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A GENERAL FIXED POINT THEOREM FOR MULTI-VALUED MAPPING IN UNIFORM SPACE

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ABSTRACT. We establish a general fixed point principle which includes known fixed point theorems in uniform spaces. Then examples show that this theorem includes known fixed point theorems and also yields a new theorem.

1. Introduction. A fixed point theorem for multi-valued contraction mappings was proved for the first time by Nadler [**15**]. Since then, many authors have given generalizations of this theorem in various forms, such as the one given by Wegrzyk. Wegrzyk has applied fixed point theorems to the proof of multi-valued functions and functional equations [**25**].

Uniform spaces form a natural extension of metric spaces. An exact analogue of the well-known Banach contraction principle in uniform spaces was obtained independently by Acharya [**1**] and Taraftar [**21**]. Since then a number of fixed point theorems for single-valued and multi-valued mappings using various contractive conditions in this setting have been obtained ([**2, 6–10, 12–14, 18–20**]). In this paper we first prove a fixed point theorem for a multi-valued map in hyperspace. Then examples show that this theorem includes known fixed point theorems and yields a new theorem.

Let $(X,\mathcal{U})$ be a uniform space. A family $P=\{d_i : i \in I\}$ of pseudometrics on X with indexing set I, is called an associated family for the uniformity $\mathcal{U}$ if the family

$$\beta = \{V(i,r) : i \in: I, r > 0\}$$

where

$$V(i,r) = \{(x,y) : x,y \in X, d_i(x,y) < r\}$$

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is a subbase for the uniformity $\mathcal{U}$. We may assume β itself to be base by adjoining finite intersections of members of β, if necessary. The corresponding family of pseudometrics is called an augmented associated family for $\mathcal{U}$. An augmented associated family for $\mathcal{U}$ will be denoted by P^*. For details, the reader it is referred to Taraftar [**21**], Thron [**22**], Acharya [**1**], Mishra [**11**], Rhoades [**19**], Türkoğlu and Fisher [**23**], Weil [**26**], Angelov [**3**] and Angelov and Donchev [**4**]. From now on, unless otherwise stated, X will denote a uniform space $(X,\mathcal{U})$ defined by P^*.

Let A be a nonempty subset of a uniform space X. Define

$$\Delta^*(A) = \sup\{d_i(x,y) : x, y \in A, i \in I\}$$

where $\{d_i(x,y) : i \in I\} = P^*$. Then $\Delta^*(A)$ is called an augmented diameter of A. Further, A is said to be P^*-bounded if $\Delta^*(A) < \infty$, see [**11**].

Let

$$2^X = \{A : A \quad \text{is a nonempty} \quad P^*\text{-bounded subset of} \quad X\}.$$

For any nonempty subsets A and B of X, define

$$\begin{aligned} d_i(x,A) &= \inf\{d_i(x,y) : a \in A, i \in I\}, \\ H_i(A,B) &= \max\{\sup_{a\in A} d_i(a,B), \sup_{b\in B} d_i(A,b)\} \\ &= \sup_{x\in X}\{|d_i(x,A) - d_i(x,B)|\}. \end{aligned}$$

It is well known that, on 2^X, H_i is a pseudometric, called the Hausdorff pseudometric, induced by $d_i, i \in I$.

Let $(X,\mathcal{U})$ be a uniform space, and let $U \in \mathcal{U}$ be an arbitrary entourage. For each subset A of X, define

$$U[A] = \{y \in X : (x,y) \in U \text{ for some } x \in A\}.$$

The Hausdorff uniformity $2^{\mathcal{U}}$ on 2^X is defined by the base

$$2^{\beta} = \{\widetilde{U} : U \in \mathcal{U}\}$$

where

$$\widetilde{U} = \{(A, B) \in 2^X \times 2^X : A \subseteq U[B], B \subseteq U[A]\}.$$

The augmented associated family P^* also induces a uniformity $\mathcal{U}^*$ on 2^X defined by the base

$$\beta^* = \{V^*(i, r) : i \in I, r > 0\}$$

where

$$V^*(i, r) = \{(A, B) \in 2^X \times 2^X : H_i(A, B) < r\}.$$

The uniformity $2^{\mathcal{U}}$ and $\mathcal{U}^*$ on 2^X are uniformly isomorphic. The space $(2^X, \mathcal{U}^*)$ is thus a uniform space called the hyperspace of $(X, \mathcal{U})$. There exist other bases which could be used to generate uniformities on X as well as 2^X, see for details, [**5, 16, 17**].

The following theorem was proved in [**25**].

Theorem 1 [**25**]. *If (Y, d) is a complete metric space and $F : X \to CL(Y)$ is a multi-valued function which satisfies the inequality*

$$D(Fx, Fy) \leq \Psi(d(x, y))$$

for all x, y in X and for some strictly increasing function Ψ such that

$$\lim_{k\to\infty} \Psi^k(t) = 0$$

for every t, then

(a) *for every $y_0 \in Y$ and for every fixed point $y \in Y$ of F, there exists a sequence of iterates of F at y_0 which converges to y.*

(b) *If*

$$\sum_{k=1}^{\infty} \Psi^k(t) < \infty,$$

for $t > 0$, then the set of fixed point of F is nonempty.

