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NORMAL NUMBERS FROM STEINHAUS'S VIEWPOINT

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ABSTRACT. In this expository paper we recall a non-standard construction of the Borel sigma-algebra $\mathcal{B}$ in $[0,1]$ and construct a family of measures (in particular, Lebesgue measure) in $\mathcal{B}$ by a completely non-topological method. This approach, that goes back to Steinhaus in 1923, is now used to flirt with natural generalizations of the concept of normal numbers and explore their intrinsic probabilistic properties. We show that, in virtually almost all of the cases, almost all real numbers in $[0,1]$ are normal (with respect to this extended concept). This procedure highlights some apparently hidden but interesting features of the Borel sigma-algebra and Lebesgue measure in $[0,1]$.

1. Introduction. The Borel sigma-algebra in $[0,1]$ is, in general, defined as the sigma-algebra generated by the (open) intervals in $[0,1]$. So, we have a natural "topological" component in the Borel sigma-algebra (and Lebesgue measure) in $[0,1]$. However, as will be shown, a completely different (non-topological) approach will be extremely useful in dealing with an extension of the concept of normal numbers and their probabilistic aspects. The proposed characterization of the Borel sigma-algebra in $[0,1]$, which goes back to 1923 with Steinhaus [**5**], in the beginning of the conception of modern probability theory, shows a natural way to consider "weighted" measures in $[0,1]$ and to define and discuss normality of numbers with respect to these measures.

The paper is organized as follows. In Section 2 we recall some background results concerning product measures, in Section 3 we characterize the Borel sigma-algebra in $[0,1]$ as a product sigma-algebra and introduce a family of "weighted" measures in $[0,1]$ and, in the last section, we apply the previous results to flirt with variations of the concept of normal numbers and explore their probabilistic behavior. The main goal of this note is to translate, to modern notation and terminology, Steinhaus's striking ideas concerning normal numbers

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and also to call attention to his contributions to the birth of modern probability theory.

Throughout, $\mathbf{R}$ denotes the set of all real numbers, $\mathbf{N}$ represents the set of all natural numbers $\{1, 2, \dots\}$ and $\mathcal{B}$ is the Borel sigma-algebra in the closed interval $[0, 1]$. If Ω is any set, 2^{Ω} denotes the set of all subsets of Ω. If $\mathcal{A} \subset 2^{\Omega}$, then $\sigma(\mathcal{A})$ represents the sigma-algebra generated by $\mathcal{A}$, $\#A$ denotes the cardinality of A and $A^{\mathbf{N}}$ represents $A \times A \times \cdots$.

2. Preliminary results. Let $\Omega \neq \phi$ be a denumerable (or finite) set, $\mathcal{A} = 2^{\Omega}$, and let ρ be a probability measure in $\mathcal{A}$.

Let

$$(\Omega^{\mathbf{N}}, \otimes\mathcal{A}, \otimes\varrho)$$

be the product space, and denote

$$\mu = \mu_{\rho} = \otimes\varrho \quad \text{and} \quad \mathcal{D} = \otimes\mathcal{A}.$$

Definition 1. $w = (a_i)_{i=1}^{\infty} \in \Omega^{\mathbf{N}}$ is simply normal, with respect to μ, if

$$\lim_{n\to\infty} \frac{S(w, r, n)}{n} = \rho(\{r\}) \quad \text{for all } r \in \Omega,$$

where $S(w, r, n)$ denotes the total of indexes $i, 1 \leq i \leq n$, such that $a_i = r$.

Definition 2. $w = (a_i)_{i=1}^{\infty} \in \Omega^{\mathbf{N}}$ is normal, with respect to μ, if

$$\lim_{n\to\infty} \frac{S(w, B_k, n)}{n} = \prod_{j=1}^{k} \rho(\{b_j\}) \quad \text{for all } k \in \mathbf{N},$$

for each word $B_k = b_1 \cdots b_k$ of k elements from Ω, where

$$S(w, B_k, n) = \#\{i \in \{1, \dots, n\}; a_{i+j-1} = b_j \quad \text{for every } j = 1, \dots, k\}.$$

The next two propositions are standard applications of the Strong law of large numbers. We sketch the proofs for the sake of completeness.

