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SOME REMARKS ON EXTREMALLY RICH C^*-ALGEBRAS

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ABSTRACT. The concept of extension of a partial isometry, which originally appeared in [**3**], is discussed more carefully. For C^*-algebras of real rank zero, an extension property is equivalent to extremal richness or purely infiniteness. We also discuss the relations between extension property and unitary lifting problem.

1. Preliminaries. Throughout this article $\mathcal{E}(A)$, or just $\mathcal{E}$, will denote the set of extreme points of A_1 the unit ball of a unital C^*-algebra A. Recall the characterization by Kadison [**6**] that elements in $\mathcal{E}$ are the partial isometries V such that $(1-VV^*)A(1-V^*V)=0$. We call them extremal partial isometries and call the projections $1-VV^*$, $1-V^*V$ defect projections. In [**3**], Brown and Pedersen defined the notion of extremal richness for a C^*-algebra A which means quasi-invertible elements are dense in A as an analogue of stable rank one for possibly infinite C^*-algebras. (We say T in A is quasi-invertible if T has closed range and the kernel projections of T^* and T are centrally orthogonal in A. For more equivalent definitions, see [**3**, Theorem 1.1].) We denote by A_q^{-1} the set of quasi-invertible elements. As a result, stable rank one C^*-algebras are characterized within the class of extremally rich C^*-algebras by the property that all extreme points of the unit ball are unitaries or $A_q^{-1}=A^{-1}$ where A^{-1} is the set of invertible elements of A. The set of unitary elements in A will be denoted by $\mathcal{U}(A)$, shortly $\mathcal{U}$. Also, A^{**} will denote the enveloping von Neumann algebra of A.

2. Extension property of extremally rich C^*-algebras. If V, W are partial isometries, we say W extends V (write $V \precsim W$) if $W^*W \geq V^*V$, and $V = WV^*V$. We say V has a unitary (respectively

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isometric, extremal) extension if W is a unitary (respectively isometry, extremal partial isometry). It has been well known that a partial isometry in $B(H)$, the set of bounded linear operators on a Hilbert space H, has a maximal extension which is an isometry or a co-isometry, and, more generally, a partial isometry in a von Neumann algebra has an extremal extension. Brown and Pedersen showed this could hold in a larger class of C^*-algebras, namely, extremally rich C^*-algebras which cover stable rank one algebras, von Neumann algebras, and purely infinite simple C^*-algebras (see Corollaries 7 and 9 below).

If we consider the polar decomposition of T as $V|T|$, we do not expect, in general, $V \in A$ (but in A^{**}). For every $\delta > 0$, let E_δ and F_δ be the spectral projection of $|T|$ and $|T^*|$, respectively, corresponding to the open interval (δ, ∞). Note that $VE_\delta = F_\delta V$ is a partial isometry. One of remarkable results of Brown and Pedersen is the following theorem about finding an extension of VE_δ.

Theorem 1 ([**3**, Theorem 2.2]). *Let* $\alpha_q = \operatorname{dist}(T, A_q^{-1})$. *If* $\delta > \alpha_q(T)$, *then* VE_δ *has an extremal extension. Furthermore, if* $\delta < \alpha_q(T)$, *then no such extension exists in* $\mathcal{E}$.

Corollary 2 ([**3**, Corollary 2.3]). *If* $T = V|T|$ *is the polar decomposition of an element of* A, *then each element of* $Vf(|T|)$ *has an extremal decomposition* $Uf(|T|) = Vf(|T|)$, *with* $U \in \mathcal{E}$, *provided that* f *is a continuous function on* $\sigma(|T|)$ *vanishing on* $[0, \delta]$ *for some* $\delta > \alpha_q(T)$.

