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ON THE SIMPLICITY AND UNIQUENESS OF POSITIVE EIGENVALUES ADMITTING POSITIVE EIGENFUNCTIONS FOR WEAKLY COUPLED ELLIPTIC SYSTEMS

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1. Introduction and preliminaries. Throughout this paper, we shall assume that Ω is a bounded domain in $\mathbf{R}^N, N \geq 1$, with $\partial\Omega$ of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$. Then, for $k = 1, 2, \ldots, r$, let L^k denote the formally self-adjoint operator on Ω given by

$$L^k w(x) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}(A^k_{ij}(x)\frac{\partial w}{\partial x_i}(x)) + A^k(x)w(x).$$

The coefficients A^k_{ij} and A^k are assumed to satisfy

(i) $(A^k_{ij}(x))^N_{i,j=1}$ is symmetric and uniformly positive definite on $\overline{\Omega}$;

(ii) $A^k(x) \geq 0$;

(iii) $A^k_{ij} \in C^{1+\alpha}(\overline{\Omega})$, i, $j = 1, 2, \ldots, N$, $0 < \alpha < 1$; and

(iv) $A^k \in C^{\alpha}(\overline{\Omega}), 0 < \alpha < 1$.

L will then denote the diagonal matrix

$$L = \begin{bmatrix} L^1 & & & \\ & L^2 & 0 & \\ & 0 & \ddots & \\ & & & L^r \end{bmatrix}.$$

In addition, the matrix $M(x) = (m_{k\ell}(x))^r_{k,\ell=1}, x \in \overline{\Omega}$ will be assumed to satisfy

(i) $m_{k\ell} \in C^{\alpha}(\overline{\Omega})$, $k, \ell = 1, 2, \ldots, r$, $0 < \alpha < 1$;

(ii) $m_{k\ell} \geq 0$ on $\overline{\Omega}$ if $k \neq \ell$; and

(iii) $m_{k\ell} = m_{\ell k}$ for $k, \ell = 1, 2, \ldots, r$.

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We will now consider the linear boundary value problem

$$
\begin{aligned} Lu &= PMu \text{ in } \Omega \\ u &\equiv 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.3}
$$

where $u = (u^1, u^2, \ldots, u^r)^t$ is viewed as an r-tuple of functions on $\overline{\Omega}$ and P is a nonnegative $r \times r$ scalar matrix with $p_{kk} > 0$ for $k = 1, 2, \ldots, r$. We are mainly interested in choices of P which admit classical solutions of (1.3) for which $u^k(x) \geq 0$ on $\overline{\Omega}, k = 1, 2, \ldots, r$.

This problem has been addressed in [**1**] and [**2**], in case $P = \lambda I, \lambda > 0$ and without the assumptions of formal self-adjointness for L and symmetry for M. The principal result of Hess [**2**] is that if $m_{kk}(x_0) > 0$ for some $k \in \{1, 2, \ldots, r\}$ and some $x_0 \in \Omega$, (1.3) has such a solution for at least one $\lambda > 0$. Some partial results on the simplicity and uniqueness of such eigenvalues are given in [**1**]. In particular, if the Hess result holds, and if $(M + \mu I)(\overline{x})$ is a nonnegative irreducible matrix for some $\overline{x} \in \Omega$, then $u^k(x)$ may be chosen *strictly positive* inside Ω for $k = 1, 2, \ldots, r$. Moreover, $\dim(\ker((L - \lambda M)^2)) = \dim(\ker(L - \lambda M)) = 1$.

However, for purposes of applications to associated nonlinear problems (as, for example, in bifurcation theory) a more relevant question is the algebraic simplicity of an eigenvalue λ of

$$
u = \lambda L^{-1} M u. \tag{1.4}
$$

As described in [**1**] and [**2**], (1.4) is equivalent to (1.3) in case $P = \lambda I$ by standard a priori estimates and embedding theorems for second-order elliptic partial differential equations. In particular, $L^{-1}M$ may be viewed as a compact linear operator on either of the Banach spaces $[C_0^{1+\alpha}(\overline{\Omega})]^r$ or $[C_0^0(\overline{\Omega})]^r$ (the choice of $[C_0^0(\overline{\Omega})]^r$ being made when it is desirable to exploit the monotone nature of the cone of positive functions in this space). To this end, it is shown in [**1**] that, in case $L^{-1}M = ML^{-1}$ and $(M + \mu I)(x_0)$ is irreducible for some $\mu > 0$ and $x_0 \in \Omega$, (1.4) has a unique algebraically simple eigenvalue admitting an eigenfunction with $u^k(x) \geq 0$, $k = 1, 2, \ldots, r$, provided $m_{k_0 k_0} > 0$ for at least one $k_0 \in \{1, 2, \ldots, r\}$. It should be noted that the commutativity assumption essentially requires that $L^1 = L^2 = \cdots = L^r$ and that M is a constant matrix, although L^1 need not be formally self-adjoint and m_{kk} can be negative for $k \neq k_0$. Partial results are given in [**1**] in case the commutativity assumption is dropped.

