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ON TRANSLATIVE COVERINGS OF CONVEX BODIES

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ABSTRACT. We introduce and study t-coverings in E^n, i.e., arrangements of proper translates of a convex body $K \subset E^n$ sufficient to cover K. First, we investigate relations between t-coverings of the whole of K and t-coverings of its boundary only. Refining the notion of t-covering in several ways, we then derive, particularly for centrally symmetric convex bodies and $n = 2$, theorems which are interesting for the geometry of normed planes. These statements are related to respective generalizations of Ţiţeica's and Miquel's theorem as well as to notions like Voronoi regions. We also compare t-coverings with coverings in the spirit of Hadwiger, using smaller homothetical copies of K instead of proper translates. This is done via a slight modification of Boltyanski's and Hadwiger's notion of illumination. Finally, we give upper bounds on the cardinalities of t-coverings.

1. Introduction. There is a large variety of covering problems in the spirit of discrete and combinatorial geometry interesting also for applied disciplines. One of the most famous and still unsettled covering problems of such a type was posed by Hadwiger: how many smaller homothetical copies of a convex body $K \subset E^n$ are needed to cover K? There are many papers and partial results about this problem (see the survey in [**5,** Chapter VI]). It is surprising that only a few results are known on the following related covering problem: How many *proper translates* of K are sufficient to cover K itself? Such a covering of K by proper translates of it is called a *translative covering* or, in short, *t-covering of K*. We should mention that the notion of translative covering already occurs in the literature but with different motivations and meanings; see, e.g., [**7, 10**].

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This notion yields, as we will show, interesting research problems and variations of problems from the mathematical literature. For example, refinements of such coverings have applications in the geometry of normed planes, and the strongly related Boltyanski-Hadwiger notion of illumination (see [**3, 14**]) is used in [**3, 4, 15**] and in many other papers as a method of attacking Hadwiger's covering problem; for a survey, again see [**5,** Chapter VI].

First we study the relation between t-coverings of a convex body $K \subset E^n$ and the coverings of its only boundary by proper translates of K. For $n = 2$ and K being centrally symmetric, these relations lead us in a natural way to new results on special, in a sense optimal t-coverings of discs and circles in normed planes, which correspond to basic theorems on circle arrangements (namely, to generalizations of Ţiţeica's and Miquel's theorem) and to notions like Voronoi regions for such planes. Here the case of strictly convex normed planes plays an essential role. Introducing the notion of t-illumination and comparing it with the Boltyanski-Hadwiger notion of h-illumination, we also clarify how t-coverings are related to "h-coverings," i.e., to coverings by smaller homothetical copies in the sense of Hadwiger. (Note that, seemingly closer to t-coverings, Levi [**18**] investigated coverings of convex bodies by the interiors of proper translates. However, it turns out that Levi's coverings are equivalent to h-coverings.) As we will see, already the comparison of h- and t-coverings yields interesting problems and results. Finally, we give upper bounds on t-covering numbers by completely clarifying the planar situation, using partial results on h-covering numbers in higher dimensions, and also showing how various notions from discrete geometry (like antipodality) are related to this framework.

Let $K \subset E^n$ denote a *convex body*, i.e., a compact, convex set with non-empty interior in E^n. We write $\mathrm{bd}\, K$ and $\mathrm{int}\, K$ for the *boundary* and *interior* of K, respectively. In addition, we use aff, conv, int and relint for *affine hull, convex hull, interior* and *relative interior*, and o denotes the *origin*. We write $h(K)$ for the *h-covering number of* K, i.e., the minimal number of smaller homothetical copies of K sufficient to cover K. Sharp upper bounds on $h(K)$ are unknown for $n \geq 3$. A family of proper translates of a convex body $K \subset E^n$ covering K itself is said to be a *t-covering* of that body. We also consider coverings of $\mathrm{bd}\, K$ by proper translates of K. We write $t(K)$ for the smallest number of proper translates that are sufficient to cover K, and it is natural to call $t(K)$ the *t-covering number of K*. Only a few results on t-coverings are known; see [**1, 11**].

A collection $\{K_i\}_{i=1}^m$ of finitely many convex bodies is called a *non-reducible covering* of a set M if $M \subset \bigcup_{i=1}^m K_i$ and if no proper subfamily of $\{K_i\}_{i=1}^m$ exists with the same property.

2. Covering the boundary by proper translates of the body.

Proposition 2.1. *If a family of m proper translates of a convex body $K \subset E^n$ covers the boundary of K, then there exists a family of m proper translates of K which covers K.*

Proof. Assume that $\operatorname{bd} K$ is covered by proper translates $K_1, \ldots, K_m$ of K. Of course, $K_i = K + v_i$, where v_i is a non-zero vector, for $i = 1, \ldots, m$. Let $x \in \operatorname{int} K$. Observe that there exists a real λ with $1 \geq \lambda > 0$ such that every set $K_i' = K + \lambda v_i$ contains x. For every $y \in K \cap K_i$, by the convexity of K and by the description of K_i', we conclude that $y \in K_i'$, which implies that $K \cap K_i \subset K_i'$.

For every $p \in K \setminus \{x\}$, take the intersection point p_x of $\operatorname{bd} K$ with the ray from x through p. By our assumption, $p_x \in K \cap K_i$ for an $i \in \{1, \ldots, m\}$. Thus, by the conclusion of the preceding paragraph, $p_x \in K_i'$. Since also $x \in K_i'$, by the convexity of K_i', we obtain that $p \in K_i'$. We see that K is covered by the translates $K_1', \ldots, K_m'$ of K. $\square$

The above proof is similar to the consideration from the paper [**16,** pages 271–272].

In [**1,** Remark 7], Asplund and Grünbaum conjectured that, in our terms, for a centrally symmetric convex body $K \subset E^n$, the following implication holds: if $\operatorname{bd} K$ is covered by $n+1$ proper translates of K, then K is covered by these translates. (For $n = 2$, they give a proof of this in [**1**]; see below.) The following example shows that this implication does not hold if the assumption of central symmetry is deleted.

