ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 5, 2009

UNIQUE DECOMPOSITION PROPERTY AND EXTREME POINTS

MIROSLAV BAČÁK

ABSTRACT. This paper presents a solution to an open problem posed by Font and Sanchis in [**1**]. We will show that the unique decomposition property of a function space is necessary to obtain a full characterization of extreme points of the unit ball in a dual space.

1. Introduction. We first introduce some notation and basic facts concerning Choquet's theory. We refer the reader to [**2**] for the details. Let X be a Hausdorff compact space, the symbol $\mathcal{C}$ denotes the set of constant functions on X. A subspace $\mathcal{H}$ of the space of continuous functions $C(X)$ is called a *function space* on X provided it separates points of X and $\mathcal{C} \subset \mathcal{H}$. Notice that the function space is not necessarily closed. The dual space $(C(X))^*$, according to the Riesz representation theorem, is considered to be the set of Radon measures on X, denoted by $\mathcal{M}(X)$. The set of probability Radon measures will be denoted by $\mathcal{M}^1(X)$. We define the positive part of the closed unit ball in the dual space $\mathcal{H}^*$ as

$$B^+_{\mathcal{H}^*} = \{\psi \in \mathcal{H}^* : 0 \leq \psi,\ \|\psi\| \leq 1\}$$

and the *state space* of $\mathcal{H}$ as

$$S(\mathcal{H}) = \{\psi \in B^+_{\mathcal{H}^*} : \|\psi\| = 1\}.$$

It is well known that $\mathcal{H}^*$ is isometrically isomorphic to the quotient space

$$\mathcal{M}(X)/\mathcal{H}^\perp$$

and that

$$S(\mathcal{H}) = \pi(\mathcal{M}^1(X)).$$

Here π stands for the quotient mapping from $\mathcal{M}(X)$ to $\mathcal{H}^*$. Further, we define a homeomorphic embedding $\phi : X \to S(\mathcal{H})$ mapping to every

Keywords and phrases. Unique decomposition property, Choquet boundary, state space.
Received by the editors on October 23, 2006, and in revised form on March 8, 2007.
DOI:10.1216/RMJ-2009-39-5-1397

$x \in X$ a functional ϕ_x by $\phi_x = \pi(\varepsilon_x)$, where ε_x is the Dirac measure at the point x. The Choquet boundary of $\mathcal{H}$ is denoted by $\mathrm{Ch}_{\mathcal{H}}(X)$. The set of extreme points of a convex set M is denoted by $\mathrm{ext}\,(M)$. For the state space we have

$$\mathrm{ext}\,(S(\mathcal{H})) = \{\phi_x \in S(\mathcal{H}) : x \in \mathrm{Ch}_{\mathcal{H}}(X)\}.$$

The symbol $\mathbf{1_X}$ stands for the function identically equal to 1 on X. We denote $\mathcal{H}_d$ the quotient space $\mathcal{H}/\mathcal{C}$ and let $\widehat{\pi}$ be the quotient mapping

$$\widehat{\pi} : \mathcal{H} \longrightarrow \mathcal{H}/\mathcal{C}.$$

We define the *diameter norm* for functions in $\mathcal{H}_d$ as

$$\|\widehat{\pi}(f)\|_d = \mathrm{diam}\,(\mathcal{R}(f)),$$

where $\mathcal{R}(f)$ stands for the range of the function f. It is easy to see that

$$\mathrm{diam}\,(\mathcal{R}(f)) = 2 \inf_{\alpha \in \mathbf{R}} \{\|f - \alpha \cdot \mathbf{1_X}\|\} = 2\|f\|$$

for every $f \in \mathcal{H}$.

Now, we introduce a lemma, which follows immediately from Theorem 4.9 in [**3**]. This lemma enables us to identify the space $(\mathcal{H}_d)^*$ with a subspace of $\mathcal{H}^*$. In what follows, we use this convention without explicit mention.

Lemma. *The space* $(\mathcal{H}_d)^*$ *is isometrically isomorphic to* $\{\psi \in \mathcal{H}^* : \psi(\mathbf{1_X}) = 0\}$.

Further, we have

$$2\|\psi\|_{d^*} = \|\psi\|$$

for every $\psi \in (\mathcal{H}_d)^*$, where $\|\cdot\|_{d^*}$ is the (diameter) norm in $(\mathcal{H}_d)^*$ defined as

$$\|\psi\|_{d^*} = \sup_{\|f\|_d \leq 1} \frac{|\psi(f)|}{\|f\|_d}.$$

Then the closed unit ball in $((\mathcal{H}_d)^*, \|\cdot\|_{d^*})$ is denoted by $B_{\mathcal{H}_d^*}$.