In this theorem $\Psi : [0, \infty) \to [0, \infty)$, D is the Hausdorff metric and

$$CL(Y) = \{A : A \text{ is closed in } Y\}.$$

Now we are going to formulate the following.

Theorem 2. *Let* $(X,\mathcal{U})$ *be a complete Hausdorff uniform space defined by* $\{d_i : i \in I\} = P^*$ *and* $(2^X,\mathcal{U}^*)$ *a hyperspace. Let* $F : X \to 2^X$ *be a continuous mapping and* Fx *compact for each* x *in* X. *Assume that*

$$H_i(Fx, Fy) \leq K(M_i(x,y)) \tag{1}$$

for all $i \in I$ *and* $x, y \in X$, *where*

$$M_i(x,y) = \max\{d_i(x,y), d_i(x,Fx), d_i(y,Fy), d_i(x,Fy), d_i(y,Fx)\},$$

$K : [0,\infty) \to [0,\infty), K(0) = 0, K(t) < t$ *for all* $t \in (0,\infty)$ *and* K *is nondecreasing. Then there exists a* z *in* X *with*

$$\sum_{n=1}^{\infty} K^n(d_i(x_0, Fx_0)) < \infty.$$

Note that in this theorem K is not assumed to be continuous and $K^n(t) = K(K^{n-1}(t))$.

Proof. If $z \in X$, then $d_i(z,Fz) = 0$, $0 = K(0) = K^2(0) = \cdots = K^n(0) = \cdots$ for each $i \in I$ and

$$\sum_{n=1}^{\infty} K^n(d_i(z, Fz)) = 0.$$

Let $x_0 \in X$ and $x_1 \in Fx_0$ be arbitrary. Suppose that there exists x_0 such that

$$\sum_{n=1}^{\infty} K^n(d_i(x_0, Fx_0)) < \infty$$

for each $i \in I$.

Let $U \in \mathcal{U}$ be an arbitrary entourage. Since β is a base for $\mathcal{U}$, there exists $V(i,r) \in \beta$ such that $V(i,r) \subseteq U$. Now $y \to d_i(x_0, y)$ is continuous on the compact Fx_0, and this implies that there exists an $x_1 \in Fx_0$ such that $d_i(x_0, x_1) = d_i(x_0, Fx_0)$. Similarly, Fx_1 is compact so there exists an $x_2 \in Fx_1$ such that $d_i(x_1, x_2) = d_i(x_1, Fx_1)$. Continuing, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \in Fx_n$ and $d_i(x_n, x_{n+1}) = d_i(x_n, Fx_n)$.

Noting that K is nondecreasing and using inequality (1), we have

$$\begin{aligned} d_i(x_n, x_{n+1}) = d_i(x_n, Fx_n) &\le H_i(Fx_{n-1}, Fx_n) \\ &\le K(\max\{d_i(x_{n-1}, x_n), d_i(x_{n-1}, Fx_{n-1}), \\ &\qquad d_i(x_n, Fx_n), d_i(x_{n-1}, Fx_n), d_i(x_n, Fx_{n-1})\}) \end{aligned}$$

which implies that

$$d_i(x_n, x_{n+1}) \le K(\max\{d_i(x_{n-1}, x_n), d_i(x_n, x_{n+1})\}). \tag{2}$$

Suppose that $d_i(x_n, x_{n+1}) > d_i(x_{n-1}, x_n)$ for some n. Then, from (2) and $K(t) < t$ for all $t \in (0, \infty)$, we have

$$d_i(x_n, x_{n+1}) \le K(d_i(x_n, x_{n+1})) < d_i(x_n, x_{n+1}),$$

which is a contradiction. Therefore, we have

$$\begin{aligned} d_i(x_n, x_{n+1}) &\le K(d_i(x_{n-1}, x_n)) \\ &= K(d_i(x_{n-1}, Fx_{n-1})) \\ &\le K(H_i(Fx_{n-2}, Fx_{n-1})) \\ &\le K^2(d_i(x_{n-2}, x_{n-1})) \\ &\le \cdots \\ &\le K^n(d_i(x_0, x_1)) = K^n(d_i(x_0, Fx_0)). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} d_i(x_n, x_{n+m}) &\le d_i(x_n, x_{n+1}) + d_i(x_{n+1}, x_{n+2}) + \cdots \\ &\quad + d_i(x_{n+m-1}, x_{n+m}) \\ &\le \sum_{k=n}^{n+m-1} K^k(d_i(x_0, Fx_0)). \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} K^n(d_i(x_0, Fx_0)) < \infty,$$

it follows that there exists a p such that $d_i(x_n, x_m) < r$ and hence $(x_n, x_m) \in U$ for all $n, m \geq p$. Therefore, the sequence $\{x_n\}$ is a Cauchy sequence in the d_i-topology on X.