Example 2.2. The boundary of every non-degenerate n-simplex $S \subset E^n$ may be covered by $n+1$ proper translates of S which do not cover S. Since we may apply an affine transformation, consider only the regular

n-simplex of height 1. As the promised $n+1$ proper translates we take translates by $(n-1)/n$ units in the directions from the centroid of S to its vertices. Then the boundary of S is covered, but the centroid of S is not.

As already mentioned, Asplund and Grünbaum proved that if K is a centrally symmetric convex body in E^2 and $\operatorname{bd} K$ is covered by three proper translates of K, then K itself is completely covered by these three translates; see [**1,** Theorem 8]. Inspired by this, we investigate in this section only t-coverings of planar, *centrally symmetric* convex bodies in the spirit of the geometry of finite-dimensional Banach spaces, also called *Minkowski geometry*. More precisely, if $K \subset E^2$ has a center of symmetry, we interpret it as the *unit disc* (and its boundary as the *unit circle*) of a normed plane and write B instead of K and C instead of $\operatorname{bd} K$. (For the geometry of normed planes and spaces we refer to the monograph [**26**] and to the survey [**22**].) Further on, speaking in the sequel about *discs* and *circles*, we mean homothetical copies of B and C, respectively. We say that a normed plane is *strictly convex* if C does not contain a non-degenerate line segment.

Remark 2.1. Two circles in a strictly convex normed plane have at most two points in common; see, e.g., [**22,** Proposition 14]. Let C_1 and C_2 be two intersecting circles of the same radius. If $C_1 \cap C_2 = \{p, q\}$ (it is possible that $p = q$), then p, q and the centers x_1, x_2 of C_1 and C_2, respectively, form a Minkowskian rhombus, i.e., a quadrangle whose sides are of the same lengths. The fact that any Minkowskian rhombus in a strictly convex normed plane is a parallelogram ([**22,** Proposition 12]) implies the equality $x_1 + x_2 = p + q$.

We define the *multiplicity of the covering of the boundary* $\operatorname{bd} K$ *of a convex body K by the interiors of convex bodies* $K_1, \dots, K_m$ as the largest number k such that every point of $\operatorname{bd} K$ belongs to at most k from amongst the sets $\operatorname{int} K_1, \dots, \operatorname{int} K_m$ (see Figures 1 and 2 for examples of coverings of multiplicity 2 and multiplicity 1, respectively). A similar notion, the multiplicity of a covering of the space by balls, is treated in [**7, 10**].

Let B be a centrally symmetric convex body, and let a family $\mathcal{B} = \{B_1, \dots, B_m\}$ consist of proper translates which cover the boundary C of B. In what follows, all coverings that we take into account consist of translates of B.

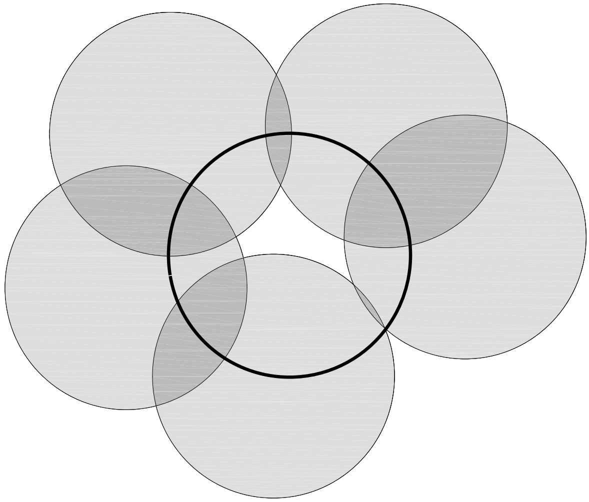

FIGURE 1. Covering the boundary of a circular disc by proper translates whose multiplicity is 2.

If B is strictly convex and $\mathcal{B}$ is a covering of C of multiplicity 1, then any point of C belongs either to the interior of exactly one element of $\mathcal{B}$ or to the boundaries of exactly two elements of $\mathcal{B}$. Indeed, if a point $x \in C$ belongs to $\operatorname{bd} B_i$ and $\operatorname{bd} B_j$ and there is a translate B_k with $B_k \ni x$, then, e.g., by [**22,** Proposition 22], an arc of C with endpoint x lies in B_k. The strict convexity of B implies that only the endpoints of this arc belong to $\operatorname{bd} B_k$. But a part of this arc also belongs to the interior of either B_i or B_j, which contradicts the multiplicity 1. Due to this fact there exist exactly m points $p_1, \ldots, p_m$ such that $p_i \in \operatorname{bd} B_i \cap \operatorname{bd} B_{i+1}$ for $i = 1, \ldots, m$, where $B_{m+1} = B_1$. We call these points the *skeleton* of the covering $\mathcal{B}$. We mention two coverings of C which have multiplicity 1. For $n = 2$ and $m = 3$, the boundaries of B_i intersect in exactly one point. This statement is known as *Ţiţeica's theorem*, and in this form it was proved by Asplund and Grünbaum in [**1**] (see also [**21**]). For the case $n = 2$ and $m = 4$, let $p_1, \ldots, p_4$ be the skeleton points of $\mathcal{B}$ such that $p_i \in \operatorname{bd} B_i \cap \operatorname{bd} B_{i+1}$. Then the second intersection points of $\operatorname{bd} B_i$ and $\operatorname{bd} B_{i+1}$ (it is also possible that such a point coincides with p_i) lie on a circle C^* of radius 1. This is *Miquel's theorem*, in this form also proved in [**1**] (see also [**25**]). The disc with the boundary C^* is said to be the *Miquel disc* of the covering $\mathcal{B}$. For

Ţiţeica's and Miquel's theorem see Figure 2.