The main aim of this paper is to provide a full characterization of the extreme points $\mathrm{ext}\,(B_{\mathcal{H}_d^*})$. In the general situation, when the function space $\mathcal{H}$ does not necessarily have any other properties, we have the following assertion.

Proposition. *Let* $\mathcal{H}$ *be a function space on a compact space* X*. Then*

$$\operatorname{ext}(B_{\mathcal{H}_d^*}) \subset \{\phi_x - \phi_y : x,\ y \in \operatorname{Ch}_{\mathcal{H}}(X),\ x \neq y\}.$$

We refer the reader to [**1**] for a proof. There is also an example showing that the inclusion can be strict. So it is clear that we have to impose some tacit assumptions on $\mathcal{H}$ in order to obtain a full characterization of $\operatorname{ext}(B_{\mathcal{H}_d^*})$.

Definition. We say that $\mathcal{H}$ satisfies the *unique decomposition property* (UDP) if for every $x, y \in \operatorname{Ch}_{\mathcal{H}}(X)$, $x \neq y$ and $\psi_1, \psi_2 \in B^+_{\mathcal{H}^*}$ such that

$$\phi_x - \phi_y = \psi_1 - \psi_2 \quad \text{and} \quad \|\phi_x - \phi_y\| = \|\psi_1\| + \|\psi_2\|,$$

there exist $z, t \in \operatorname{Ch}_{\mathcal{H}}(X)$ such that $\psi_1 = \phi_z$, $\psi_2 = \phi_t$.

Remarks. Let us note two important things about (UDP).

- Such decomposition of $\phi_x - \phi_y$ always exists: For a Hahn-Banach extension $\Phi \in (C(X))^*$ of the functional $\phi_x - \phi_y \in \mathcal{H}^*$ we can consider its positive and negative variations Φ^+, Φ^-. Let us denote the restrictions of these two functionals on $\mathcal{H}$ by ψ^+, ψ^-. Then we have

$$\phi_x - \phi_y = \psi^+ - \psi^-$$

and

$$\|\phi_x - \phi_y\| = \|\psi^+\| + \|\psi^-\|.$$

- If $\mathcal{H}$ has (UDP), then $\|\phi_x - \phi_y\| = 2$. Indeed, there are $z, t \in \operatorname{Ch}_{\mathcal{H}}(X)$ such that $\|\phi_x - \phi_y\| = \|\phi_z\| + \|\phi_t\| = 1 + 1$.

2. Main result. Now we introduce the main theorem. The first proved implication is due to Font and Sanchis [**1**]; we present a little different proof for the sake of completeness. The second proved implication answers the question, posed in [**1**], whether the unique decomposition property is necessary for (1) to hold.

Theorem. *Let $\mathcal{H}$ be a function space on a compact space X. Then*

$$\operatorname{ext}(B_{\mathcal{H}_d^*}) = \{\phi_x - \phi_y : x,\ y \in \operatorname{Ch}_{\mathcal{H}}(X),\ x \neq y\}, \tag{1}$$

if and only if the function space $\mathcal{H}$ enjoys (UDP).

Proof. (i) Suppose that $\mathcal{H}$ has (UDP). According to the above-mentioned proposition it remains to show that for every $x,\ y \in \operatorname{Ch}_{\mathcal{H}}(X)$, $x \neq y$, the functional $\phi_x - \phi_y$ is an extreme point of the closed unit ball of $(\mathcal{H}_d)^*$. From the previous remarks it follows that the diameter norm of $\phi_x - \phi_y$ is equal to 1. Let us write

$$\phi_x - \phi_y = \frac{1}{2}(\omega + \psi),$$

where ω and ψ are from the closed unit ball of $(\mathcal{H}_d)^*$, $\|\omega\|_{d^*} = \|\psi\|_{d^*} = 1$. Then for Hahn-Banach extensions $\Omega,\ \Psi \in (C(X))^*$ of the functionals $\omega,\ \psi$, we can take their positive and negative variations $\Omega^+,\ \Psi^+,\ \Omega^-,\ \Psi^-$. Then

$$\phi_x - \phi_y = \frac{1}{2}\left[(\omega^+ + \psi^+) - (\omega^- + \psi^-)\right],$$

where $\omega^+,\ \psi^+,\ \omega^-,\ \psi^-$ stand for the restrictions of $\Omega^+,\ \Psi^+,\ \Omega^-,\ \Psi^-$ on $\mathcal{H}$. We have

$$\begin{aligned} 2 = \|\phi_x - \phi_y\| &= \frac{1}{2}\left\|(\omega^+ + \psi^+) - (\omega^- + \psi^-)\right\| \\ &\leq \frac{1}{2}\left(\|\omega^+ + \psi^+\| + \|\omega^- + \psi^-\|\right) \\ &\leq \frac{1}{2}\left(\|\omega^+\| + \|\psi^+\| + \|\omega^-\| + \|\psi^-\|\right) \\ &= \frac{1}{2}\left(\|\omega\| + \|\psi\|\right) = 2, \end{aligned}$$

which yields

$$\|\phi_x - \phi_y\| = \frac{1}{2}\left(\|\omega^+ + \psi^+\| + \|\omega^- + \psi^-\|\right).$$

