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THE CLIQUE NUMBER OF $\Gamma(\mathbf{Z}_{p^n}(\alpha))$

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ABSTRACT. The zero-divisor graph of a commutative ring with one (say R) is a graph whose vertices are the nonzero zero-divisors of this ring, with two distinct vertices are adjacent in case their product is zero. This graph is denoted by $\Gamma(R)$. We study the zero-divisor graph $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ where p is a prime number, $\mathbf{Z}_{p^n}$ is the set of integers modulo p^n, and $\mathbf{Z}_{p^n}(\alpha) = \{a + bx : a, b \in \mathbf{Z}_{p^n}$ and $x^2 = 0\}$. We find the clique number of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ and the complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ that achieve this clique number.

1. Introduction. Zero-divisor graphs were first introduced by Beck, see [**6**]. Beck was mainly interested in graph coloring. In his work, he let the elements of a commutative ring R be the vertices of the graph, and he let two distinct vertices x and y be adjacent if $xy = 0$. In a subsequent work, Anderson and Livingston introduced the zero-divisor graph of a commutative ring R, see [**3**]. In their definition, they let the nonzero zero-divisors of R be the set of vertices for the graph and two distinct vertices x and y be adjacent if $xy = 0$. Usually the set of zero-divisors of R is denoted by $Z(R)$ and the set of nonzero zero-divisors of R is denoted by $Z^*(R) = Z(R) - \{0\}$. The zero-divisor graph of R, $\Gamma(Z^*(R))$, is usually written $\Gamma(R)$. The definition of the zero-divisor graph that was given by Anderson and Livingston is the one that is used in the literature now. Also in this paper we will use their definition.

Many articles have been done on zero-divisor graphs; the reader is advised to consult [**1, 2, 3, 9**] for more details. Some researchers generalized the idea to commutative semigroups, see [**7, 8**]. Others worked on the noncommutative case where they introduces the directed graph related to the zero-divisors of noncommutative rings. For more information see [**11, 12, 14**].

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A clique in a graph Γ is a complete subgraph of Γ that has the maximum cardinality among all complete subgraphs of Γ. The clique number of a graph Γ is the size of a clique in Γ. In this paper, our goal is to find the clique number of the graph $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. Where p is a prime number, $\mathbf{Z}_{p^n}$ is the set of integers modulo p^n, and $\mathbf{Z}_{p^n}(\alpha) = \{a + bx : a, b \in \mathbf{Z}_{p^n} \text{ and } x^2 = 0\}$. Also we find all complete subgraphs in $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ that achieve the clique number.

Note that the ring $\mathbf{Z}_{p^n}(\alpha)$ can also be described using Nagata's principle of idealization. In general, for a commutative ring R and an R-module M, the idealization of M is the ring $R(+)M$ formed from $R \times M$ using $(r, a) + (s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa)$. Hence, $\mathbf{Z}_{p^n}(\alpha) \approx \mathbf{Z}_{p^n}(+)\mathbf{Z}_{p^n}$. Zero-divisor graph for rings formed using idealization was studied in [**5**]. The diameter and girth of zero-divisor graph of idealization were investigated in [**4, 5**].

Next, we will present an algorithm that computes the zero-divisors of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ and shows how these zero-divisors are adjacent. Basically, this algorithm can be used to build the zero-divisor graph of $\mathbf{Z}_{p^n}(\alpha)$. Part of this algorithm was given by Shaqboua, see [**13**].

2. An algorithm for building $\Gamma(\mathbf{Z}_{p^n}(\alpha))$.

We present the following lemma; one can find the proof in [**1**]. This lemma will characterize the zero-divisors of $R(\alpha)$ where R is a commutative ring with one.

Lemma 1. *Let R be a commutative ring with one not necessarily finite; then $a + bx$ is a zero-divisor in $R(\alpha)$ if and only if a is a zero-divisor in R.*

According to Lemma 1, one can characterize the zero-divisors of $\mathbf{Z}_{p^n}(\alpha)$ by finding the zero-divisors of $\mathbf{Z}_{p^n}$. In [**10**], Joan Krone presented an algorithm to compute the zero-divisor graph of $\mathbf{Z}_k$ for some cases of k. One can find the zero-divisors of $\mathbf{Z}_{p^n}$ by taking the numbers $1, 2, p^{n-1} - 1$ then multiplying those numbers by p. One can divide the zero-divisors into $n - 1$ sets according to how many factors of p each divisor has. We can categorize the zero-divisors of $\mathbf{Z}_{p^n}(\alpha)$ into three types. In the first type we have $n - 1$ sets, and these sets are $S_p, S_{p^2}, \ldots, S_{p^{n-1}}$ where $S_{p^i} = \{sp^i : \gcd(s, p) = 1\}$. The use of Euler's phi-function gives the sizes of the S_{p^i}'s, and one will get $|S_{p^i}| = p^{n-i} - p^{n-(i+1)}$ for $1 \leq i \leq n-1$. The second type consists of n sets, and these sets are $E_{p^0}, E_{p^1}, \ldots, E_{p^{n-1}}$ where $E_{p^i} = \{bx : b \in S_{p^i}\}$. Note that S_{p^0} is the set of units in $\mathbf{Z}_{p^n}$, i.e., the set of elements in $\mathbf{Z}_{p^n}$ that are not divisible by p. We have $|E_{p^i}| = |S_{p^i}| = p^{n-i} - p^{n-(i+1)}$ for $1 \leq i \leq n-1$ and $|E_{p^0}| = p^n - p^{n-1}$. The third type consists of $n(n-1)$ sets. These sets are S_{p^i,p^j}, $i \in \{1, 2, \ldots, n-1\}$ and $j \in \{0, 1, \ldots, n-1\}$ where $S_{p^i,p^j} = \{sp^i + tp^j x : \text{gcd }(s,p) = 1 \text{ and } \text{gcd }(t,p) = 1\}$. Again the use of the Euler's phi-function gives the sizes of the S_{p^i,p^j}'s and one will get $|S_{p^i,p^j}| = (p^{n-i} - p^{n-(i+1)})(p^{n-j} - p^{n-(j+1)})$ for $1 \leq i \leq n-1$ and $0 \leq j \leq n-1$.