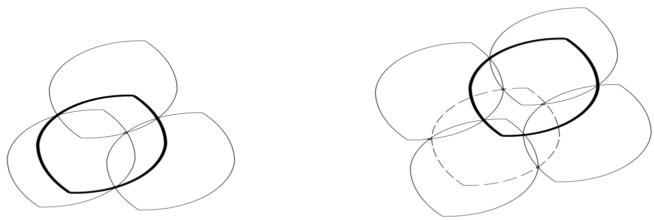

FIGURE 2. Covering the boundary of a centrally symmetric body by 3 translates (Ţiţeica's theorem) and by 4 translates (Miquel's theorem), both of multiplicity 1.

Theorem 8 in [**1**] says that a covering of the boundary C of B consisting of three translates of B is always a covering of B. In Theorem 2.1 we prove that a covering $\{B_1, \dots, B_m\}$ of C with multiplicity 1 does not cover B for $m > 3$.

Let p_1 and p_2 lie on a circle C with center x. These two points determine two arcs. That one which does not lie in the half plane bounded by the line through p_1 and p_2 and containing x is called the *smaller arc* of C with endpoints p_1 and p_2 and denoted by $\mathrm{arc}\,(p_1, p_2; c)$.

Remark 2.2. Any translate B_i covers the smaller arc of C determined by p_i and p_{i+1} and does not cover the larger one. This fact follows, e.g., from [**22,** Proposition 22]. It implies that proper translates $B_1, \dots, B_m$ of B form a covering of the boundary of B of multiplicity 1 if and only if $\mathrm{conv}\,\{p_1, \dots, p_m\}$ contains the center of B.

Lemma 2.1. *In a strictly convex normed plane with unit circle C centered at the origin o, let there be given three points p_1, p_2, p_3 on C. Let C_1 be a translate of C passing through p_1 and p_2, and let C_2 be a translate of C passing through p_2 and p_3. Let $q_2 \in C_1 \cap C_2$. Then we have:*

(i) *If $o \notin \mathrm{conv}\,\{p_1, p_2, p_3\}$ and p_2 belong to the half-plane bounded by the line through p_1 and p_3 and not containing o, then $\|q_2\| > 1$.*

(ii) *If* $o \in [p_1, p_3]$, *then* $q_2 = p_2$.
(iii) *If* $o \in conv\{p_1, p_2, p_3\}$, *then* $\|q_2\| < 1$.

Proof. Let us fix p_1 and p_2, and let p_3 move along the semicircle A of C with endpoints $-p_2$ and p_2 that do not contain p_1. Thus, if p_3 belongs to the arc of C with endpoints $-p_2$ and $-p_1$, then the origin $o \in \operatorname{conv}\{p_1, p_2, p_3\}$, i.e., we have (iii) from Lemma 4.1. If $p_3 = -p_2$, then we have (ii) from above, and when p_3 runs from $-p_1$ to p_2, we have (i) from above. One can see that, if p_3 moves along A, the locus of the midpoints of the segments $[p_3, p_2]$ is a semicircle of the circle C_1 with center $(1/2)p_2$ and radius $1/2$. More precisely, the semicircle A_1 of C_1 that lies in the half plane bounded by the line through o and p_2 does not contain p_1. Then the locus of the centroids of the triangles $p_1p_2p_3$ is the semicircle A_2 of the circle with center $(1/3)(p_1 + p_2)$ and of radius $1/3$, which is the image of A_1 under the homothety φ with fixed point p_1 and ratio $2/3$. Note that the endpoints of A_2 are $(1/3)p_1$ and $(1/3)p_1 + (2/3)p_2$. Moreover, if $p_3 = -p_1$, then $\varphi((1/2)(p_2 + p_3)) = (1/3)p_2$. In other words, if p_3 moves from $-p_2$ to $-p_1$, then the centroid $(1/3)(p_1 + p_2 + p_3)$ moves from $(1/3)p_1$ to $(1/3)p_2$ through the part A_2' of A_2. If p_3 moves from $-p_1$ to p_2, then the point $(1/3)(p_1 + p_2 + p_3)$ moves from $(1/3)p_2$ to $(1/3)p_1 + (2/3)p_2$. Thus, again applying [**22,** Proposition 22], we get that the only part of A_2 which belongs to $(1/3)B$ is A_2'. If y_i $(i = 1, 2)$ is the center of C_i, then $y_1 = p_1 + p_2$ and $y_2 = p_2 + p_3$; see Remark 2.1. Hence, again by Remark 2.1, we obtain $q_2 = y_1 + y_2 - p_2 = p_1 + p_2 + p_3$ which completes the proof. □

Theorem 2.1. *Let* $B \subset E^2$ *be a centrally symmetric, strictly convex body, and let* $\mathcal{B}$ *be a family of proper translates of it. Assume that* $\mathcal{B}$ *forms a non-reducible covering of* $\operatorname{bd} B$ *whose multiplicity is* 1, *and that* $\mathcal{B}$ *is a covering of* B. *We claim that* $\mathcal{B}$ *consists of exactly three translates.*

Proof. Let $p_1, \dots, p_m$ be the skeleton of the covering $\mathcal{B} = \{B_1, \dots, B_m\}$. Assume that $\cup_{i=1}^m B_i$ covers B. Due to the strict convexity of B, we have that $m > 2$. Consider the pair B_i and B_{i+1}. Then $C_i = \operatorname{bd} B_i$ and $C_{i+1} = \operatorname{bd} B_{i+1}$ have two intersection points (it is also possible that they coincide), and one of them is p_i. Denote the second one by q_i. If there exists $i \in \{1, \dots, m\}$ such that $\|q_i\| \geq 1$, then $\cup_{i=1}^m B_i$ does not cover B. Thus we get that, for all $i \in \{1, \dots, m\}$, the point q_i belongs to $\operatorname{int} B$. Then Lemma 4.1 implies that $o \in \operatorname{int} \operatorname{conv}\{p_1, p_2, p_3\}$. But the same lemma also implies that the origin has to belong to $\operatorname{int} \operatorname{conv}\{p_1, p_m, p_{m-1}\}$. This is only possible for $m = 3$. $\square$