Let $S_p = \{x_n : n \geq p\}$ for all positive integers p, and let β be the filter basis $\{S_p : p = 1, 2, \ldots\}$. Then since $\{x_n\}$ is a d_i-Cauchy sequence for each $i \in I$, it is easy to see that the filter basis β is Cauchy filter in the uniform space $(X, \mathcal{U})$. To see this we first note that the family $\{V(i, r) : i \in I\}$ is a base for $\mathcal{U}$ as $P^* = \{d_i : i \in I\}$. Now, since $\{x_n\}$ is a d_i-Cauchy sequence in X, there exists a positive integer p such that $d_i(x_n, x_m) < r$ for $m > p, n \geq p$. This implies that $S_p \times S_p \subset V(i, r)$. Thus, given any $U \in \mathcal{U}$, we can find an $S_p \in \beta$ such that $S_p \times S_p \subset U$. Hence, β is a Cauchy filter in $(X, \mathcal{U})$. Since $(X, \mathcal{U})$ is a complete Hausdorff space, the Cauchy filter $\beta = \{S_p\}$ converges to a unique point $z \in X$. Consequently, $FS_p \to Fz$ (follows from the continuity of F). Also,

$$S_{p+1} \subseteq F(S_p) = \bigcup\{Fx_n : n \geq p\}$$

for $p = 1, 2, \ldots$. It follows that $z \in Fz$. Hence, z is a fixed point of F. This completes the proof. $\square$

To apply Theorem 2, one needs a nondecreasing function K and x in X with

$$\sum_{n=1}^{\infty} K^n(d_i(x, Fx)) < \infty.$$

The following examples satisfy these conditions and therefore illustrate the generality of Theorem 2. Let X denote a complete Hausdorff uniform space defined by $\{d_i : i \in I\} = P^*$ and

$$M_i(x, y) = \max\{d_i(x, y), d_i(x, Fx), d_i(y, Fy), d_i(x, Fy), d_i(y, Fx)\}$$

for all x, y in X.

Example 1. Suppose $0 < \lambda_i < 1$. Let $K(M_i(x, y)) = \lambda_i M_i(x, y)$ for all $x, y \in X$. Then $H_i(Fx, Fy) \leq K(M_i(x, y)) = \lambda_i M_i(x, y)$ and $K^n(d_i(x, Fx)) = \lambda_i^n d_i(x, Fx)$ for any x in X. It is known that there exists a z with $z \in Fz$ without assuming that Fx is compact.

Example 2. Suppose that F satisfies

$$H_i(Fx, Fy) \leq \Phi(M_i(x,y))M_i(x,y)$$

for all x, y in X, where $\Phi : [0,\infty) \to [0,\infty)$ and Φ is nondecreasing. Then $K(M_i(x,y)) = M_i(x,y)\Phi(M_i(x,y))$, K is nondecreasing, and $K : [0,\infty) \to [0,\infty)$. It follows by induction that $K^n(t) \leq t[\Phi(t)]^n$, since $\Phi(t) < 1$ and

$$\sum_{n=1}^{\infty} K^n(t) < \infty.$$

Example 3. Consider $K(M_i(x,y)) = M_i(x,y)\Phi(M_i(x,y))$ for all x, y in X, where $\Phi : [0,\infty) \to [0,\infty), \Phi(t) < t$ for $t \leq 1$. It follows that $K^n(t) \leq t[\Phi(t)]^n$. If K is nondecreasing, then Theorem 2 can be applied.

Example 4. $K(M_i(x,y)) = M_i(x,y)\Phi(M_i(x,y))$ for all x, y in X, where $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(\alpha M_i(x,y)) \leq \alpha\Phi(M_i(x,y))$ for $q \in (0,1)$. If $\Phi(t) < 1$, then $K^n(t) \leq K(t)[\Phi(t)]^n$ for all $n \geq 2$.

Assume that K is nondecreasing, K is convex on $[0,1)$ and $K(M_i(x,y)) < M_i(x,y)$ for all x, y in X. If $t < 1$, $K(t) < t$ for all $0 < t < 1$. Then $K(t) = \alpha t$ for some $0 < \alpha < 1$. It can be shown that $K^n(t) \leq \alpha^n t$ for all n and thus

$$\sum_{n=1}^{\infty} K^n(t) < \infty.$$

Theorem 3. *Let $(X, \mathcal{U})$ be a complete Hausdorff uniform space defined by $\{d_i : i \in I\} = P^*$, let $F : X \to 2^X$ be a continuous multi-valued mapping and Fx compact for each x in X. Assume that*

$$H_i(Fx, Fy) \leq [M_i(x,y)]^q$$

where $q > 1$, then F has a fixed point in X.

Proof. Let $K(M_i(x,y)) = [M_i(x,y)]^q$ for all x, y in X. Then $K(0) = 0$ and K is increasing, $K(t) < t$ if $t < 1$ and K is convex.

If $t = d_i(x, Fx) < 1$, then

$$\sum_{n=1}^{\infty} K^n(t) < \infty$$

from the previous example. Also F is continuous, so Theorem 2 applies.

Remark 1. If we replace with $M_i(x, y)$ in Theorem 2, Theorem 3 and Example 1—5 in the uniform space $(X, \mathcal{U})$ by $d_i(x, y)$, then the result of Türkoğlu et al. [**24**] will follow as special cases of our results.

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