Proof. Note that $VE_{\delta'}f(|T|) = Vf(|T|)$ for $\delta > \delta' > \alpha_q(T)$. By applying Theorem 1 to $VE_{\delta'}$, we get the conclusion. □

Corollary 3 ([**3**, Proposition 2.6]). *If* V *is a partial isometry in* A, *then* $\alpha_q(V) = 1$, *or else* $\alpha_q(V) = 0$, *in which case* $V = UV^*V = VV^*U$ *for some* $U \in \mathcal{E}$.

Proof. If $\alpha_q(V) < \delta < 1$, then let $f(t) = \max\{(t-\delta)/(1-\delta), 0\}$. Also, let $P = V^*V$. By Corollary 2, there is a $U \in \mathcal{E}$ such that

$$V = Vf(P) = Uf(P) = UV^*V.$$

Since $U(P + \varepsilon I) \in A_q^{-1}$ for any ε, it follows that $\alpha_q(V) = 0$. □

Corollary 4. *If a unital C^*-algebra A is extremally rich, then every partial isometry in A has an extremal extension.*

Proof. Note that when A is extremally rich, $\alpha_q(T) = 0$ for every $T \in A$. Thus the conclusion follows from Corollary 3. □

Corollary 5. *If a unital C^*-algebra A has stable rank one, then every partial isometry has a unitary extension.*

Proof. Note that a C^*-algebra A has stable rank one if and only if it is extremally rich and $\mathcal{E} = \mathcal{U}(A)$ [**10**, Corollary 3.4]. Thus it follows from Corollary 4. □

Recall that a unital C^*-algebra A has real rank zero if and only if it has IP property [**1**]: If p and q are projections A^{**} such that p is compact, q is closed and $pq = 0$, then there is a projection r in A such that $p \le r \le 1-q$. The following result was originally proved by Brown and Pedersen (unpublished). However, since this extension property, in our opinion, is a more powerful condition than the original definition of extremal richness at least for C^*-algebras of real rank zero, we give a proof of this result. Note that a version of this theorem also appeared in [**7**, Proposition 3.4].

Theorem 6 [**2**]. *A unital C^*-algebra A of real rank zero is extremally rich if and only if every partial isometry in A has an extremal extension.*

Proof. "Only if" was proved in Corollary 4.

For "if," suppose A is a C^*-algebra of real rank zero. First, we show that, given $\delta > 0$, $T \in A$, there is an S in A such that S has closed range and $\|T - S\| < \delta$. Let p be the spectral projection of $|T|$ corresponding to $[\delta, \|T\|]$ and q the spectral projection of $|T|$ corresponding to $[0, \delta/2]$ in A^{**}. Then p is compact, q is closed and $pq = 0$. Thus, there is a projection r in A such that $p \le r \le 1 - q$. If we define S as Tr, then $S^*S \ge pT^*Tp \ge \delta^2 p$. It follows that S has closed range and $\|S - T\| \le \|(1 - p)|T|)\| < \delta$. In this case, 0 is an isolated point $\sigma(|S|)$. Therefore, $(0, \varepsilon) \cap \sigma(|S|) = \varnothing$ for some $\varepsilon > 0$. Let $e(t) = 1/t$

if $t > \varepsilon$ with $e(0) = 0$. Then $V = Se(|S|) \in A$ is a partial isometry and $S = V|S|$. By the assumption, then we have an extremal extension $U \in \mathcal{E}$ of V. In addition, we have

$$S = V|S| = VV^*V|S| = UV^*V|S| = U|S|.$$

Note that $U(|S| + \delta I) \in A_q^{-1}$ for any $\delta > 0$; hence, $\alpha_q(T) < 2\delta$ for any $\delta > 0$. Thus we have shown that A_q^{-1} is dense in A and we are done. □

Corollary 7. *If A is a von Neumann algebra, it is extremally rich.*

Proof. It follows from [**2**, Proposition 1.3] and [**10**, Proposition 3.6]. □

Following Cuntz a simple C^*-algebra A is said to be *purely infinite* if it has real rank zero and every non-zero projection is Murray-von Neumann equivalent to a proper projection [**5**]. This implies that for any pair P, Q of non-zero projections, there is a partial isometry V in A such that $V^*V = P$ and $VV^* < Q$. It is well known that a purely infinite simple C^*-algebra is extremally rich [**10**]. We re-prove this fact by showing purely infinite simple C^*-algebras satisfy isometric or co-isometric extension property.

