ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 44, Number 3, 2014

APPROXIMATION BY HÖLDER CONTINUOUS FUNCTIONS IN A SOBOLEV SPACE

DAVID SWANSON

ABSTRACT. We prove that a function $u \in W^{1,1}(\mathbf{R}^n)$ may be simultaneously approximated by a Hölder continuous function both pointwise and in the Sobolev norm.

1. Introduction. Given an open set $\Omega \subset \mathbf{R}^n$ we denote by $W^{1,p}(\Omega)$, $p \geq 1$, the Sobolev space consisting of all functions $u \in L^p(\Omega)$ whose first order distributional partial derivatives also belong to $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,p;\Omega} = \|u\|_p + \|Du\|_p.$$

A fundamental property of functions $u \in W^{1,p}(\mathbf{R}^n)$ is quasicontinuity: any function $u \in W^{1,p}(\mathbf{R}^n)$ has a representative (also denoted by u) with the property that, for any $\varepsilon > 0$ there exists a continuous function v defined on $\mathbf{R}^n$ with the property that the set $\{u \neq v\}$ has small p-capacity (see e.g., [**1**]). This is a strengthening of the classical Lusin theorem regarding the approximation of measurable functions continuous functions. Functions $u \in W^{1,p}(\mathbf{R}^n)$ are identified up to a set with zero p-capacity with their quasicontinuous representatives.

Malý [**8**] observed that when $p > 1$, if the condition on the capacity is relaxed somewhat, then the approximator v may be chosen to be Hölder continuous, a property now commonly called *Hölder quasicontinuity.* In addition, Malý showed that the approximator v may be chosen in the space $W^{1,p}(\mathbf{R}^n)$ with norm arbitrarily close to u. Hajłasz and Kinnunen [**5**], working with Hausdorff content in place of capacity, extended Malý's result to the Sobolev space $M^{1,p}(X)$ of functions defined on a metric measure space X, and recently the result has been

2010 AMS *Mathematics subject classification.* Primary 46E35, Secondary 26B35, 41A28.
Keywords and phrases. Sobolev functions, Hölder approximation, quasi-continuity.
Received by the editors on January 19, 2012.
DOI:10.1216/RMJ-2014-44-3-1027

further extended to variable-exponent Sobolev spaces with exponent $1 < p(x) < n$ by Harjulehto, Kinnunen and Tuhkanen [**6**].

Although the results mentioned above omit the case $p = 1$, Kinnunen and Tuominen [**7**] have extended the result in [**5**] to the Sobolev space $M^{1,1}(X)$. Since $M^{1,1}(\mathbf{R}^n) \neq W^{1,1}(\mathbf{R}^n)$ this leaves a small gap concerning the Euclidean case. The purpose of this note is to supplement the above-mentioned results and obtain approximation by Hölder continuous functions in the space $W^{1,1}(\mathbf{R}^n)$ as well.

It was proved in [**9**] that if $u \in W^{1,1}(\mathbf{R}^n)$, then for any $\varepsilon > 0$ the continuous approximator satisfying v may be chosen so that $\|u-v\|_{1,1} < \varepsilon$ in addition to the set $\{u \neq v\}$ having 1-capacity less than ε. We adapt the methods developed in [**5**] and [**9**] to obtain the following main result.

Theorem 1.1. *Let $u \in W^{1,1}(\mathbf{R}^n)$, and let $0 < \lambda < 1$. Then, for any $\varepsilon > 0$, there exists a Hölder continuous function $v \in W^{1,1}(\mathbf{R}^n)$ with exponent λ with the property that*

(1) $H_\infty^{n-1+\lambda}(\{x : \overline{u}(x) \neq v(x)\}) < \varepsilon$, *and*
(2) $\|u - v\|_{1,1} < \varepsilon$,

where $\overline{u}$ is the precise representative of u.

2. Preliminaries. Throughout the paper $C_{a,b,c,\ldots}$ will denote a constant whose precise value depends only on the parameters $a, b, c, \ldots$ but may change between occurrences.

Definition 2.1. Let $\alpha \geq 0$. The α-dimensional Hausdorff content of a set $E \subset \mathbf{R}^n$ is the quantity

$$H_\infty^\alpha(E) = \inf\left\{ \sum_{k=1}^\infty (\operatorname{diam} E_k)^\alpha,\ E \subset \textstyle\bigcup_{k=1}^\infty E_k \right\}.$$

In light of the elementary inequality,

$$\sum_{k=1}^\infty a_k^q \leq \left(\sum_{k=1}^\infty a_k \right)^q$$

whenever $a_k \geq 0$ and $q \geq 1$, we have

$$H_\infty^{\alpha_2}(E)^{\alpha_1} \leq H_\infty^{\alpha_1}(E)^{\alpha_2}, \tag{2.1}$$

whenever $0 \leq \alpha_1 \leq \alpha_2$. In particular, there is a constant c_n with the property that the Lebesgue outer measure satisfies $|E| \leq cH_\infty^\alpha(E)^{n/\alpha}$ whenever $0 \leq \alpha \leq n$.