Let B be a centrally symmetric convex body. Let $\mathcal{B} = \{B_1, \dots, B_m\}$ be a covering of its boundary by translates of B whose multiplicity is 1. As our Theorem 2.1 shows, for $m \geq 4$, this is not a covering of B. The set of points from B which do not belong to $\cup_{i=1}^m B_i$ is called the *gray area of $\mathcal{B}$ for B*. If the points x_i, $i \in \{1, \dots, m\}$, are the centers of B_i, then the set of points whose distance to the origin does not exceed the distance to any point x_i is said to be the *Voronoi region* of $\mathcal{B}$. It is now our aim to investigate the gray area of a covering of the boundary of B by translates of B. Note that the notions of gray area and Voronoi region of the covering were introduced in [**2**], but only for a covering of the plane by Euclidean discs.

Theorem 2.2. *Let $B \subset E^2$ be a centrally symmetric strictly convex body, and let $\mathcal{B}$ be a family of proper translates of B. Assume that $\mathcal{B}$ forms a covering of* $\operatorname{bd} B$ *whose multiplicity is* 1. *Then we have $G \subset V \subset B$, where G and V denote the gray area and the Voronoi region of $\mathcal{B}$ for B, respectively. Moreover, if $m = 4$, then G is contained in the Miquel disc of the covering $\mathcal{B}$.*

Proof. Let $\beta(x, y)$ be the bisector of different points x and y, i.e., $\beta(x, y) := \{p : \|x - p\| = \|y - p\|\}$. Remember that, for strictly convex norms, any bisector is an unbounded curve without points of self-intersection. Note also that, if $z \in \beta(x, y)$, then $\beta(x, y)$ belongs to the double cone spanned by x and y with apex z; see, e.g., [**22,** Proposition 17]. Let $p_1, \dots, p_m$ be the skeleton of $\mathcal{B}$, and let x_i be the center of B_i. Denote by β_i the part of $\beta(0, x_i)$ between p_i and p_{i+1}. Since $o \in \operatorname{conv}\{p_1, \dots, p_m\}$, the statement cited above implies that all β_i form a curvilinear polygon with vertices $p_1, \dots, p_m$ (the sides of the polygon intersect only in their endpoints). Then this polygon is the boundary of the Voronoi region V of $\mathcal{B}$. For any three non-collinear points x, y, z and a point $u \in \operatorname{conv}\{x, y, z\}$, the inequality $\|x - z\| + \|z - y\| > \|x - u\| + \|u - y\|$ holds; see, e.g., [**22,** Corollary 28]. Therefore $\beta_i \in B \cap B_i$, yielding $G \subset V \subset B$. For the rest of the proof, denote the second intersection points of $\operatorname{bd} B_i$ and $\operatorname{bd} B_{i+1}$ by q_i

(the first one is p_i). Then q_i lies on the boundary of the Miquel disc of the covering $\mathcal{B}$. According to Lemma 4.1, exactly two of the points q_i, $i = 1, \dots, 4$, belong to the interior of B, say q_1 and q_2. Then the gray area (it can be connected or disconnected) is determined by the arcs $\mathrm{arc}\,(q_1, q_2; x_1)$, $\mathrm{arc}\,(q_2, p_3; x_2)$, $\mathrm{arc}\,(p_3, p_4; x_3)$ and $\mathrm{arc}\,(p_4, q_1; x_4)$. Since the arcs $\mathrm{arc}\,(q_1, q_2; x_1)$, $\mathrm{arc}\,(q_2, q_3; x_2)$, $\mathrm{arc}\,(q_2, q_3; x_3)$ and $\mathrm{arc}\,(q_3, q_4; x_4)$ belong to M, it follows that $G \subset M$. □

Corollary 2.1. *If $\mathcal{B} = \{B_1, \dots, B_4\}$ is a 1-multiplicity t-covering of the boundary of a centrally symmetric, strictly convex body B in the plane, and M is the Miquel disc of $\mathcal{B}$, then $\cup_{i=1}^4 B_i \cup M$ is a covering of B.*

3. On translative coverings in terms of illumination. A suitable notion of illumination, introduced in **[3, 14]**, permits the expression of the h-covering problem of Hadwiger in terms of illumination. Below we introduce an illumination type somehow related to t-coverings. Based on this, it is easy to observe new results which are certainly stimulating for further research on t-coverings.

We say that a boundary point x of a convex body $K \subset E^n$ *is t-illuminated by a direction δ* if there exists a different point $y \in K$ such that the vector $\vec{xy}$ has direction δ. And we note that this definition still makes sense if the word "boundary" is omitted. The related illumination of the boundary of $K \subset E^n$ introduced in **[3, 14]**, referring to Hadwiger's covering problem, and the number $h(K)$ are defined as follows: A boundary point x of K is *h-illuminated by a direction δ* if there is some interior point y of K such that the vector $\vec{xy}$ has direction δ. The comparison of both definitions shows that differences in the illumination of boundary parts of K occur only in one situation, namely, when K has non-degenerate segments in its boundary which are parallel to the illumination direction. More precisely, if the direction δ is parallel to a non-degenerate segment $I \subset \mathrm{bd}\, K$, say, then all $x \in I$ are not h-illuminated, but all of them, except for one of the two endpoints of I, are t-illuminated. For all other boundary points of K, both illumination (or covering) types are equivalent. Various further types of illumination and visibility, discussed in the expository paper **[20]**, might also be compared with t-illumination.

For the following theorem, we denote by $i(K)$ the smallest number of directions sufficient to t-illuminate the whole of $\operatorname{bd} K$.