Theorem 8. *Let A be a unital C^*-algebra. A is simple and purely infinite if and only if it has real rank zero and every partial isometry in A has an isometric or a co-isometric extension.*

Proof. If A is purely infinite and simple, it has real rank zero. In addition, if V is a non-zero partial isometry in A, and if we let $P = V^*V$ and $Q = VV^*$, then $I-P$ and $I-Q$ are non-zero projections (if not, we are done). Hence, there is a partial isometry W such that $W^*W = I-P$ and $WW^* < I - Q$. It is easily checked that $V + W$ is an isometry which extends V.

For the converse, it is enough to show that every non-zero projection is infinite. Let P be a non-zero projection in A. Since P itself is a partial isometry, by the assumption, it can have an isometric extension W but not co-isometric in A. Then

$$P = WP = PW^* = PW = W^*P.$$

It follows that $PWW^*P(\precsim P)$ is a projection which is Murray-von Neumann equivalent to $W^*PW = P$, and the other case is similar. □

Corollary 9 [**2**]. *If a unital C^*-algebra is purely infinite and simple, then it is extremally rich.*

Proof. It follows from Theorem 6 and Theorem 8. □

3. Some examples of lifting problems. If A is a C^*-algebra and I is a closed ideal of A, we denote by $\partial_1 : K_1(A/I) \to K_0(I)$ the index map in K-theory. In this section, we observe that either extremal richness or extension property plays a role in lifting unitaries to extremal partial isometries. Since certain extremal rich C^*-algebras have good non-stable K-theoretic properties as stable rank one or purely infinite simple C^*-algebras do [**4**], the following results are also expected under the same spirit.

Proposition 10. *Let A be a* (*non-simple*) *extremally rich C^*-algebra, and let I be a* (*σ-unital*) *ideal of A. Then any unitary u in A/I is liftable to an extremal partial isometry in A. Moreover, if A is a* (*non-simple*) *C^*-algebra of stable rank one, then any unitary u in A/I is liftable to a unitary in A. Consequently, $\partial_1([u]) = 0$ in this case.*

Proof. This result was also pointed out in [**9**, Theorem 3.6] and Nistor also proved the latter statement (see [**8**, Lemma 3]). However, we give our proof emphasizing the extension property. From Theorem 6.1 in [**3**], since A is extremally rich, any extremal partial isometry in A/I can be lifted to a partial isometry. Thus a unitary u which is an extremal partial isometry in A/I can be lifted to a partial isometry V in A. By Theorem 6, there is an extremal partial isometry W in A such that $V \precsim W$. If $\pi : A \to A/I$ is the natural quotient map, $u = \pi(V) = \pi(WV^*V) = \pi(W)u^*u = \pi(W)$.

If u can be lifted to a partial isometry V in A, then it is a standard fact that the index can be computed as $\partial_1([u]) = [1 - V^*V] - [1 - VV^*]$. Thus if u is liftable to a unitary, $\partial_1([u]) = 0$. □

Proposition 11. *Let I be an ideal of an isometrically rich C^p*-algebra A. Assume A has a (strong) cancelation property. If a unitary u in A/I satisfies that $\partial_1([u]) = 0$, then u lifts to a unitary in A.*

Proof. Since A is isometrically rich which is equal to extremally richness for prime C^*-algebras, then any unitary u in A/I is liftable to an extremal partial isometry V in A and V is either an isometry or a co-isometry. Thus $\partial_1([u]) = 0$ and the cancelation property implies that V must be a unitary. □

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