Using (UDP) we get

$$\frac{1}{2}\left(\omega^+ + \psi^+\right) = \phi_z, \qquad \frac{1}{2}\left(\omega^- + \psi^-\right) = \phi_t,$$

for some $z,\ t \in \mathrm{Ch}_{\mathcal{H}}(X)$. Further,

$$\omega^+ = \psi^+ = \phi_z, \qquad \omega^- = \psi^- = \phi_t,$$

because ϕ_z and ϕ_t are extreme points of $S(\mathcal{H})$. Then $\omega = \psi = \phi_z - \phi_t$, and we see that $\phi_x - \phi_y$ is an extreme point of $B_{\mathcal{H}_d^*}$. This finishes the proof of the first implication.

(ii) If we suppose that $\mathcal{H}$ lacks (UDP), we can find $x,\ y \in \mathrm{Ch}_{\mathcal{H}}(X)$, $x \neq y$, and $\psi_1,\ \psi_2 \in B^+_{\mathcal{H}^*}$ such that

$$\phi_x - \phi_y = \psi_1 - \psi_2, \qquad \|\phi_x - \phi_y\| = \|\psi_1\| + \|\psi_2\|, \tag{2}$$

and $\psi_1 \neq \phi_z$ for every $z \in \mathrm{Ch}_{\mathcal{H}}(X)$ or $\psi_2 \neq \phi_z$ for every $z \in \mathrm{Ch}_{\mathcal{H}}(X)$. Without loss of generality let $\psi_1 \neq \phi_z$ for every $z \in \mathrm{Ch}_{\mathcal{H}}(X)$. Notice that from (2) it follows that

$$\|\psi_1\| = \psi_1 \mathbf{1_X} = \psi_2 \mathbf{1_X} = \|\psi_2\|. \tag{3}$$

If $\|\phi_x - \phi_y\| < 2$, then clearly the functional $\phi_x - \phi_y$ is not an extreme point of $B_{\mathcal{H}_d^*}$, so we can assume $\|\phi_x - \phi_y\| = 2$. Now (2) and (3) yield

$$\psi_1,\ \psi_2 \in S(\mathcal{H}).$$

Further, ψ_1 is not an extreme point of $S(\mathcal{H})$. So there exist $\xi_1,\ \xi_2 \in S(\mathcal{H})$ and $\xi_1 \neq \xi_2$ such that

$$\psi_1 = \frac{1}{2}(\xi_1 + \xi_2).$$

Hence,

$$\phi_x - \phi_y = \psi_1 - \psi_2 = \frac{1}{2}\left(\xi_1 + \xi_2\right) - \psi_2 = \frac{1}{2}\left[(\xi_1 - \psi_2) + (\xi_2 - \psi_2)\right]. \tag{4}$$

Since

$$(\xi_1 - \psi_2)\mathbf{1_X} = 0, \qquad (\xi_2 - \psi_2)\mathbf{1_X} = 0,$$

and

$$\|\xi_1 - \psi_2\| \le (\xi_1 + \psi_2)\mathbf{1}_{\mathbf{X}} = 2, \qquad \|\xi_2 - \psi_2\| \le (\xi_2 + \psi_2)\mathbf{1}_{\mathbf{X}} = 2,$$

the functionals $\xi_1 - \psi_2$ and $\xi_2 - \psi_2$ are (different) elements of the closed unit ball of $(\mathcal{H}_d)^*$. Having a nontrivial combination in (4), we can conclude that $\phi_x - \phi_y$ is not an extreme point of $B_{\mathcal{H}_d^*}$. $\square$

Acknowledgments. The author wishes to thank Professors J. Lukeš and J. Spurný for many useful remarks concerning this paper.

REFERENCES

1. J.J. Font and M. Sanchis, *Extreme points and the diameter norm*, Rocky Mountain J. Math. **34** (2004), 1325–1331.

2. R.R. Phelps, *Lectures on Choquet's theorem*, 2nd edition, Springer-Verlag, Berlin, 2001.

3. W. Rudin, *Functional analysis*, McGraw Hill, New York, 1979.

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University in Prague, Prague, Czech Republic
Email address: bacak@karlin.mff.cuni.cz

Current address: School of Mathematical and Physical Sciences, University of Newcastle, Australia
Email address: miroslav.bacak@newcastle.edu.au