Proposition 1. *The measure of the set*

$$\{w = (a_i)_{i=1}^{\infty} \in \Omega^{\mathbf{N}}; w \textit{ is simply normal}\}$$

is 1.

Proof. Let $\mathcal{B}_{\mathbf{R}}$ denote the Borel sigma-algebra on $\mathbf{R}$ and $r \in \Omega$. Define, for every $n \in \mathbf{N}$,

$$\begin{aligned} X_n^r : (\Omega^{\mathbf{N}}, \mathcal{D}, \mu) &\longrightarrow (\mathbf{R}, \mathcal{B}_{\mathbf{R}}) \\ w = (a_i)_{i=1}^{\infty} &\longmapsto \begin{cases} 0 & \text{if } a_n \neq r \\ 1 & \text{if } a_n = r \end{cases} \end{aligned} \tag{2.1}$$

It is easy to see that each X_n^r is measurable, and hence $(X_n^r)_{n=1}^{\infty}$ is a sequence of real random variables. Moreover, $(X_n^r)_{n=1}^{\infty}$ is an independent, integrable and identically distributed sequence.

Note that

$$\int_{\Omega^{\mathbf{N}}} X_n^r d\mu = \mu\left((X_n^r)^{-1} (\{1\}) \right) = \rho(\{r\}).$$

Hence, a well-known result due to Kolmogorov asserts that the sequence $(X_n^r)_{n=1}^{\infty}$ satisfies the Strong law of large numbers, and so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i^r(w) = \rho(\{r\}) \quad (\mu\text{-a.e}).$$

Therefore,

$$\lim_{n\to\infty} \frac{S(w,r,n)}{n} = \rho(\{r\}) \quad (\mu\text{-a.e}).$$

Denoting

$$M^r = \left\{ w \in \Omega^{\mathbf{N}}; \lim_{n\to\infty} \frac{S(w,r,n)}{n} = \rho(\{r\}) \right\},$$

we have that the set composed by the simply normal sequences on $\Omega^{\mathbf{N}}$ is precisely

$$\bigcap_{r\in\Omega} M^r.$$

Since $\mu(\Omega^{\mathbf{N}} \setminus M^r) = 0$ and Ω is, at most, denumerable, we have

$$\mu\Bigg(\bigcap_{r\in\Omega} M^r\Bigg) = 1. \qquad \square$$

Proposition 2. *The measure of the set* $\{w = (a_i)_{i=1}^{\infty} \in \Omega^{\mathbf{N}};\ w$ *is normal*$\}$ *is* 1.

Proof. Let $r_1 \cdots r_k$ be a word with k elements from Ω, with $k \in \mathbf{N}$. Using the notation from (2.1), define, for every $n \in \mathbf{N}$,

$$\begin{cases} Y_{(1),n}^{r_1\cdots r_k}(w) = X_{kn-(k-1)}^{r_1}(w)\cdot X_{kn-(k-2)}^{r_2}(w)\cdots X_{kn}^{r_k}(w) \\ Y_{(2),n}^{r_1\cdots r_k}(w) = X_{kn-(k-2)}^{r_1}(w)\cdot X_{kn-(k-3)}^{r_2}(w)\cdots X_{kn+1}^{r_k}(w) \\ \vdots \\ Y_{(k),n}^{r_1\cdots r_k}(w) = X_{kn}^{r_1}(w)\cdot X_{kn+1}^{r_2}(w)\cdots X_{kn+(k-1)}^{r_k}(w). \end{cases}$$

It is plain that, for every $j = 1, \dots, k$, $(Y_{(j),n}^{r_1\cdots r_k})_{n=1}^{\infty}$ are integrable, independent and identically distributed sequences. We also have, for every $j = 1, \dots, k$,

$$\int_{\Omega^{\mathbf{N}}} Y_{(j),n}^{r_1\cdots r_k}\, d\mu = \mu\left(\left(Y_{(j),n}^{r_1\cdots r_k}\right)^{-1}(\{1\})\right) = \prod_{j=1}^{k} \rho(\{r_j\}).$$