Definition 2.2. Let $u \in L^1_{\text{loc}}(\mathbf{R}^n)$ and let $x \in \mathbf{R}^n$. For $r > 0$, we define

$$\overline{u}_r(x) = \fint_{B(x,r)} u(y)\, dy.$$

The precise representative of u is defined by

$$\overline{u}(x) = \lim_{r\to 0^+} \overline{u}_r(x)$$

at all points $x \in \mathbf{R}^n$ where this limit exists.

It is well known that, if $u \in W^{1,1}(\mathbf{R}^n)$, then $\overline{u}(x)$ exists for all x outside a set N with $H_\infty^{n-1}(N) = 0$ and $\overline{u}(x)$ is quasicontinuous. Moreover, we have the following uniform behavior (see, e.g., [**4,** pages 160–162]):

Proposition 2.3. *If $u \in W^{1,1}(\mathbf{R}^n)$, then for every $\varepsilon > 0$, there exists an open set U with $H_\infty^{n-1}(U) < \varepsilon$ such that*

$$\lim_{r\to 0^+} \fint_{B(x,r)} |u(y) - \overline{u}(x)|\, dy = 0$$

uniformly for $x \in \mathbf{R}^n \setminus U$.

Definition 2.4. Let $u \in L^1_{\text{loc}}(\mathbf{R}^n)$, and let $\lambda \geq 0$. The fractional maximal function $M^\lambda u$ is defined by

$$\sup_{r>0} r^\lambda \fint_{B(x,r)} |u(y)|\, dy.$$

The following weak-type estimate is well known.

Proposition 2.5. *If $u \in L^1(\mathbf{R}^n)$ and $t > 0$, then the set $\{x : M^\lambda u(x) > t\}$ is open and*

$$H_\infty^{n-\lambda}(\{x : M^\lambda u(x) > t\}) \leq \frac{C_{n,\lambda}}{t}\|u\|_1.$$

Proposition 2.5 is straightforward to prove using a covering argument and the observation that, for any point x with $M^\lambda u(x) > t$, there is a number $r = r_x$ satisfying

$$r^{n-\lambda} < t^{-1}\int_{B(x,r)} |u(y)|\, dt.$$

The fractional maximal function is used in the following inequalities which may be proved as corollaries of the Bojarski-Hajłasz inequality [**5,** Theorem 2].

Proposition 2.6. *Suppose that $u \in W^{1,1}(\mathbf{R}^n)$. Then*

$$|\overline{u}(x) - \overline{u}(y)| \leq C_{n,\lambda}|x-y|^\lambda \left(M^{1-\lambda}|Du|(x) + M^{1-\lambda}|Du|(y)\right)$$

and

$$⨍_{B(x,r)} |u(y) - \overline{u}(x)|\, dy \leq C_{n,\lambda} r^\lambda M^{1-\lambda}|Du|(x)$$

at all points x where $\overline{u}(x)$ is defined.

We denote by $W_0^{1,1}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the $W^{1,1}(\Omega)$ norm. The following characterization of $W_0^{1,1}(\Omega)$ was obtained in [**9**]:

Proposition 2.7. *Let $\Omega \subset \mathbf{R}^n$ be an arbitrary open set, and let $u \in W^{1,1}(\Omega)$. Then $u \in W_0^{1,1}(\Omega)$ if and only if*

$$\lim_{r\to 0^+} r^{-n}\int_{B(x,r)\cap\Omega} |f(y)|\, dy = 0$$

for H_∞^{n-1}-almost every $x \in \partial\Omega$.

Given a function $u : \Omega \to \mathbf{R}$, denote by u^* its zero extension off Ω:

$$u^*(x) = \begin{cases} u(x) & x \in \Omega, \\ 0 & x \notin \Omega. \end{cases} \tag{2.2}$$

We make the trivial observation that $u \in W_0^{1,1}(\Omega)$ implies $u^* \in W^{1,1}(\mathbf{R}^n)$.