We will state how the elements of these types are adjacent. Elements of S_{p^i} are adjacent to elements of $S_{p^{i'}}$ if $i + i' \geq n$. Also elements of S_{p^i} are adjacent to elements of $E_{p^{i'}}$ if $i + i' \geq n$. Again elements of S_{p^i} are adjacent to elements of $S_{p^{i'},p^{j'}}$ if $i + i' \geq n$ and $i + j' \geq n$. For the E_{p^i}'s, the elements of E_{p^i} are adjacent to all the elements of $E_{p^{i'}}$ for any i', and hence $\cup_{i=0}^{n-1} E_{p^i}$ forms a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. Again, elements of E_{p^i} are adjacent to elements of $S_{p^{i'},p^{j'}}$ if $i + i' \geq n$. Observe that no conditions are required on j'. We want to see when the elements of $S_{p^{i'},p^{j'}}$ are adjacent to each other. Consider the two sets S_{p^i,p^j} and S_{p^k,p^m}. Observe that an essential condition in order that some elements in the set S_{p^i,p^j} be connected to some elements in the set S_{p^k,p^m} is that $i + k \geq n$. Now consider the following subcases where $i + k \geq n$ and $p \neq 2$ (we will talk about case $p = 2$ at the end of this section).

1) Suppose that $i + m \geq n$ and $k + j < n$, and take $y_1 \in S_{p^i,p^j}$ and $y_2 \in S_{p^k,p^m}$, say $y_1 = a_1 p^i + b_1 p^j x$ and $y_2 = a_2 p^k + b_2 p^m x$, where gcd $(a_1, p) =$ gcd $(b_1, p) =$ gcd $(a_2, p) =$ gcd $(b_2, p) = 1$. We have $y_1 \cdot y_2 = a_1 a_2 p^{i+k} + a_1 b_2 p^{i+m} x + a_2 b_1 p^{k+j} x = a_2 b_1 p^{k+j} x \neq 0 \pmod{p^n}$. Hence, no element in the set S_{p^i,p^j} is connected to any element in the set S_{p^k,p^m}. Similarly, if $i + m < n$ and $k + j \geq n$ then no element in the set S_{p^i,p^j} is connected to any element in the set S_{p^k,p^m}.

2) If $i + m \geq n$ and $k + j \geq n$, then it is clear that all the elements of S_{p^i,p^j} are connected to all the elements in the set S_{p^k,p^m}.

3) Suppose that $i + m < n$ and $k + j < n$ with $i + m < k + j$. Take $y_1 \in S_{p^i,p^j}$ and $y_2 \in S_{p^k,p^m}$, say $y_1 = a_1 p^i + b_1 p^j x$ and $y_2 = a_2 p^k + b_2 p^m x$ where $\text{gcd}(a_1, p) = \text{gcd}(b_1, p) =$

$\gcd(a_2,p) = \gcd(b_2,p) = 1$. We have $y_1 \cdot y_2 = a_1a_2p^{i+k} + a_1b_2p^{i+m}x + a_2b_1p^{k+j}x = p^{i+m}(a_1b_2 + a_2b_1p^{k+j-i-m})x \pmod{p^n}$. If $p^{i+m}(a_1b_2 + a_2b_1p^{k+j-i-m}) = 0 \pmod{p^n}$, then $p^{n-(i+m)}$ divides $(a_1b_2 + a_2b_1p^{k+j-i-m})$, and hence p divides a_1b_2. So we get p divides a_1 or p divides b_2. But this is impossible because $\gcd(a_1,p) = \gcd(b_2,p) = 1$. Thus, $y_1 \cdot y_2 \neq 0 \pmod{p^n}$. Hence no element in the set S_{p^i,p^j} is connected to any element in the set S_{p^k,p^m}. Similarly, if $i+m < n$ and $k+j < n$ with $i+m > k+j$, then no element in the set S_{p^i,p^j} is connected to any element in the set S_{p^k,p^m}.