Hence, for every $j = 1, \dots, k$, $(Y_{(j),n}^{r_1\cdots r_k})_{n=1}^{\infty}$ satisfies the Strong law of large numbers and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_{(j),i}^{r_1\cdots r_k}(w) = \prod_{j=1}^{k} \rho(\{r_j\})(\mu\text{-a.e.}).$$

We thus have

$$\lim_{n\to\infty} \frac{S(w, r_1\cdots r_k, n)}{n} = \prod_{j=1}^{k} \rho(\{r_j\})(\mu\text{-a.e.}).$$

Denoting

$$M^{r_1\cdots r_k} = \left\{ w \in \Omega^{\mathbf{N}};\ \lim_{n\to\infty} \frac{S(w, r_1\cdots r_k, n)}{n} = \prod_{j=1}^{k} \rho(\{r_j\}) \right\},$$

the set composed by the normal sequences on $\Omega^{\mathbf{N}}$ is precisely

$$\bigcap_{k\in\mathbf{N}} \Bigg(\bigcap_{r_1\cdots r_k \in \Omega^k} M^{r_1\cdots r_k} \Bigg).$$

Since $\mu(\Omega^{\mathbf{N}} \setminus M^{r_1\cdots r_k}) = 0$ and Ω is, at most, denumerable, we have

$$\mu\Bigg(\bigcap_{k\in\mathbf{N}} \Bigg(\bigcap_{r_1\cdots r_k \in \Omega^k} M^{r_1\cdots r_k} \Bigg)\Bigg) = 1. \qquad \square$$

3. A family of measures in $[0,1]$. The present section can be regarded, in some sense, as a translation of Steinhaus's ideas from the roots of the modern probability theory (see [**5**]).

Henceforth, $\Omega = \{0, \ldots, 9\}$, $\mathcal{A} = 2^{\Omega}$ and ρ is a probability measure on $(\Omega, \mathcal{A})$. If $(\Omega^{\mathbf{N}}, \mathcal{D}, \mu_\rho)$ is the product space, then $\mathcal{D} = \sigma(\mathcal{C})$, with

$$\mathcal{C} = \Bigg\{ \Omega \times \cdots \times \Omega \times \overset{\text{position } n}{\{a\}} \times \Omega \times \cdots ; a \in \Omega \text{ and } n \in \mathbf{N} \Bigg\}.$$

Consider the mapping

$$\begin{aligned} \Psi : \Omega^{\mathbf{N}} &\longrightarrow [0,1] \\ w = (a_j)_{j=1}^{\infty} &\longmapsto \sum_{j=1}^{\infty} a_j 10^{-j}. \end{aligned} \tag{3.1}$$

Note that Ψ is "almost injective," in the sense that, for every $(a_j)_{j=1}^{\infty} \in \Omega^{\mathbf{N}}$,

$$\begin{aligned} \#(\{(b_j)_{j=1}^{\infty} \in \Omega^{\mathbf{N}}; (b_j)_{j=1}^{\infty} \neq (a_j)_{j=1}^{\infty} \quad &\text{and} \\ \Psi((b_j)_{j=1}^{\infty}) = \Psi((a_j)_{j=1}^{\infty})\}) \\ = 0 \text{ or } 1. \end{aligned}$$

Besides,

$$\begin{aligned}\#(\{(a_j)_{j=1}^{\infty} \in \Omega^{\mathbf{N}}; \exists (b_j)_{j=1}^{\infty} &\neq (a_j)_{j=1}^{\infty} \quad \text{with} \\ \Psi((b_j)_{j=1}^{\infty}) &= \Psi((a_j)_{j=1}^{\infty})\}) \\ &= \#(\mathbf{N}).\end{aligned}$$

From now on, sometimes we will write μ in the place of μ_ρ.