3. Proof of Theorem 1. Let $u \in W^{1,1}(\mathbf{R}^n)$, $0 < \lambda < 1$, and $\varepsilon > 0$ be given. Let $\theta > 0$ be a quantity whose precise value will be specified later. Since

$$H_\infty^{n+1-\lambda}(\{x : M^{1-\lambda}|Du|(x) > T\}) < \frac{C_{\lambda,n}}{T}\|Du\|_1,$$

we may take T sufficiently large and refer to Proposition 2.3 and equation (2.1) to find an open set U with $H^{n-1+\lambda}(U) < \theta$ such that $\overline{u}(x)$ is defined and $M^{1-\lambda}|Du(x)| \le T$ for all $x \in \mathbf{R}^n \setminus U$. Define $K = \mathbf{R}^n \setminus U$. Then K is closed, and by Proposition 2.6, we have

$$|\overline{u}(x) - \overline{u}(y)| \le C_{n,\lambda} T |x-y|^\lambda \tag{3.1}$$

for all $x, y \in K$ and

$$⨍_{B(x,r)} |u(y) - \overline{u}(x)|\, dy \le C_{n,\lambda} T r^\lambda \tag{3.2}$$

for all $x \in K$. Since $|U| \le C_n \theta^{(n-1+\lambda)/n}$, we may assume without loss of generality that $|U| < 1$. The definition of the approximator v will rely on a smoothing procedure introduced in **[3]**.

Proposition 3.1. **[3,** Lemma 3.2] *There exists a function $\delta \in C^\infty(U)$ with the property that*

$$C_n \operatorname{dist}(x, \partial U) \le \delta(x) \le \operatorname{dist}(x, \partial U) \tag{3.3}$$

for all $x \in U$ and

$$\sup_{x \in U} |D\delta(x)| \le C_n. \tag{3.4}$$

At this point, we fix a regularizing kernel $\phi \in C_0^\infty(B(0,1))$ and a function $\delta \in C^\infty(U)$ as in Proposition 3.1. For each $\varepsilon > 0$ write $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$. For each $z \in U$, define $\psi_z(x) = \phi_{\delta(z)/2}(x - z)$. We define a smoothing Su of u on U by

$$Su(z) = \int_{\mathbf{R}^n} \psi_z(x) u(x)\, dx, \quad z \in U. \tag{3.5}$$

We will require the following estimate on the $W^{1,1}(U)$ norm of Su:

Proposition 3.2. **[9,** Lemma 4.5] *$Su \in C^\infty(U)$ and $\|Su\|_{1,1;U} \leq C_n\|u\|_{1,1;U}$.*

We are now in a position to define the approximator v:

$$v(x) = \begin{cases} Su(x), & x \in U, \\ \overline{u}(x), & x \in K. \end{cases} \tag{3.6}$$

By construction, we have

$$H_\infty^{n-1+\lambda}(\{x : \overline{u}(x) \neq v(x)\}) < \theta,$$

which can be made arbitrarily small by choosing θ sufficiently close to zero.

We may show that v is Hölder continuous with exponent λ by considering several cases. Throughout the argument, for any point $\xi \in U$, we denote by ξ' a point in K satisfying $|\xi - \xi'| = \operatorname{dist}(\xi, K)$. We refer to a generic constant simply as C.

First, if $x, y \in K$ then by (3.1) we have $|v(x) - v(y)| \leq CT|x - y|^\lambda$.

Second, if $x \in U$ and $y \in K$, then $|v(x') - v(y)| \leq CT|x' - y|^\lambda$ since $x' \in K$. By (3.5), we have

$$v(x') - v(x) = \int_{\mathbf{R}^n} \psi_z(w)[\overline{u}(x') - u(w)]\, dw$$

so that

$$|v(x') - v(x)| \leq C \fint_{B(x',2|x-x'|)} |\overline{u}(x') - u(w)|\, dw \leq CT|x - x'|^\lambda.$$

Since $|x - x'| \leq |x - y|$, we have $|x' - y| \leq 2|x - y|$ so that

$$|x - x'|^\lambda + |x' - y|^\lambda \leq C|x - y|^\lambda.$$

The triangle inequality yields $|v(x) - v(y)| \leq CT|x - y|^\lambda$.

Third, if $x \in U$, $y \in U$, and $|x-x'| < 2|x-y|$, then $|y-x'| < 3|x-y|$. Since

$$\begin{aligned} |v(x) - v(y)| &\le |v(x) - v(x')| + |v(y) - v(x')| \\ &\le CT \left(|x - x'|^\lambda + |y - x'|^\lambda \right), \end{aligned}$$

we have $|v(x) - v(y)| \le CT|x - y|^\lambda$.

