;\,\alpha,\,\beta)} - \beta F_{D,p}^{(3;\,\alpha-1,\,\beta)}$$
$$+ \frac{pB(a-1,e-a-1)}{B(a,e-a)} F_{D,p}^{(3;\,\alpha,\,\beta+1)}(a-1,b,c,d;e-2;x,y,z) = 0.$$

From (5.5), the following hold:

$$(\alpha-\beta+1)F_1^{(\alpha,\,\beta)} - \alpha F_1^{(\alpha+1,\,\beta)} - (1-\beta)F_1^{(\alpha,\,\beta-1)} = 0,$$
$$(\alpha-\beta+1)F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')} - \alpha F_2^{(\alpha+1,\,\beta,\,\alpha',\,\beta')} - (1-\beta)F_2^{(\alpha,\,\beta-1,\,\alpha',\,\beta')} = 0,$$

and

$$(\alpha-\beta+1)F_{D,p}^{(3;\,\alpha,\,\beta)} - \alpha F_{D,p}^{(3;\,\alpha+1,\,\beta)} - (1-\beta)F_{D,p}^{(3;\,\alpha,\,\beta-1)} = 0.$$

Based on (5.6), we can establish

$$\alpha\beta F_1^{(\alpha,\,\beta)} - \alpha\beta F_1^{(\alpha+1,\,\beta)} - \frac{p\beta B(a-1,d-a-1)}{B(a,d-a)}$$
$$F_1^{(\alpha,\,\beta)}(a-1,b,c;d-2;x,y;p)$$
$$+ \frac{p(\beta-\alpha)B(a-1,d-a-1)}{B(a,d-a)} F_1^{(\alpha,\,\beta+1)}(a-1,b,c;d-2;x,y;p) = 0,$$
$$\alpha\beta F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')} - \alpha\beta F_2^{(\alpha+1,\,\beta,\,\alpha',\,\beta')}$$
$$- \frac{p\beta B(b-1,d-b-1)}{B(b,d-b)} F_2^{(\alpha,\,\beta,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p)$$

$$
\begin{aligned}
&+\frac{p(\beta-\alpha)B(b-1,d-b-1)}{B(b,d-b)}\\
&F_2^{(\alpha,\,\beta+1,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p)=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\alpha\beta F_{D,p}^{(3;\,\alpha,\,\beta)}-\alpha\beta F_{D,p}^{(3;\,\alpha+1,\,\beta)}-\frac{p\beta B(a-1,e-a-1)}{B(a,e-a)}\\
&\quad\times F_{D,p}^{(3;\,\alpha,\,\beta)}(a-1,b,c,d;e-2;x,y,z)+\frac{p(\beta-\alpha)B(a-1,e-a-1)}{B(a,e-a)}\\
&\quad\times F_{D,p}^{(3;\,\alpha,\,\beta+1)}(a-1,b,c,d;e-2;x,y,z)=0.
\end{aligned}
$$

From (5.7), we have

$$
\begin{aligned}
&(\alpha-\beta+1)(\beta-\alpha)F_1^{(\alpha-1,\,\beta)}+\alpha(\alpha-1)F_1^{(\alpha+1,\,\beta)}+(\beta-1)(\beta-2\alpha)\\
&\quad F_1^{(\alpha,\,\beta-1)}-\frac{p\alpha B(a-1,d-a-1)}{B(a,d-a)}F_1^{(\alpha+1,\,\beta)}(a-1,b,c;d-2;x,y;p)\\
&\quad+\frac{p(\beta-1)B(a-1,d-a-1)}{B(a,d-a)}F_1^{(\alpha,\,\beta-1)}(a-1,b,c;d-2;x,y;p)=0,\\
&(\alpha-\beta+1)(\beta-\alpha)F_2^{(\alpha-1,\,\beta,\,\alpha',\,\beta')}+\alpha(\alpha-1)\\
&\quad\times F_2^{(\alpha+1,\,\beta,\,\alpha',\,\beta')}+(\beta-1)(\beta-2\alpha)F_2^{(\alpha,\,\beta-1,\,\alpha',\,\beta')}\\
&\quad-\frac{p\alpha B(b-1,d-b-1)}{B(b,d-b)}F_2^{(\alpha+1,\,\beta,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p)\\
&\quad+\frac{p(\beta-1)B(b-1,d-b-1)}{B(b,d-b)}\\
&\quad\times F_2^{(\alpha,\,\beta-1,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p)=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\alpha-\beta+1)(\beta-\alpha)F_{D,p}^{(3;\,\alpha-1,\,\beta)}+\alpha(\alpha-1)F_{D,p}^{(3;\,\alpha+1,\,\beta)}\\
&\quad+(\beta-1)(\beta-2\alpha)F_{D,p}^{(3;\,\alpha,\,\beta-1)}\\
&\quad-\frac{p\alpha B(a-1,e-a-1)}{B(a,e-a)}F_{D,p}^{(3;\,\alpha+1,\,\beta)}(a-1,b,c,d;e-2;x,y,z)\\
&\quad+\frac{p(\beta-1)B(a-1,e-a-1)}{B(a,e-a)}
\end{aligned}
$$