Note that Ψ is a random variable. In fact, if $A \in \Psi(\mathcal{C})$, then

$$A = \Psi(\Omega \times \cdots \times \Omega \times \{b\} \times \Omega \times \cdots)$$

for some $b \in \Omega$, and thus

$$\Psi^{-1}(A) = (\Omega \times \cdots \times \Omega \times \{b\} \times \Omega \times \cdots) \cup D,$$

where $D \in \mathcal{D}$ is a denumerable set. So,

$$\Psi^{-1}(A) \in \mathcal{D},$$

and, denoting $\mathcal{S} = \sigma(\Psi(\mathcal{C}))$, we can conclude that $\Psi : \Omega^{\mathbf{N}} \to ([0,1], \mathcal{S})$ is measurable. The probability measure μ_ρ (which is a probability measure on $(\Omega^{\mathbf{N}}, \mathcal{D})$) induces via Ψ a measure λ_ρ on $([0,1], \mathcal{S})$, called distribution.

The measures λ_ρ on $\mathcal{S}$ present an interesting behavior since, in general, these measures are "weighted," i.e., they "protect" some digits. For example, it is not hard to see that if

$$\rho(\{9\}) = 3/10,$$

then

$$\lambda_\rho\left(\left[\frac{9}{10}, 1\right]\right) = 3/10.$$

Next, we will prove the following results, that are probably folkloric, but, as far as we know, are (at least) very difficult to be found in the literature:

• $\Psi(\mathcal{D}) = \mathcal{S} = \mathcal{B}$.

• If $\rho(\{r\}) = 1/10$ for every $r \in \Omega$, then λ_ρ is precisely the Lebesgue measure, i.e., when the measure ρ is "non-weighted," λ_ρ coincides with the Lebesgue measure.

Theorem 1. $\mathcal{S} = \mathcal{B}$.

Proof. Recall that

$$\mathcal{B} = \sigma\left(\{[a,b]\,;a<b \text{ and } a,b\in[0,1[\}\right).$$

The following notation will be convenient:

$$\begin{aligned}\{>a\} &= \{x\in\Omega; x>a\}\\ \{<a\} &= \{x\in\Omega; x<a\}\\ \{\geq a\} &= \{x\in\Omega; x\geq a\}\\ \{\leq a\} &= \{x\in\Omega; x\leq a\}\\ \{>a,<b\} &= \{x\in\Omega; x>a \text{ and } x<b\}\,.\end{aligned}$$

We also define $\{>a,\leq b\}$, $\{\geq a,<b\}$ and $\{\geq a,\leq b\}$ in a similar way.

If

$$a=\sum_{j=1}^{\infty}a_j10^{-j} \quad \text{and} \quad b=\sum_{j=1}^{\infty}b_j10^{-j},$$

let us consider the unique way to write the expansion so that the sequence of digits does not stabilize at an endless string of 9s. Note that

$$[a,b]=\bigcap_{n=n_0+1}^{\infty}\left[\sum_{j=1}^{n}a_j10^{-j},\sum_{j=1}^{n}b_j10^{-j}+\sum_{j=n+1}^{\infty}9.10^{-j}\right],$$

where n_0 is the smallest index for which $a_{n_0}\neq b_{n_0}$.

Since, for each $n\geq n_0$,

$$\begin{aligned}&\left[\sum_{j=1}^{n}a_j10^{-j},\sum_{j=1}^{n}b_j10^{-j}+\sum_{j=n+1}^{\infty}9.10^{-j}\right]\\ &=\Psi\left(\{a_1\}\times\{a_2\}\times\cdots\times\{a_{n_0-1}\}\right.\\ &\qquad\left.\times\{>a_{n_0},<b_{n_0}\}\times\Omega\times\Omega\times\cdots\right)\\ &\quad\bigcup\Psi\Bigg(\bigcup_{k=n_0+1}^{n}\{a_1\}\times\{a_2\}\times\cdots\end{aligned}$$