Theorem 3.1. *For every convex body $K \subset E^n$ we have $i(K) \leq t(K)$.*

Proof. Let $K_1, \dots, K_m$ be proper translates of K satisfying $K \subset \cup_{i=1}^m \operatorname{int} K_i \subset \cup_{i=1}^m K_i$, and denote by $v_i \neq o$ the translation vector defined by $K + v_i = K_i$, $i = 1, \dots, m$. (By (2) below in Section 4, m is finite.) For $x \in \operatorname{int} K_i \cap \operatorname{bd} K$, we have $x + v_i \in \operatorname{int} K$, and therefore $x \in \operatorname{bd} K$ is t-illuminated by $-v_i$. Thus, the whole of $\operatorname{int} K_i \cap \operatorname{bd} K$ is t-illuminated by $-v_i$, and the vector system $\{-v_1, \dots, -v_m\}$ illuminates the whole of $\operatorname{bd} K = \cup_{i=1}^m (\operatorname{int} K_i \cap \operatorname{bd} K) = \cup_{i=1}^m (K_i \cap \operatorname{bd} K)$. Hence, $i(K) \leq t(K)$. □

The proof of the next theorem is done by modifying the proof of Theorem 34.3 in [**5**].

Theorem 3.2. *Let $K \in E^n$ be a convex body, and let $\delta_1, \dots, \delta_m$ be directions such that the subsets of* $\operatorname{bd} K$ *t-illuminated by them are open in* $\operatorname{bd} K$ *and that the union of these subsets is* $\operatorname{bd} K$. *Then there exist non-zero vectors $w_1, \dots, w_m$ opposite to $\delta_1, \dots, \delta_m$, respectively, for which K is covered by the translates $\{K + w_i\}$, $i = 1, \dots, m$.*

Proof. Denote by W_i the set of boundary points of K t-illuminated by δ_i, $i = 1, \dots, m$. We will show the existence of open sets $V_i, \dots, V_m \subset \operatorname{bd} K$ satisfying

$$\text{(1)} \qquad \operatorname{cl} V_i \subset W_i \quad (i = 1, \dots, m) \text{ and } \bigcup_{i=1}^{m} V_i = \operatorname{bd} K,$$

using induction over $k \in \{1, \dots, m\}$. Assume that, for any such k, we have sets $V_1, \dots, V_{k-1}$ with $\operatorname{cl} V_i \subset W_i$ $(i = 1, \dots, k-1)$ and $V_1 \cup \dots \cup V_{k-1} \cup W_k \cup \dots \cup W_m = \operatorname{bd} K$ (the case $k = 1$ is trivial). In order to construct V_k, we consider the sets

$$F_k = \operatorname{bd} K \setminus (V_1 \cup \dots \cup V_{k-1} \cup W_{k+1} \cup \dots \cup W_m),$$

and $H_k = \operatorname{bd} K \setminus W_k$. Since F_k and H_k are closed and disjoint, we may consider their distance $h_k = \min\{\|x - y\| : x \in F_k, y \in H_k\}$ and choose some positive $\varepsilon < h_k$ with setting $V_k = U_\varepsilon(F_k) \cap \operatorname{bd} K$, where U_ε denotes the ε-environment. Then, V_k is open in $\operatorname{bd} K$ with $\operatorname{cl} V_k \cap H_k = \emptyset$. Thus,

$$\operatorname{cl} V_k \subset W_k; \quad V_1 \cup \cdots \cup V_k \cup W_k \cup \cdots \cup W_m = \operatorname{bd} K,$$

confirming the existence of sets $V_1, \dots, V_m$ which satisfy (1).

For each $x \in W_i$, let $l_i(x)$ be the ray with starting point x in direction δ_i. Since x is t-illuminated by $-\delta_i$, $l_i(x) \setminus \{x\} \cap K \neq \emptyset$, and the point of this set having largest distance to x is denoted by $y_i \neq x$. Thus, the segment $[x, y_i]$ has length $f_i(x) > 0$, yielding a positive function on W_i, also continuous by the convexity of K. The compactness of $\operatorname{cl} V_i \subset W_i$ implies that there is some $q_i > 0$ such that, for all $x \in \operatorname{cl} V_i$, the relation $f_i(x) > q_i$ holds. Thus, denoting the translation via $-\delta_i$, with $\|v_i\| = q_i$, by π_i, we have $\pi_i(\operatorname{cl} V_i) \subset K$ and also $K_i := \pi_i^{-1}(K) \supset \operatorname{cl} V_i$ for $i = 1, \dots, m$. Now let y_0 be an arbitrary interior point of K. Then q_i from above can be chosen sufficiently small such that $y_0 \in \pi_i^{-1}(K) = K_i$ for all $i \in \{1, \dots, m\}$. Thus, the closed set $\operatorname{cl} V_i \cup \{y_0\}$ is contained in K_i. To show $K \subset \cup_{i=1}^m K_i$, we choose for any $z \in K$ a boundary point x of K such that z belongs to the segment $[x, y_0]$. We choose some $i \in \{1, \dots, m\}$ such that $x \in \operatorname{cl} V_i$ (note that $\cup_{i=1}^m V_i = \operatorname{bd} K$), and since y_0 and $x \in \operatorname{cl} V_i$ lie in the convex set $K_i = \pi_i^{-1}(K)$, we get $[x, y_0] \subset K_i$, yielding $z \in \cup_{i=1}^m K_i$, i.e., $K \subset \cup_{i=1}^m K_i$. $\square$

Corollary 3.1. *Let $K \subset E^n$ be a convex body, and assume that a system of a minimum number of directions that t-illuminates* $\operatorname{bd} K$ *has the property that the subsets of* $\operatorname{bd} K$ *t-illuminated by them are open in* $\operatorname{bd} K$. *Then $t(K) = i(K)$.*

Corollary 3.2. *Let $K \subset E^n$ be a strictly convex body. Then $i(K) = t(K)$.*

In the planar situation, we get even more.