4) Suppose that $i+m < n$ and $k+j < n$ with $i+m = k+j$. In this case some elements of the set S_{p^i,p^j} will be connected to some elements in the set S_{p^k,p^m}. To explain that, take $y_1 \in S_{p^i,p^j}$ and $y_2 \in S_{p^k,p^m}$, say $y_1 = a_1p^i + b_1p^jx$ and $y_2 = a_2p^k + b_2p^mx$ where $\gcd(a_1,p) = \gcd(b_1,p) = \gcd(a_2,p) = \gcd(b_2,p) = 1$. We have $y_1 \cdot y_2 = a_1a_2p^{i+k} + a_1b_2p^{i+m}x + a_2b_1p^{k+j}x = p^{i+m}(a_1b_2 + a_2b_1)x \pmod{p^n}$. If $a_1b_2 + a_2b_1 = 0 \pmod{p^{n-i-m}}$, then y_1 is adjacent to y_2 and otherwise y_1 and y_2 are not adjacent. For instance, if $a_1 = a_2 = 1$, $b_1 = p^{n-i-m}+1$, $b_2 = p^{n-i-m}-1$, then $a_1b_2 + a_2b_1 = 0 \pmod{p^{n-i-m}}$, and hence y_1 and y_2 are adjacent. On the other hand, if $a_1 = a_2 = b_1 = b_2 = 1$, then $a_1b_2 + a_2b_1 = 1 \pmod{p^{n-i-m}}$, and hence y_1 and y_2 are not adjacent.

The following simple example explains the zero-divisor graph of $\mathbf{Z}_{3^3}(\alpha)$.

Example 1. Consider the zero-divisor graph $\Gamma(\mathbf{Z}_{3^3}(\alpha))$. We have $S_{3^1} = \{s3^1 : \gcd(s,3) = 1\} = \{3,6,12,15,21,24\}$ and $S_{3^2} = \{s3^2 : \gcd(s,3) = 1\} = \{9,18\}$. We have the following E_{3^i}'s: $E_{3^1} = \{bx : b \in S_{3^1}\} = \{3x,\ 6x,\ 12x,\ 15x,\ 21x,\ 24x\}$, $E_{3^2} = \{bx : b \in S_{3^2}\} = \{9x, 18x\}$, and $E_{3^0} = \{bx : b \text{ is not divisible by } 3\} = \{x,\ 2x,\ 4x,\ 5x,\ 7x,\ 8x,\ 10x,\ 11x,\ 13x,\ 14x,\ 16x,\ 17x,\ 19x,\ 20x,\ 22x,\ 23x,\ 25x,\ 26x\}$. Now, we will state the $S_{3^i,3^j}$'s. We have $S_{3,3^0} = \{3s+tx : \gcd(s,3) = 1 \text{ and } \gcd(t,3) = 1\} = \{a+bx : a \in S_{3^1} \text{ and } b \in B = \{1,\ 2,\ 4,\ 5,\ 7,\ 8,\ 10,\ 11,\ 13,\ 14,\ 16,\ 17,\ 19,\ 20,\ 22,\ 23,\ 25,\ 26\}\}$, $S_{3^2,3^0} = \{9s+tx : \gcd(s,3) = 1 \text{ and } \gcd(t,3) = 1\} = \{a+bx : a \in S_{3^2} \text{ and } b \in B\}$, $S_{3,3} = \{3s+3tx : \gcd(s,3) = 1 \text{ and } \gcd(t,3) = 1\} = \{a+bx : a \in S_{3^1} \text{ and } b \in S_{3^1}\}$, $S_{3^2,3} = \{9s+3tx : \gcd(s,3) = 1 \text{ and } \gcd(t,3) = 1\} = \{a+bx : a \in S_{3^2} \text{ and } b \in S_{3^1}\}$, $S_{3,3^2} = \{3s+9tx : \gcd(s,3) = 1 \text{ and } \gcd(t,3) = 1\} = \{a+bx : a \in S_{3^1} \text{ and } b \in$

$S_{3^2}\}$, and $S_{3^2,3^2} = \{9s + 9tx : \gcd(s,3) = 1 \text{ and } \gcd(t,3) = 1\} = \{a + bx : a \in S_{3^2} \text{ and } b \in S_{3^2}\}$. Observe that $E_{3^0} \cup E_{3^1} \cup E_{3^2}$ form a complete subgraph, the elements of S_{3^2} are adjacent to all the elements of

$$\bigcup_{i=1}^{2} S_{3^i} \cup \bigcup_{i=1}^{2} E_{3^i} \cup \bigcup_{i,j=1}^{2} S_{3^i,3^j},$$

and the elements of S_{3^1} are adjacent to all the elements of $S_{3^2} \cup E_{3^2} \cup S_{3^2,3^2}$. Now, we state the adjacency between the elements of the $S_{3^i,3^j}$'s. We have the following: the elements of $S_{3^2,3^2}$ are adjacent to all the elements of

$$\bigcup_{i=1}^{2} S_{3^i} \cup \bigcup_{i=1}^{2} E_{3^i} \cup \bigcup_{i,j=1}^{2} S_{3^i,3^j},$$