In the fourth and final case we consider $x \in U$ and $y \in U$ with $2|x-y| \le |x-x'|$. In this case the line segment adjoining x and y lies entirely within U. For any point $z \in U$ and real number a we have

$$v(z) - a = \int_{\mathbf{R}^n} \psi_z(x)[u(w) - a]\, dw$$

so that

$$Dv(z) = \int_{\mathbf{R}^n} D\psi_z(x)[u(w) - a]\, dw,$$

and therefore

$$|Dv(z)| \le \int_{\mathbf{R}^n} |D\psi_z(w)||u(w) - a|\, dw.$$

In light of (3.3) and (3.4) we have $|D\phi_z| \le C_n \delta(z)^{-n-1}$, so that

$$\begin{aligned} |Dv(z)| &\le C\delta(z)^{-1} \fint_{B(z,\delta(z)/2)} |u(w) - a|\, dw \\ &\le C\delta(z)^{-1} \fint_{B(z',2|z-z'|)} |u(w) - a|\, dw. \end{aligned}$$

With the special choice $a = v(z')$, we may appeal to (3.2) to conclude that

$$|Dv(z)| \le C\delta(z)^{-1+\lambda} M^{1-\lambda} |Du|(z') \le CT\delta(z)^{-1+\lambda} \le CT|z - z'|^{-1+\lambda}$$

for all $z \in U$. In particular, for any point z belonging to the line segment adjoining x and y and satisfying

$$|v(x) - v(y)| = |Dv(z)||x - y|$$

we have the estimate

$$|v(x) - v(y)| \le CT|z - z'|^{-1+\lambda}|x - y|.$$

The geometry of the situation implies $|x-y| \leq |z-z'|$, since otherwise we would have $|x-x'| \leq |x-z'| \leq |x-z| + |z-z'| < 2|x-y|$. Thus,

$$|v(x) - v(y)| \leq CT|x-y|^{\lambda},$$

as required.

By construction, we have $v, u \in W^{1,1}(U)$. Since

$$r^{-n} \int_{B(x,r)\cap U} |v(y) - u(y)|\, dy \leq C_n \fint_{B(x,r)} |v(y) - v(x)|\, dy + C_n \fint_{B(x,r)} |u(y) - \overline{u}(x)|\, dy$$

for all $x \in K$, the fact that v is continuous and (3.2) holds on K, we obtain

$$\lim_{r\to 0^+} r^{-n} \int_{B(x,r)\cap U} |v(y) - u(y)|\, dy = 0$$

for all $x \in \partial U$. Proposition 2.7 implies that $v - u \in W_0^{1,1}(U)$, and consequently $(v-u)^* \in W^{1,1}(\mathbf{R}^n)$. It follows that

$$v = (v-u)^* + u \in W^{1,1}(\mathbf{R}^n).$$

Moreover,

$$\|u - v\|_{1,1} = \|u - v\|_{1,1;U} \leq \|u\|_{1,1;U} + \|v\|_{1,1;U} \leq C_n \|u\|_{1,1;U}$$

by Proposition 3.2. Since $|U| \to 0$ as $\theta \to 0$, we may choose θ sufficiently small to guarantee $\|u - v\|_{1,1} < \varepsilon$.

REFERENCES

1. D. Adams and L.I. Hedberg, *Function spaces and nonlinear potential theory*, Grundl. Math. Wissen. **314**, Springer-Verlag, Berlin, 1996.

2. B. Bojarski and P. Hajłasz, *Pointwise inequalities for Sobolev functions and some applications*, Stud. Math. **106** (1993), 77–92.

3. A.P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Stud. Math. **20** (1961), 171–225.

4. L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, Stud. Adv. Math., CRC Press, Boca Raton, 1992.

5. P. Hajłasz and J. Kinnunen, *Hölder quasicontinuity of Sobolev functions on metric spaces*, Rev. Matem. Iberoamer. **14** (1998), 601–622.

6. P. Harjulehto, J. Kinnunen and K. Tuhkanen, *Hölder quasicontinuity in variable exponent Sobolev spaces*, J. Inequal. Appl. **2007**, Art. ID 32324, 18 pages.

7. J. Kinnunen and H. Tuominen, *Pointwise behaviour of* $M^{1,1}$ *Sobolev functions*, Math. Z. **257** (2007), 613–630.

8. J. Malý, *Hölder type quasicontinuity*, Potential Anal. **2** (1993), 249–254.

9. D. Swanson, *Area, coarea, and approximation in* $W^{1,1}$, Ark. Math. **45** (2007), 381–399.

Department of Mathematics, 328 Natural Science Building, University of Louisville, Louisville KY 40292
Email address: **david.swanson@louisville.edu**