Corollary 3.3. *Let $K \subset E^2$ be a convex body. Then $i(K) = t(K)$.*

Proof. We have $t(K) = 2$ if and only if $\operatorname{bd} K$ contains two parallel segments (see Proposition 5.2 below), and it is obvious that only in this case also $i(K) = 2$. By Proposition 4.3 below, we have $h(K) = t(K)$ if and only if $\operatorname{bd} K$ does not contain parallel segments. We also have $h(K) \geq 3$ (see **[5]**) and, obviously, $h(K) \geq t(K) \geq i(K)$ as well as

$4 > h(K)$ for all non-parallelograms (which satisfy $t(K) = 2$). Thus, $t(K) = i(K) = 3$ for all convex bodies $K \subset E^2$ without parallel boundary segments. $\square$

Unfortunately, for $n \geq 3$, in general the equality $i(K) = t(K)$ does not hold. We wish to thank Christian Richter (FSU Jena) for bringing the following counterexample to our attention.

Let $K = \operatorname{conv}(S \cup (0, \ldots, 0, 1) \cup (0, \ldots, 0, -1))$ with $S = \{(x_1, \ldots, x_{n-1}, 0) : x_1^2 + \cdots + x_{n-1}^2 = 1\}$ be a compact double cone over an $(n-1)$-sphere S in E^n, $n \geq 3$. Then $\operatorname{bd} K$ is t-illuminated by $n+1$ directions, namely: the vector $(1, 0, \ldots, 0, 1)$ t-illuminates (also) all $x \in \operatorname{bd} K$ with $x_n < 0$ as well as $(-1, 0, \ldots, 0)$ (the upper apex lies beyond it), and the vector $(-1, 0, \ldots, 0, -1)$ t-illuminates analogously all $x \in \operatorname{bd} K$ with $x_n > 0$ as well as $(1, 0, \ldots, 0)$.

For the t-illumination of $S \setminus \{(1, 0, \ldots, 0), (-1, 0, \ldots, 0)\}$ we consider a regular simplex with centroid $(0, \ldots, 0)$ and vertices $v_1, \ldots, v_{n-1}$ which is embedded in the equatorial $(n-2)$-plane $x_1 = x_n = 0$. The vectors $v_1, \ldots, v_{n-1}$ t-illuminate the equator and therefore S, except for the poles $(1, 0, \ldots, 0), (-1, 0, \ldots, 0)$ which are already t-illuminated. Thus, we get $i(K) \leq 2 + (n-1) = n+1$.

For t-covering $(0, \ldots, 0, 1)$ we need a translate $K + v$, where the nth coordinate v_n of v is positive and the slope angle between v and $\operatorname{aff} S$ is $\geq 45°$. Then $(K + v) \cap \operatorname{aff} S$ is an $(n-1)$-ball of radius $1 - v_n < 1$ (degenerate for $v_n \geq 1$), which is completely contained in $\operatorname{conv} S$. This $(n-1)$-ball intersects S in at most one point, i.e., $K + v$ covers at most one point from S. Analogously, a second translate of K is needed to cover $(0, \ldots, 0, 1)$, and this intersects S again in at most one point. The remaining needed translates of K have to cover S, except for two points of S. Since they are closed, they cover S completely even. If we would have only $\leq n-1$ translate of K for this, then it would have to cover a pair of diametrical points of S (Borsuk-Ulam theorem). This is only possible with the translation vector $(0, \ldots, 0)$, a contradiction. Thus, at least n translates are needed for covering S, and therefore $t(K) \geq n+2$.

One can easily prove various further theorems on t-illumination. Here is an example. If x denotes a non-extreme boundary point of a convex body $K \subset E^n$ which is not strictly convex, then x is from the relative interior of some boundary segment yz with z as extreme point of C. If z is t-illuminated by some direction v, then there is some point $z+\lambda v \in K, \lambda > 0$. Since then also the triangle with vertices $y, z, \lambda v$ is contained in K, there is some $x + \nu v, 0 < \nu < \lambda$, also belonging to K, and so x is also t-illuminated by v. Therefore, we have

Proposition 3.1. *The boundary of a convex body $K \subset E^n$ is completely t-illuminated by a system V of directions if and only if the set of extreme points of K is t-illuminated by V.*

4. Comparison of two covering numbers and some consequences. Since any smaller homothetical copy of a convex body K is contained in a translate of K, we obviously have:

(2) $$t(K) \leq h(K) \quad \text{for every convex body } K \subset E^n.$$

Lemma 4.1. *Let $K \subset E^n$ be a convex body, let v be a vector and let $0 \leq \lambda \leq 1$. We have $K \cap (K+v) \subset K + \lambda v$. Moreover, if K is strictly convex and $0 < \lambda < 1$, then $K \cap (K+v) \subset \operatorname{int}(K+\lambda v)$.*

Proof. Take any $x \in K \cap (K+v)$. Since $x \in (K+v)$, there exists a point $z \in K$ such that $x = z+v$. By the convexity of K, the segment $zx = z(z+v)$ is contained in K. Put $y = z + \lambda v$. From $z \in K$, we obtain $y \in (K+\lambda v)$. Since $z(z+v) \subset K$, we have $y(y+v) \subset (K+\lambda v)$. This and $x \in y(y+v)$, which follows from $y = z+\lambda v$, $x = z+v$ and $y+v = z+v+\lambda v$, imply $x \in (K+\lambda v)$.