the elements of $S_{3^2,3^1}$ are adjacent to all the elements of

$$S_{3^2} \cup \bigcup_{i=1}^{2} E_{3^i} \cup S_{3^2,3^1} \cup S_{3^2,3^2},$$

the elements of $S_{3^1,3^2}$ are adjacent to all the elements of $S_{3^2} \cup E_{3^2} \cup S_{3^2,3^2}$, the elements of $S_{3^1,3^1}$ are adjacent to all the elements of $S_{3^2} \cup E_{3^2} \cup S_{3^2,3^2}$, the elements of $S_{3^2,3^0}$ are adjacent to all the elements of $\cup_{i=1}^{2} E_{3^i}$, and the elements of $S_{3^1,3^0}$ are adjacent to all the elements of E_{3^1}. Also we have some of the elements of $S_{3^2,3^0}$ are adjacent to each other and some of the elements of $S_{3^1,3^0}$ are adjacent to some of the elements of $S_{3^2,3^1}$. For instance, the element $9 + 3x \in S_{3^2,3^1}$ is adjacent to all the elements in the set $A = \{(3 + 9i) + (2 + 3j)x : i \in \{0, 1, 2\} \text{ and } j \in \{0, \ldots, 8\}\} \cup \{(6+9i)+(1+3j)x : i \in \{0, 1, 2\} \text{ and } j \in \{0, \ldots, 8\}\} \subseteq S_{3^1,3^0}$.

Similarly, we divide the zero-divisors of $\mathbf{Z}_{2^n}(\alpha)$ into E_{2^i}'s, S_{2^i}'s and $S_{2^i,2^j}$'s. The adjacency between different elements of E_{2^i}'s, S_{2^i}'s and $S_{2^i,2^j}$'s in $\mathbf{Z}_{2^n}(\alpha)$ is similar to the adjacency between different elements of E_{p^i}'s, S_{p^i}'s and S_{p^i,p^j}'s in $\mathbf{Z}_{p^n}(\alpha)$ except for one case. We will explain this case next. Take $S_{2^i,2^j}$ and $S_{2^l,2^m}$ with $i + l \geq n$ and $i + m = j + l = n - 1$. Take $y_1 \in S_{2^i,2^j}$ and $y_2 \in S_{2^i,2^j}$, say $y_1 = a_1 2^i + a_2 2^j x$ and $y_2 = b_1 2^l + b_2 2^m x$ where a_1, a_2, b_1 and b_2 are odd numbers. We have $y_1 \cdot y_2 = (a_1 b_2 2^{i+m} + a_2 b_1 2^{j+l})x =$

$(a_1b_2 + a_2b_1)2^{i+m}x = (a_1b_2 + a_2b_1)2^{n-1}x \pmod{2^n}$. Since a_1, a_2, b_1 and b_2 are odd numbers, we get $a_1b_2 + a_2b_1$ is an even number, and hence $(a_1b_2 + a_2b_1)2^{n-1} = 0 \pmod{2^n}$. So $y_1 \cdot y_2 = 0 \pmod{2^n}$, and hence y_1 and y_2 are adjacent. Thus, the elements of the set $S_{2^i,2^j}$ are adjacent to all the elements of $S_{2^l,2^m}$. We give the following example on $\mathbf{Z}_{2^n}(\alpha)$.

Example 2. Consider the graph $\Gamma(\mathbf{Z}_{2^3}(\alpha))$; one can divide the zero-divisors of $\mathbf{Z}_{2^3}(\alpha)$ into E_{2^i}'s, S_{2^i}'s and $S_{2^i,2^j}$'s. Observe that the elements of $S_{2^2,2^0}$ form a complete subgraph, and the elements of $S_{2^1,2^0}$ are adjacent to all the elements of $S_{2^2,2^1}$. Other than that, the adjacency between the elements in $\Gamma(\mathbf{Z}_{2^3}(\alpha))$ is similar to that of the previous example.

3. The clique number of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. The following lemma will be used in calculating the clique number of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$, where p is an odd prime.

Lemma 2. *In $R = \Gamma(\mathbf{Z}_{p^n}(\alpha))$ where p is an odd prime, suppose that $y_1 \in S_{p^i,p^j}$, $y_2 \in S_{p^l,p^m}$ and $y_3 \in S_{p^u,p^v}$, with $i + l \geq n$, $i + u \geq n$, $l + u \geq n$ and $i + m = j + l = i + v = u + j = l + v = u + m \leq n - 1$. Moreover, assume that y_1, y_2 and y_3 are distinct elements, and S_{p^i,p^j}, S_{p^l,p^m} and S_{p^u,p^v} are not necessarily distinct sets. Then $\{y_1, y_2, y_3\}$ cannot form a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$.*