$$\times F_{D,p}^{(3;\,\alpha,\,\beta-1)}(a-1,b,c,d;e-2;x,y,z)=0.$$

By appealing to (5.8), the next three recurrence relations can be derived:

$$\begin{aligned}
&\alpha\beta(\beta-1)F_1^{(\alpha+1,\,\beta)}+\alpha\beta(1-\beta)F_1^{(\alpha,\,\beta-1)}\\
&\quad-\frac{p\alpha\beta B(a-1,d-a-1)}{B(a,d-a)}F_1^{(\alpha+1,\,\beta)}(a-1,b,c;d-2;x,y;p)\\
&\quad-\frac{p(1-\beta)\beta B(a-1,d-a-1)}{B(a,d-a)}F_1^{(\alpha,\,\beta-1)}(a-1,b,c;d-2;x,y;p)\\
&\quad-\frac{p(\alpha-\beta)(\alpha-\beta+1)B(a-1,d-a-1)}{B(a,d-a)}\\
&\quad\times F_1^{(\alpha,\,\beta+1)}(a-1,b,c;d-2;x,y;p)=0\,,\\
&\alpha\beta(\beta-1)F_2^{(\alpha+1,\,\beta,\,\alpha',\,\beta')}+\alpha\beta(1-\beta)F_2^{(\alpha,\,\beta-1,\,\alpha',\,\beta')}\\
&\quad-\frac{p\alpha\beta B(b-1,d-b-1)}{B(b,d-b)}F_2^{(\alpha+1,\,\beta,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p)\\
&\quad-\frac{p(1-\beta)\beta B(b-1,d-b-1)}{B(b,d-b)}\\
&\quad\times F_2^{(\alpha,\,\beta-1,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p)\\
&\quad-\frac{p(\alpha-\beta)(\alpha-\beta+1)B(b-1,d-b-1)}{B(b,d-b)}\\
&\quad\times F_2^{(\alpha,\,\beta+1,\,\alpha',\,\beta')}(a,b-1,c;d-2,e;x,y;p)=0,
\end{aligned}$$

and

$$\begin{aligned}
&\alpha\beta(\beta-1)F_{D,p}^{(3;\,\alpha+1,\,\beta)}+\alpha\beta(1-\beta)F_{D,p}^{(3;\,\alpha,\,\beta-1)}\\
&\quad-\frac{p\alpha\beta B(a-1,e-a-1)}{B(a,e-a)}F_{D,p}^{(3;\,\alpha+1,\,\beta)}(a-1,b,c,d;e-2;x,y,z)\\
&\quad-\frac{p(1-\beta)\beta B(a-1,e-a-1)}{B(a,e-a)}F_{D,p}^{(3;\,\alpha,\,\beta-1)}(a-1,b,c,d;e-2;x,y,z)\\
&\quad-\frac{p(\alpha-\beta)(\alpha-\beta+1)B(a-1,e-a-1)}{B(a,e-a)}\\
&\quad\times F_{D,p}^{(3;\,\alpha,\,\beta+1)}(a-1,b,c,d;e-2;x,y,z)=0.
\end{aligned}$$

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School of Science, Dalian Nationalities University, Dalian 116600, P.R. China
Email address: **liuhm7911@163.com**