If, in addition, K is strictly convex and $0 < \lambda < 1$, then the chosen point $x \in K \cap (K+v)$ hast to be an interior point of $K + \lambda v$. □

Proposition 4.1. *For every strictly convex body $K \subset E^n$, we have $h(K) = t(K)$.*

Proof. Assume that K is covered by proper translates $K + v_k$ for $k = 1, \ldots, m$, where every v_k is a non-zero vector. Then K is covered by proper translates $K + (1/2)v_k$ for $k = 1, \ldots, m$. By Lemma 4.1 the set $P_k = K \cap (K + v_k)$ is contained in the interior of $K + (1/2)v_k$. Hence, every P_k is covered by a homothetical copy of $K + (1/2)v_k$ with a positive ratio smaller than 1. Thus, it is also covered by a homothetical copy of K with a positive ratio smaller than 1. We conclude that $h(K) \leq t(K)$. This together with (2) finishes the proof. $\square$

As a stronger statement, we even have the following:

Proposition 4.2. *If a convex body $K \subset E^n$ does not have parallel boundary segments, then $h(K) = t(K)$.*

Proof. Look at the proof of Proposition 4.1. Now it may happen that P_k is *not* contained in the interior of $K + (1/2)v_k$ (but still $P_k \subset K + (1/2)v_k$). Observe that there exists a non-zero vector w_k such that P_k is contained in the interior of $K + (1/2)v_k - w_k$. Such a vector w_k should be well chosen: if v_k is parallel to a boundary segment S of K, then we may take as w_k a sufficiently short vector with its initial point in S, directed to an interior point of K (we apply here the fact that the boundary of K does not contain a segment parallel to S). In the opposite case, instead of w_k, take the zero vector. $\square$

Thus, we get the following problem.

Problem 4.1. Characterize the class of convex bodies $K \subset E^n$ for which $h(K) = t(K)$.

For $n = 2$, this is solved by

Proposition 4.3. *Let $K \subset E^2$ be a convex body. We have $h(K) = t(K)$ if and only if the boundary of K does not contain parallel segments.*

Proof. By Proposition 4.2, we have $h(K) = t(K)$ provided C does not have a pair of boundary segments. If, on the other hand, the boundary of K contains a pair of parallel segments, by Proposition 5.2 below, we have $t(K) = 2$. On the other hand, $h(K) \geq 3$ (see, e.g., [**5**]). Consequently, $h(K) \neq t(K)$ for this case. $\square$

The statement of Proposition 4.3 does not hold true for $n \geq 3$. This follows from the example of the double cone $D \subset E^n$ whose base is an $(n-1)$-dimensional ball. It is easy to see that $h(D) = n+2 = t(D)$. On the other hand, the boundary of D contains a pair of parallel segments (in fact, infinitely many such pairs).

5. Bounds on translative covering numbers. In this section we give some upper bounds on minimum cardinalities of t-coverings.

Proposition 5.1. *For every strictly convex body $K \subset E^n$, where $n \geq 2$, we have*

$$t(K) \leq n^n - (n-2)^n.$$

Proof. Let P_1 be a parallelotope of maximum volume contained in K. Then a parallelotope P_2, being a homothetic copy of P_1 with ratio n, contains K (see **[17]**).

A particular case of Corollary 2 from **[16]** (when we take $p_1 = \cdots = p_n = n-1$ there) says the following. Assume that an n-dimensional parallelotope P is dissected into n^n equal, n-times smaller parallelotopes (being homothetical copies of P with ratio $1/n$) by n families of hyperplanes, each consisting of $n-1$ hyperplanes parallel to a successive pair of opposite facets of P. Then, for an arbitrary convex body $K \subset P$, there exists a family $\mathcal{F}$ of at most $n^n - (n-2)^n$ of the obtained n-times smaller parallelotopes which covers the boundary of K.

Taking into account both of these facts, where $P_2 = P$, we conclude that $\operatorname{bd} K$ is covered by a family $\mathcal{F}$ of at most $n^n - (n-2)^n$ translates of P_1.

If $P_1 \notin \mathcal{F}$, then $\mathcal{F}$ consists only of proper translates of P_1.

If $P_1 \in \mathcal{F}$, then we may omit P_1, and the remaining translates of P_1 from $\mathcal{F}$ still cover $\operatorname{bd} K$. Let us explain why. Observe that the strict convexity of K and $P_1 \subset K \subset P_2$ imply that P_1 has empty intersection with the boundary of P_2 and that $\operatorname{bd} P_1 \cap \operatorname{bd} K$ does not contain boundary points of P besides some vertices of P_1. Consequently, again from the strict convexity of K and since the union of parallelotopes from $\mathcal{F}$ covers $\operatorname{bd} K$, we conclude that each of these vertices is in at least one parallelotope from $\mathcal{F}$ different to P_1. So parallelotopes from $\mathcal{F}$ different to P_1 cover $\operatorname{bd} K$.

We see that $\operatorname{bd} K$ is always covered by at most $n^n - (n-2)^n$ proper translates of P_1. Since $P_1 \subset K$, we conclude that $\operatorname{bd} K$ is covered by at most $n^n - (n-2)^n$ proper translates of K. By Proposition 2.1, K is also covered by some $n^n - (n-2)^n$ proper translates of K. $\square$

Remark 5.1. The estimate given in this proposition is only slightly better than the estimate $(n+1)^n - (n-1)^n$ (concerning Hadwiger's number $h(K)$ for any convex body K) in [**16,** line 7, page 272], and also better than the estimate $(n+1)n^{n-1} - (n-1)(n-2)^{n-1}$ in Corollary 4 there. From Proposition 5.1 and Proposition 4.1, we immediately deduce also that $h(K) \leq n^n - (n-2)^n$ for every strictly convex body $K \subset E^n$.

Remark 5.2. Another upper bound on $h(K)$ which is interesting for our purpose is presented in [**6,** Theorem 9.15.1]; see also [**5,** Section 34]. It implies that, for an arbitrary convex body $K \subset \mathbf{R}^n$, $n \geq 2$, we have $t(K) \leq 5n\, 4^n \ln n$. Our bound in Proposition 5.1 is better only for $n \leq 8$.