Proof. We have $y_1 = a_1p^i + b_1p^jx$, $y_2 = a_2p^l + b_2p^mx$ and $y_3 = a_3p^u + b_3p^vx$ where $\gcd(a_1, p) = 1$, $\gcd(a_2, p) = 1$, $\gcd(a_3, p) = 1$, $\gcd(b_1, p) = 1$, $\gcd(b_2, p) = 1$ and $\gcd(b_3, p) = 1$. Suppose that $\{y_1, y_2, y_3\}$ form a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$; then $y_1 \cdot y_2 = 0 \pmod{p^n}$, $y_1 \cdot y_3 = 0 \pmod{p^n}$ and $y_2 \cdot y_3 = 0 \pmod{p^n}$. Since $y_1 \cdot y_3 = 0 \pmod{p^n}$, we get $a_1b_3 + a_3b_1 = 0 \pmod{p^{n-(i+v)} = p^{n-(i+m)}}$. Since b_1 is invertible in $\mathbf{Z}_{p^{n-(i+m)}}$, we get $a_3 = -a_1b_3b_1^{-1} \pmod{p^{n-(i+m)}}$. In the same way, since $y_2 \cdot y_3 = 0 \pmod{p^n}$, we get $a_3 = -a_2b_3b_2^{-1} \pmod{p^{n-(l+v)} = p^{n-(i+m)}}$. So, we have $a_3 = -a_1b_3b_1^{-1} = -a_2b_3b_2^{-1} \pmod{p^{n-(i+m)}}$, and since b_3 is invertible in $\mathbf{Z}_{p^{n-(i+m)}}$, we get $a_1b_2 = a_2b_1 \pmod{p^{n-(i+m)}}$. Since, $y_1 \cdot y_2 = 0 \pmod{p^n}$, we get $a_1b_2 + a_2b_1 = 0 \pmod{p^{n-(i+m)}}$, and hence $a_1b_2 = -a_2b_1 \pmod{p^{n-(i+m)}}$. So we have $a_1b_2 = a_2b_1 \pmod{p^{n-(i+m)}}$ and $a_1b_2 = -a_2b_1 \pmod{p^{n-(i+m)}}$. Combining these together we get $2a_1b_2 = 0 \pmod{p^{n-(i+m)}}$. Since p does not divide 2, a_1 or b_2, we get a contradiction. Thus, $\{y_1, y_2, y_3\}$ cannot form a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. □

Basically, this lemma tells us that we cannot have a complete subgraph of order greater than two in $S_{p^i,p^j} \cup S_{p^l,p^m} \cup S_{p^u,p^v} \subseteq \Gamma(\mathbf{Z}_{p^n}(\alpha))$, where $i+l \geq n$, $i+u \geq n$, $l+u \geq n$ and $i+m = j+l = i+v = u+j = l+v = u+m \leq n-1$. In the next lemma we state a similar result for $\Gamma(\mathbf{Z}_{2^n}(\alpha))$.

Lemma 3. *Suppose that $y_1 \in S_{2^i,2^j} \subseteq \Gamma(\mathbf{Z}_{2^n}(\alpha))$, $y_2 \in S_{2^l,2^m} \subseteq \Gamma(\mathbf{Z}_{2^n}(\alpha))$ and $y_3 \in S_{2^u,2^v} \subseteq \Gamma(\mathbf{Z}_{2^n}(\alpha))$, with $i+l \geq n$, $i+u \geq n$, $l+u \geq n$ and $i+m = j+l = i+v = u+j = l+v = u+m \leq n-2$. Moreover, assume that y_1, y_2 and y_3 are distinct elements, and S_{p^i,p^j}, S_{p^l,p^m} and S_{p^u,p^v} are not necessarily distinct sets. Then $\{y_1, y_2, y_3\}$ cannot form a complete subgraph of $\Gamma(\mathbf{Z}_{2^n}(\alpha))$.*

The proof is similar to that of Lemma 2, and we will omit it. □

We give some complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$, where $p \geq 3$, and n is an integer that is greater than one. Let

$$L_m = \bigcup_{i=\lceil n/2\rceil+m}^{n-1} \bigcup_{j=\lfloor n/2\rfloor-m}^{n-1} S_{p^i,p^j} \cup \bigcup_{k=\lceil n/2\rceil+m}^{n-1} S_{p^k} \cup \bigcup_{l=\lfloor n/2\rfloor-m}^{n-1} E_{p^l}$$

with $m \in \{0, 1, \ldots, \lfloor n/2\rfloor - 1\}$ and $N = \cup_{l=0}^{n-1} E_{p^l}$, where $\lfloor\ \rfloor$ is the greatest integer function, and $\lceil\ \rceil$ is the least integer function. Observe that each one of the L_m's is a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$, and N is also a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. Ultimately, we will show that each one of the L_m's and N have the maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. First, we want to calculate the sizes of the L_m's and N. One can view the elements of L_m in the plane $\mathbf{Z}_{p^n} \times \mathbf{Z}_{p^n}$. The entire set consists of the nonzero points of the form (ap^s, bp^t) where $s = \lceil n/2\rceil + m$, $0 \leq a < p^{n-s}$, $t = \lfloor n/2\rfloor - m$, and $0 \leq b < p^{n-t}$, (but not both a and b are zeros). Adding $(0,0)$ gives a rectangular grid. Thus, $|L_m| = p^{2n-(s+t)} - 1 = p^n - 1$. For N, it is clear that N corresponds to the nonzero elements of $\mathbf{Z}_{p^n}$ and hence $|N| = p^n - 1$.