$$
\begin{aligned}
&\times \{a_{k-1}\} \times \{> a_k\} \times \Omega \times \cdots \Big) \\
&\bigcup \Psi \left(\{a_1\} \times \{a_2\} \times \cdots \times \{a_n\} \times \Omega \times \Omega \times \cdots\right) \\
&\bigcup \Psi \Bigg(\bigcup_{k=n_0+1}^{n} \{b_1\} \times \{b_2\} \times \cdots \\
&\qquad \times \{b_{k-1}\} \times \{< b_k\} \times \Omega \times \cdots \Bigg) \\
&\bigcup \Psi \left(\{b_1\} \times \{b_2\} \times \cdots \times \{b_n\} \times \Omega \times \cdots\right),
\end{aligned}
$$

we can easily conclude that

$$\left[\sum_{j=1}^{n} a_j 10^{-j}, \sum_{j=1}^{n} b_j 10^{-j} + \sum_{j=n+1}^{\infty} 9.10^{-j}\right] \in \sigma(\Psi(\mathcal{C})) = \mathcal{S},$$

and hence

$$\mathcal{B} \subset \sigma(\Psi(\mathcal{C})) = \mathcal{S}.$$

Now, we must show that $\mathcal{S} \subset \mathcal{B}$.

It suffices to show that $\Psi(\mathcal{C}) \subset \mathcal{B}$. Let $A \in \Psi(\mathcal{C})$. Hence,

$$
\begin{aligned}
A &= \Psi(\Omega \times \cdots \times \Omega \times \overset{\text{position } n}{\{b\}} \times \Omega \times \cdots) \\
&= \bigcup_{\substack{a_j \in \{0,\dots,9\} \\ j=1,\dots,n-1}} \left[\sum_{j=1}^{n-1} a_j 10^{-j} + \frac{b}{10^n}, \sum_{j=1}^{n-1} a_j 10^{-j} \right. \\
&\qquad \left. + \frac{b}{10^n} + \sum_{j=n+1}^{\infty} 9.10^{-j}\right] \in \mathcal{B}. \qquad \square
\end{aligned}
$$

Theorem 2. $\Psi(\mathcal{D}) = \mathcal{B}$.

Proof. Using the previous result, all we need to show is that $\mathcal{S} = \Psi(\mathcal{D})$.

The proof that $\Psi(\mathcal{D})$ is a sigma-algebra needs a little bit of hard work, but it is standard and we omit.

It is plain that

$$\mathcal{S} = \sigma(\Psi(\mathcal{C})) \subset \Psi(\mathcal{D}).$$

We will prove the converse inclusion.

It is not difficult to show that

$$\mathcal{R} = \{A \in \mathcal{D}; \Psi(A) \in \sigma(\Psi(\mathcal{C}))\}$$

is a sigma-algebra and, since $\mathcal{C} \subset \mathcal{R}$, we have

$$\mathcal{D} = \sigma(\mathcal{C}) \subset \mathcal{R}.$$

From the definition of $\mathcal{R}$, we conclude that

$$\Psi(\mathcal{D}) \subset \sigma(\Psi(\mathcal{C})) = \mathcal{S},$$

and the proof is done. $\square$

Finally we have:

Theorem 3. *If $\rho(\{a\}) = 1/10$ for every $a \in \Omega$, the distribution of the random variable Ψ is the Lebesgue measure.*

Proof. Let $\overline{\mu}$ be the distribution of Ψ and λ the Lebesgue measure on the Borel sigma-algebra on $[0, 1]$. In order to show that $\overline{\mu}$ and λ coincide, it suffices to show that they coincide over the intervals of $[0, 1]$. In fact, in this case, they will coincide in the algebra $\mathcal{U}$ composed by the finite union of disjoint intervals and, by invoking the Carathéodory extension theorem, the measures $\overline{\mu}$ and λ will coincide in $\mathcal{B} = \sigma(\mathcal{U})$.

Note that the set

$$J = \Big\{x \in [0, 1]; \exists m \in \mathbf{N} \text{ so that } x = \sum_{j=1}^{m} x_j \cdot 10^{-j}, \\ 0 \le x_j \le 9 \text{ (for all } j = 1, \dots, m)\Big\}$$

is dense in $[0, 1]$. So, we just need to show that $\overline{\mu}$ and λ coincide over the intervals $[a, b] \subset [0, 1]$, with $a, b \in J$.