Proposition 5.2. *Let $K \subset E^n$ be a convex body. We have $t(K) = 2$ if and only if there is a direction u such that the intersection of the boundary of K and any supporting line of K parallel to u is a segment of length at least ε, where $\varepsilon > 0$ (or, equivalently, K has a non-degenerate segment summand in the sense of Minkowski addition).*

Proof. Assume that $\operatorname{bd} K$ contains parallel segments of lengths at least ε, where $\varepsilon > 0$. Observe that K is covered by $K + (1/2)(K + v_\varepsilon)$ and $K - (1/2)(K + v_\varepsilon)$, where v_ε is one of the two vectors whose starting and end-points are on a segment of length ε meant as in the assumption. Hence, $t(K) = 2$.

Assume that $t(K) = 2$. This means that K can be covered by two proper translates $K + v_1$ and $K + v_2$. Here v_1 and v_2 are some oppositely directed vectors. Assume that this is not true. Then the union of $K + v_1$ and $K + v_2$ does not contain the point of support of K by a hyperplane with the property that the inner products $\langle v_1, v \rangle$ and $\langle v_2, v \rangle$ both are positive, where v denotes a normal vector of this hyperplane. This contradicts $t(K) = 2$. We see that v_1 and v_2 are oppositely directed vectors. Hence, $\operatorname{bd} K$ contains parallel segments of lengths at least ε, where ε is the sum of lengths of v_1 and v_2. $\square$

Together with (1) (see also Proposition 4.3), this proposition implies:

Corollary 5.1. *Let $K \subset E^2$ be a convex body. We have $2 \le t(K) \le 3$ with $t(K) = 2$ if and only if K is as described in Proposition* 5.2.

According to (1) every upper bound of $h(K)$ is also an upper bound on $t(K)$. But not all known upper bounds on $h(K)$ are interesting in view of $t(K)$, as is the case for *zonotopes*, i.e., vector sums of finitely many line segments. Namely, the best known upper bound on $h(Z)$ for Z being an arbitrary n-dimensional zonotope is $(5/8) \cdot 2^{n-3}$, and for a few types of zonotopes even larger, as in the case $h(Z) = 2^n$ for Z the affine n-cube; see [**4**]. But since any zonotope Z has segment summands, in view of Proposition 5.2, we clearly have $t(Z) = 2$ for any zonotope. Thus, only a certain selection of bounds on $h(K)$ is interesting for estimation also of $t(K)$. In the following (see also the survey in [**5**]) we give a list of such selected results in terms of $t(K)$.

For every centrally symmetric convex body $K \subset E^n$, we have $h(K) \le 5n2^n \ln n$ (see [**6,** Theorem 9.15.1]); this estimate also holds for $t(K)$. If $K \subset E^3$ is a convex body, then analogously the estimate $t(K) \le 16$ holds (see [**23**]). If $K \subset E^3$ is an arbitrary centrally symmetric convex body, then the estimate $h(K) \le 8$ (see [**15**]) analogously implies $t(K) \le 8$. If K is a smooth convex body in E^n, then $2 \le t(K) \le n+1$ (whereas $h(K) = n+1$), and if, in particular, K is smooth and strictly convex, then clearly $t(K) = n+1$. We have also $2 \le t(K) \le n+1$ if $K \subset E^n$ has at most n non-regular boundary points, and for $n = 3$ even four non-regular boundary points still yield $2 \le t(K) \le 4$. On the other hand, $t(K) \le n+1$ still holds if K has arbitrarily many non-regular boundary points which, however, have to be "not too acute," or if K has at least one shadow boundary consisting only of regular boundary points; see [**5,** pages 271–272]. All the bounds on $h(K)$ hold for bodies of constant width or for convex bodies with certain symmetry properties (see again [**5,** pages 271–272]) and also yield upper bounds on $t(K)$ for such bodies.

Now we turn to lower bounds for the unknown upper bounds on $t(K)$ for $n \ge 3$, i.e., we ask for realizations of convex bodies $K \subset \mathbf{E}^n$ with $t(K)$ being as large as possible. It turns out that strictly antipodal sets in $\mathbf{E}^n$ yield such lower bounds. A pair of points x, y in a set $X \subset \mathbf{E}^n$ is called *strictly antipodal* if X lies in the slab between the parallel hyperplanes $H_x \ni x$ and $H_y \ni y$ with $X \cap H_x = \{x\}$, $X \cap H_y = \{y\}$. By this definition, it is clear that no proper translate of $\operatorname{conv} X$ can cover x and y simultaneously. Now denote by a_n the maximum cardinality of a finite set $X \subset \mathbf{E}^n$ with the property that any two points of X are strictly antipodal. Thus, we have with Proposition 3.1, that $a_n = t(\operatorname{conv} X)$. Danzer and Grünbaum **[8]** introduced the notion of strictly antipodal points in finite sets $X \subset \mathbf{E}^n$, and they posed the question on the upper bound for a_n. Grünbaum **[12]** proved that $a_3 = 5$, and in **[8]**, a set of $2n - 1$ points in $\mathbf{E}^n$ is constructed, any two of these points being strictly antipodal. For a long time it was believed that $a_n = 2n - 1$, but Erdős and Füredi **[9]** showed that $a_n \geq \lfloor (2/\sqrt{3})^n/2 \rfloor$. Thus, there are convex polytopes $P \subset \mathbf{E}^n$ satisfying $t(P) \geq \lfloor (2/\sqrt{3})^n/2 \rfloor$. The exact values for a_n, $n \geq 4$, are still unknown, and the best known lower bound is $3^{n/3}$, due to Talata; see **[6,** Section 9.11**]**.

If $a_n(X_m)$ denotes the number of strictly antipodal pairs in a set $X_m \subset \mathbf{E}^n$ of cardinality m, and $a_n(m)$ stands for the maximum of $a_n(X_m)$ taken over all sets X_m, then these numbers are also interesting for our purpose, since obviously $2a_n(X_m) = t(\operatorname{conv} X_m)$, and $2a_n(m)$ denotes maximum over all numbers $t(\operatorname{conv} X_m)$, for all sets X_m of cardinality m. Results on the numbers $a_n(X_m)$ and $a_n(m)$ are summarized in **[19,** Section 4**]**.

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