We will present a sequence of lemmas that will be used in showing that the L_i's and N have a maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$.

Lemma 4. *Suppose that H is a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ that contains an element of $\cup_{k=1}^{\lceil n/2\rceil-1} S_{p^k}$, where p is an odd prime and n is an integer that is greater than one. Then $|H| < p^n - 1$. Moreover, H cannot have maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$.*

Proof. Suppose that $y \in S_{p^k}$ where $1 \le k \le \lceil n/2\rceil - 1$ (implying $n > 2$) and $y \in H$; then $y = ap^k$ where $\gcd(a,p) = 1$. Since H is a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$, then H is a subset of $\{y\} \cup \mathrm{Ann}\,(y)\backslash\{0\}$. We have $\mathrm{Ann}\,(y) = \mathrm{Ann}\,(ap^k) = \{bp^{n-k} + cp^{n-k}x :\ 0 \le b < p^k,\ 0 \le c < p^k\}$. Thus, $|H| \le |\mathrm{Ann}\,(y)| = p^{2k} < p^n - 1$ (with $1 \le k \le \lceil n/2\rceil - 1$). Since $|L_m| = p^n - 1$ and each one of the L_m's is complete, then H cannot have the maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. $\square$

Lemma 5. *Suppose that H is a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ that contains an element of $\cup_{i=1}^{\lceil n/2\rceil-1} \cup_{j=0}^{n-1} S_{p^i,p^j}$, where p is an odd prime and n is an integer that is greater than one. Then $|H| < p^n - 1$. Moreover, H cannot have the maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$.*

Proof. Suppose that $z \in H$ and $z = ap^i + bp^jx$ where $\gcd(a,p) = \gcd(b,p) = 1$ and $1 \le i \le \lceil n/2\rceil - 1$ (implying $n > 2$). If $j > i$, then $\mathrm{Ann}\,(z) = \{cp^{n-i} + dp^{n-i}x :\ 0 \le c < p^i,\ 0 \le d < p^i\}$. Thus, $|H| \le |\mathrm{Ann}\,(z)| = p^{2i} \le p^{n-1} < p^n - 1$.

For the other case, if $j < i$, then there are $p^{i+j}-1$ $(< p^{n-1}-1)$ nonzero annihilators of z of the form $y = c'p^{n-j} + d'p^{n-i}x$ with $0 \le c' < p^j$ and $0 \le d' < p^i$. The other type of annihilator is an element of the form $w = ep^k + fp^lx$ where $i+k \ge n > i+l = j+k$, $\gcd(e,p) = \gcd(f,p) = 1$, and $af + be = 0 \pmod{p^{n-(i+l)}}$. By Lemma 2, H contains at most one such w. Hence, at most p^{i+j} nonzero annihilators of z are in H. As $n-1 \geq 2$, $|H| \leq p^{i+j}+1 < p^{n-1}$. In both cases $|H| < p^n - 1$, and therefore H cannot have the maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. $\square$

Lemma 6. *Suppose that H is a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ (p is an odd prime and n is an integer that is greater than one) that contains an element of the form $z = ap^i + bp^j x$ with $\gcd(a,p) = \gcd(b,p) = 1$, $i \geq \lceil n/2 \rceil$, and $i+j < n$. Then $|H| < p^n - 1$. Moreover, H cannot have the maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$.*

Proof. Suppose that $z \in H$ and $z = ap^i + bp^j x$ where $\gcd(a,p) = \gcd(b,p) = 1$, $i \geq \lceil n/2 \rceil$ and $i+j < n$. Note that $j < i$ and the number of annihilators of the form $y = cp^{n-j} + dp^{n-i}x$ with $0 \leq c < p^j$ and $0 \leq d < p^i$ is $p^{i+j} \leq p^{n-1}$. By Lemma 2, z has at most one other annihilator in H. Thus $H \leq p^{n-1} + 1 < p^n - 1$ and H cannot have the maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. $\square$

In the next lemma, we show that if M is a subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ that contains N or any of the L_i's properly, then M is not complete.

Lemma 7. *If $M = N \cup \{y\}$ where $y \notin N$ or $M = L_m \cup \{y\}$ where $y \notin L_m$ and $m \in \{0, 1, \dots, \lfloor n/2 \rfloor - 1\}$, then M is not a complete subgraph of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ (p is an odd prime).*

Proof. The elements of E_{p^0} are not adjacent to any element of $\cup_{i,j} S_{p^i,p^j}$ or $\cup_{k=1}^{n-1} S_{p^k}$. Hence, we cannot add any elements to N and get a complete subgraph. For the L_i's, suppose that

$$\begin{aligned} M &= L_m \cup \{y\} \\ &= \bigcup_{i=\lceil n/2 \rceil + m}^{n-1} \ \bigcup_{j=\lfloor n/2 \rfloor - m}^{n-1} S_{p^i,p^j} \cup \bigcup_{k=\lceil n/2 \rceil + m}^{n-1} S_{p^k} \cup \bigcup_{l=\lfloor n/2 \rfloor - m}^{n-1} E_{p^l} \cup \{y\} \end{aligned}$$

where $y \notin L_m$ and $m \in \{0, 1, \dots, \lfloor n/2 \rfloor - 1\}$. If $y \in \cup_{k=1}^{\lfloor n/2 \rfloor - m - 1} E_{p^k}$, then y is not adjacent to any element of $S_{p^{\lceil n/2 \rceil + m}}$, and if $y \in \cup_{k=1}^{\lceil n/2 \rceil + m - 1} S_{p^k}$, then y is not adjacent to any element of $E_{p^{\lfloor n/2 \rfloor - m}}$.