Moreover, there is no loss of generality in dealing with intervals of the form

$$I = \left[\sum_{j=1}^{n} a_j 10^{-j}, \sum_{j=1}^{n} b_j 10^{-j} \right].$$

Let n_0 be the smallest index such that $a_{n_0} \neq b_{n_0}$. Hence,

$$\lambda(I) = \sum_{j=n_0}^{n} (b_j - a_j) 10^{-j}.$$

Consider $A \in \mathcal{D}$ given by

$$\begin{aligned}
A = & (\{a_1\} \times \{a_2\} \times \cdots \times \{a_{n_0-1}\} \times \{> a_{n_0}, < b_{n_0}\} \times \Omega \times \cdots) \\
& \bigcup \left(\bigcup_{k=n_0+1}^{n} \{a_1\} \times \{a_2\} \times \cdots \times \{a_{k-1}\} \times \{> a_k\} \times \Omega \times \cdots \right) \\
& \bigcup (\{a_1\} \times \{a_2\} \times \cdots \times \{a_n\} \times \Omega \times \Omega \times \cdots) \\
& \bigcup \left(\bigcup_{k=n_0+1}^{n} \{b_1\} \times \{b_2\} \times \cdots \times \{b_{k-1}\} \times \{< b_k\} \times \Omega \times \cdots \right) \\
& \bigcup (\{b_1\} \times \{b_2\} \times \cdots \times \{b_n\} \times \{0\} \times \{0\} \times \cdots).
\end{aligned}$$

Hence, $\Psi^{-1}(I) = A \cup D$ with $\mu_\rho(D) = 0$ and

$$\begin{aligned}
\mu_\rho(A) = & \frac{1}{10^{n_0}} (b_{n_0} - a_{n_0} - 1) + \sum_{k=n_0+1}^{n} \frac{1}{10^k} (9 - a_k) \\
& + \frac{1}{10^n} + \sum_{k=n_0+1}^{n} \frac{b_k}{10^k} + 0,
\end{aligned}$$

and straightforward calculations show that

$$\mu_\rho(A) = \lambda(I).$$

We thus have

$$\overline{\mu}(I) = \mu_\rho \left(\Psi^{-1}(I) \right) = \mu_\rho (A \cup D) = \mu_\rho(A) = \lambda(I). \qquad \square$$

4. A more general approach to normal numbers. The notion of normal numbers (with respect to Lebesgue measure) was introduced by Borel [**1**] in 1909, and, since then, several interesting questions on normal numbers have been investigated and various intriguing problems remain open (for example, the normality of $\sqrt{2}$). The results of the previous sections turn natural to considering the concept of normal numbers with respect to other measures than the Lebesgue measure on $[0,1]$. In this section, as an application of the previous results, we generalize the concept of normal numbers and obtain the measure of the sets of normal numbers (with this generalized concept). In particular, we give an alternative simple proof (essentially due to Steinhaus [**5**]) for the fact that almost all real numbers in $[0,1]$ are normal, with respect to Lebesgue measure (different proofs of this result can be found, for example, in [**1**–**4**]).

If $\Omega = \{0,\dots,9\}$, ρ is a probability measure on $\mathcal{A} = 2^{\Omega}$ and λ_ρ is the distribution of the random variable Ψ defined in (3.1), a number $\eta \in [0,\infty[$, represented in the decimal scale by

$$\eta = [\eta] + \sum_{j=1}^{\infty} a_j 10^{-j}, a_j \in \{0,\dots,9\}, \quad \text{for all } j \in \mathbf{N}, \tag{4.1}$$

with $[\eta] = \sup\{r \in \mathbf{N}; r \leq \eta\}$, and so that the sequence of digits does not stabilize at an endless string of 9s, is said to be *simply normal* (*with respect to* λ_ρ) when

$$\lim_{n\to\infty} \frac{S(\eta,r,n)}{n} = \rho(\{r\}) \quad \text{for all } r \in \{0,\dots,9\}, \tag{4.2}$$

where $S(\eta,r,n)$ denotes the total of indexes $i, 1 \leq i \leq n$ such that $a_i = r$. A number η, as in (4.1), is said to be *normal* (*with respect to* λ_ρ) if

$$\lim_{n\to\infty} \frac{S(\eta,B_k,n)}{n} = \prod_{j=1}^{k} \rho(\{b_k\}) \quad \text{for all } k \in \mathbf{N}, \tag{4.3}$$

for each word $B_k = b_1 \cdots b_k$ of k digits, where

$$S(\eta,B_k,n) = \#\left\{i \in \{1,\dots,n\}; a_{i+j-1} = b_j \text{ for every } j = 1,\dots,k\right\}.$$

In particular, if $\rho(\{a\}) = 1/10$ for every $a \in \Omega$, this concept is precisely Borel's original concept of normal numbers, with respect to Lebesgue measure.

Remark 1. This generalized concept arises in some interesting situations. For example, for "degenerate" cases, in which $\rho(\{a\}) = 1$ for some $a \in \{0, 1, \ldots, 9\}$, normal numbers are very special numbers, with a strong preference to the digit $\{a\}$. For example, if $\rho(\{a\}) = 1$ for some $a \in \{0, 1, \ldots, 9\}$, then

$$0, a0aa0aaa0aaaa0aaaaa0...$$

is normal with respect to λ_ρ. To the best of our knowledge, it is not known if, for example, $\sqrt{2}$ is not normal with respect to any of these degenerate cases.

The next result shows that, in virtually all cases, the sets N_{μ_ρ}, of normal numbers in $[0, 1]$, with respect to λ_ρ, are so that $\lambda_\rho(N_{\mu_\rho}) = 1$, but there is one situation in which $\lambda_\rho(N_{\mu_\rho}) = 0$.

Theorem 4. *The measure of the set of normal numbers in* $[0, 1]$, *with respect to* λ_ρ *is*:

(a) 0, *if* $\rho(\{9\}) = 1$.

(b) 1, *if* $\rho(\{9\}) < 1$.

Proof. In the following, N_{μ_ρ} denotes the set of all normal numbers in $[0, 1]$, with respect to λ_ρ and M_{μ_ρ} represents the set of all normal sequences in $\Omega^{\mathbf{N}}$, with respect to μ_ρ.

(a) If $\rho(\{9\}) = 1$, then $\mu_\rho(\{9\}^{\mathbf{N}}) = 1$. We have

$$\Psi^{-1}(N_{\mu_\rho}) = D_1,$$

with

$$D_1 = M_{\mu_\rho} \setminus \left\{(a_j)_{j=1}^{\infty};\ \exists N \in \mathbf{N} \text{ such that } a_n = 9 \text{ for every } n \geq N\right\}.$$

Hence, $N_{\mu_\rho} \in \mathcal{B}$ and, since

$$D_1 \cap \{9\}^{\mathbf{N}} = \phi,$$

we have

$$\lambda_\rho(N_{\mu_\rho}) = \mu_\rho\,(D_1) = 0.$$

(b) If $\rho(\{9\}) < 1$, note that

$$\Psi^{-1}(N_{\mu_\rho}) = M_{\mu_\rho} \cup D_2,$$

with $\mu_\rho(D_2) = 0$. Hence,

$$N_{\mu_\rho} \in \Psi(\mathcal{D}) = \mathcal{B},$$

and, by invoking Proposition 2, we conclude that

$$\lambda_\rho(N_{\mu_\rho}) = \mu_\rho(M_{\mu_\rho}) = 1. \qquad \square$$

Corollary 1 [**1**]. *The set of normal numbers in* $[0,1]$*, with respect to the Lebesgue measure, has measure* 1*.*

Remark 2. The concept of normal numbers can be naturally considered in basis different from 10, and the previous results can be adapted straightforwardly. A number η is said to be absolutely normal if η is normal (with respect to the Lebesgue measure) in any basis $g \in \mathbf{N} \setminus \{1\}$. The extension of this notion by considering different weighted measures in different basis may be an interesting subject of investigation.

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