If

$$y \in \left(\bigcup_{i,j} S_{p^i,p^j} \right) - \left(\bigcup_{i=\lceil n/2 \rceil + m}^{n-1} \bigcup_{j=\lfloor n/2 \rfloor - m}^{n-1} S_{p^i,p^j} \right),$$

then either $y \in S_{p^i,p^j}$ where $i < \lceil n/2 \rceil + m$ or $y \in S_{p^i,p^j}$ where $i \geq \lceil n/2 \rceil + m$ and $i + j < n$. In the former case y is not adjacent to any element of $E_{p^{\lfloor n/2 \rfloor - m}}$, and in the latter case y is not adjacent to any element of $S_{p^{\lceil n/2 \rceil + m}}$. □

In the next theorem, we will show that every one of the complete subgraphs

$$L_m = \bigcup_{i=\lceil n/2 \rceil + m}^{n-1} \bigcup_{j=\lfloor n/2 \rfloor - m}^{n-1} S_{p^i,p^j} \cup \bigcup_{k=\lceil n/2 \rceil + m}^{n-1} S_{p^k} \cup \bigcup_{l=\lfloor n/2 \rfloor - m}^{n-1} E_{p^l}$$

where $m \in \{0, 1, \ldots, \lfloor n/2 \rfloor - 1\}$ and $N = \cup_{l=0}^{n-1} E_{p^l}$ has the maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$.

Theorem 1. *If p is an odd prime and n is an integer that is greater than one, then every one of the complete subgraphs L_m where $m \in \{0, 1, \ldots, \lfloor n/2 \rfloor - 1\}$ and $N = \cup_{l=0}^{n-1} E_{p^l}$ has the maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$.*

Proof. Suppose that M is a complete subgraph that has a maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. We will show that either M is N or M is one of the L_i's. Observe that $E_{p^{n-1}} \subseteq M$, and this is because M is a complete subgraph with maximum cardinality and the elements of $E_{p^{n-1}}$ are adjacent to any element of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$. Hence, M intersects $\cup_{l=0}^{n-1} E_{p^l}$. Let $l_0 = \min\{l : 0 \leq l \leq n-1 \text{ and } M \cap E_{p^l} \neq \varnothing\}$. If $l_0 = 0$, then M contains some elements of E_{p^0}. Hence, M contains no elements of

$$\bigcup_{i=1}^{n-1} \bigcup_{j=0}^{n-1} S_{p^i,p^j} \cup \bigcup_{k=1}^{n-1} S_{p^k},$$

and this is because M is a complete subgraph and no elements of

$$\bigcup_{i=1}^{n-1} \bigcup_{j=0}^{n-1} S_{p^i,p^j} \cup \bigcup_{k=1}^{n-1} S_{p^k}$$

are adjacent to any element of E_{p^0}. Thus, M is a subset of N. Since M is a complete subgraph with maximum cardinality we get $M = N$. If $0 < l_0 \leq \lfloor n/2 \rfloor$, then the use of Lemmas 5, 6 and 7 ensures that M will be

$$\bigcup_{i=n-l_0}^{n-1} \bigcup_{j=l_0}^{n-1} S_{p^i,p^j} \cup \bigcup_{k=n-l_0}^{n-1} S_{p^k} \cup \bigcup_{l=l_0}^{n-1} E_{p^l} = L_{(\lfloor n/2 \rfloor - l_0)}.$$

If $l_0 > \lfloor n/2 \rfloor$, then the use of Lemmas 4, 5, and 6 ensures that M is a subset of one of the elements of the set

$$\left\{ \bigcup_{i=\lceil n/2 \rceil + r}^{n-1} \bigcup_{j=\lfloor n/2 \rfloor - r}^{n-1} S_{p^i,p^j} \cup \bigcup_{k=\lceil n/2 \rceil + r}^{n-1} S_{p^k} \cup \bigcup_{l=l_0}^{n-1} E_{p^l} : r \in \left\{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1 \right\} \right\}.$$

And these elements are proper subsets of $L_0, L_1, \ldots, L_{\lfloor n/2 \rfloor - 1}$. Since each one of the L_i's is a complete subgraph and M is a complete subgraph with maximal cardinality then this case will not happen. Hence, either $l_0 = 0$ and we get $M = N$ or $0 < l_0 \leq \lfloor n/2 \rfloor$ and we get that M is one of the L_i's. □

Hence the clique number of $\Gamma(\mathbf{Z}_{p^n}(\alpha))$ is $p^n - 1$, and this clique number is attained by each one of the L_i's and N, where p is an odd prime and n is an integer that is greater than one. Indeed, Theorem 1 shows that the only complete subgraphs with maximum cardinality are N and L_m where $m \in \{0, 1, \ldots, \lfloor n/2 \rfloor - 1\}$.

If $S_{2^i,2^j} \subseteq \Gamma(\mathbf{Z}_{2^n}(\alpha))$ with n is an integer that is greater than one, $i \geq \lceil n/2 \rceil$ and $i + j = n - 1$, then using Lemma 3, $S_{2^i,2^j}$ is a complete subgraph of $\Gamma(\mathbf{Z}_{2^n}(\alpha))$. This will increase the number of complete subgraphs with maximum cardinality in $\Gamma(\mathbf{Z}_{2^n}(\alpha))$. Take

$$L_m = \bigcup_{i=\lceil n/2 \rceil + m}^{n-1} \bigcup_{j=\lfloor n/2 \rfloor - m}^{n-1} S_{2^i,2^j} \cup \bigcup_{k=\lceil n/2 \rceil + m}^{n-1} S_{2^k} \cup \bigcup_{l=\lfloor n/2 \rfloor - m}^{n-1} E_{2^l}$$

where $m \in \{0, 1, \dots, \lfloor n/2 \rfloor - 1\}$,

$$I_s = S_{2^{(\lceil n/2\rceil + s-1)}, 2^{(\lfloor n/2\rfloor - s)}} \cup \bigcup_{i=\lceil n/2\rceil + s}^{n-1} \bigcup_{j=\lfloor n/2\rfloor - s+1}^{n-1} S_{2^i,2^j} \cup \bigcup_{k=\lceil n/2\rceil + s}^{n-1} S_{2^k} \cup \bigcup_{l=\lfloor n/2\rfloor - s+1}^{n-1} E_{2^l}$$

where $s \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$, and $N = \cup_{l=0}^{n-1} E_{2^l}$. Each one of the L_i's, I_s's and N is a complete subgraph of $\Gamma(\mathbf{Z}_{2^n}(\alpha))$. In the same way as the p^n case, one can show that the cardinality of each one of the L_i's and N is equal to $2^n - 1$. With regard to the sizes of the sets I_s, the size of the set $S_{2^{(\lceil n/2\rceil + s-1)}, 2^{(\lfloor n/2\rfloor - s)}}$ is equal to $(2^{(n-\lceil n/2\rceil - s+1)} - 2^{(n-\lceil n/2\rceil - s+1-1)})(2^{(n-\lfloor n/2\rfloor + s)} - 2^{(n-\lfloor n/2\rfloor + s-1)}) = 2^{n-1}$. The rest of I_s (plus $(0,0)$) forms a rectangular grid with 2^{n-1} points. Hence, $|I_s| = 2^n - 1$. In a similar fashion to the p^n case, one can show that each one of the L_i's, I_s's and N has a maximum cardinality among all complete subgraphs of $\Gamma(\mathbf{Z}_{2^n}(\alpha))$. Hence, the clique number of $\Gamma(\mathbf{Z}_{2^n}(\alpha))$ is equal to $2^n - 1$, and this clique number is achieved by each one of the L_i's, I_s's and N. Next we will give two examples.

Example 3. In $\Gamma(\mathbf{Z}_{2^2}(\alpha)$, we have $E_{2^0} = \{x, 3x\}$, $E_{2^1} = \{2x\}$, $S_{2^1} = \{2\}$, $S_{2^1,2^0} = \{2+x, 2+3x\}$, $S_{2^1,2^1} = \{2+2x\}$. The subgraphs $N = E_{2^0} \cup E_{2^1}$, $L_0 = S_{2^1,2^1} \cup S_{2^1} \cup E_{2^1}$ and $I_1 = S_{2^1,2^0} \cup E_{2^1}$ are the complete subgraphs with maximum cardinality among all complete subgraphs in $\Gamma(\mathbf{Z}_{2^2}(\alpha))$. The clique number of $\Gamma(\mathbf{Z}_{2^2}(\alpha))$ is equal to $2^2 - 1 = 3$.

Example 4. The subgraphs $N = E_{2^0} \cup E_{2^1} \cup E_{2^2}$, $L_0 = S_{2^2,2^1} \cup S_{2^2,2^2} \cup S_{2^2} \cup E_{2^1} \cup E_{2^2}$ and $I_1 = S_{2^2,2^0} \cup E_{2^1} \cup E_{2^2}$ are the complete subgraphs with maximum cardinality among all complete subgraphs in $\Gamma(\mathbf{Z}_{2^3}(\alpha))$. The clique number of $\Gamma(\mathbf{Z}_{2^3}(\alpha))$ is equal to $2^3 - 1 = 7$, and this clique number is achieved by N, L_0 and I_1.

We did not mention anything about the case $\Gamma(\mathbf{Z}_{p^1}(\alpha))$, i.e., the case when $n = 1$. In this case $\Gamma(\mathbf{Z}_{p^1}(\alpha)) = \{ax : a \in \mathbf{Z}_{p^1}^*\}$ which is a complete subgraph, and hence the clique number is $p - 1$.

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