In this article we shall show that the simplicity and uniqueness results obtain as above without the commutativity assumption provided that L is formally self-adjoint and M is symmetric. (These results extend to systems the results of [**3**].) To this end, in §2, we prove a basic simplicity theorem, which covers a number of cases, including $P = \lambda I$. Corresponding uniqueness results are presented in §3, making strong use of the results of [**1**].

2. Simplicity results.

THEOREM 2.1. *Consider* (1.4), *where* L, M, *and* P *are as described in* §1. *In addition, invertible matrix;*

(i) P *is a symmetric, invertible matrix*

(ii) $P^{-1}L = LP^{-1}$;

(iii) *If* $A = PM$ *and* $A = (a_{k\ell})_{k,\ell=1}^{r}$, *then* $a_{k\ell} \geq 0$ *if* $k \neq \ell$ *and* $(A+\delta I)(\overline{x})$ *is nonnegative irreducible, for some* $x \in \Omega$ *and some* $\delta > 0$;

(iv) *The map* $Q : [C_0^2(\overline{\Omega})]^r \to \mathbf{R}$ *given by*

$$Q(w) = \langle w, P^{-1}Lw\rangle$$

is positive definite, where $\langle , \rangle$ *is the inner product for* $[L^2(\overline{\Omega})]^r$.

Then, if (1.4) *has a nontrivial solution* u *in* $[C_0^{2+\alpha}(\overline{\Omega})]^r$ *with* $u^k \geq 0$ *on* $\overline{\Omega}$ *for* $k = 1,2,\ldots,r$, $u^k(x) > 0$ *for* $x \in \Omega$ *and* $\frac{\partial u^k}{\partial \nu}(x) < 0$ *on* $\partial\Omega$, *where* $\frac{\partial}{\partial \nu}$ *denotes the outward normal derivative. Moreover,* $N((I - PL^{-1}M)^2) = N(I - PL^{-1}M) = \mathit{span}\,(u)$.

PROOF. That u^k is as described for $k = 1,2,\ldots,r$ and that $N(I - PL^{-1}M) = \operatorname{span}(u)$ follow from (iii) as in [**1**; §3]. Suppose now that $(I - PL^{-1}M)^2x = 0$. Then $(I - PL^{-1}M)x = cu$, where $c \in \mathbf{R}$. Consequently

$$\begin{aligned} 0 &= \langle (I - PL^{-1}M)^2x, y\rangle \\ &= \langle (I - PL^{-1}M)x, (I - PL^{-1}M)^*y\rangle \\ &= c\langle u, (I - ML^{-1}P)y\rangle \end{aligned}$$

for any $y \in [C_0^\alpha(\overline{\Omega})]^r$. In particular, if $y = P^{-1}$ Lx

$$\begin{aligned} 0 &= c\langle u, (I - ML^{-1}P)(P^{-1}Lx)\rangle \\ &= c\langle u, P^{-1}Lx - Mx\rangle. \end{aligned}$$

But now $x = PL^{-1}Mx + cu$ and $PL^{-1} = L^{-1}P$ imply $P^{-1}Lx - Mx = cP^{-1}Lu$. Hence

$$0 = c^2\langle u, P^{-1}Lu\rangle.$$

Since $u^k > 0$ on Ω for $k = 1, 2, \ldots, r$, (iv) implies $c^2 = 0$.

REMARK. Hypothesis (iv) of Theorem 2.1 may be omitted provided it is known that $\langle u, P^{-1}Lu\rangle = \langle u, Mu\rangle \neq 0$. However, we have chosen to present the result with hypothesis (iv) included, as there are two important cases in which the hypotheses of Theorem 2.1 may be verified.

COROLLARY 2.2. *Suppose that L, M, and P are as in §1. In addition, assume that $p_{k\ell} = 0$ if $k \neq \ell$ and that $(M + \delta I)(\overline{x})$ is nonnegative irreducible for some $\overline{x} \in \Omega$ and some $\delta > 0$. Then the conclusion of Theorem* 2.1 *obtains.*

PROOF. That hypotheses (i)-(iii) of Theorem 2.1 are satisfied is immediate. Suppose now that $w \in [C_0^2(\overline{\Omega})]^r$. Then

$$\begin{aligned} Q(w) &= \sum_{k=1}^{r} \frac{1}{p_{kk}} \int_\Omega w^k L^k w^k \\ &= \sum_{k=1}^{r} \frac{1}{p_{kk}} \Big[\int_\Omega \sum_{i,j=1}^{N} A_{ij}^k(x) \frac{\partial w^k}{\partial x_i}\frac{\partial w^k}{\partial x_j} dx + \int_\Omega A^k(x)[w^k]^2 dx\Big] \end{aligned}$$

by the formal self-adjointness of L^k, $k = 1, \ldots, r$. Consequently, $Q(w) > 0$ unless $w \equiv 0$.

REMARK. In particular, Corollary 2.2 includes, of course, the case $P = \lambda I$.

COROLLARY 2.3. *Suppose that L, M, and P are as in* §1. *In addition, assume that P and M satisfy hypotheses* (i) *and* (iii) *of* Theorem 2.1 *and that P is positive definite. Then if $L^1 = L^2 = \cdots = L^r$, the conclusion of* Theorem 2.1 *obtains.*

PROOF. Again, we need only verify hypothesis (iv). Since P^{-1} is a symmetric positive definite matrix, it is well-known that there is a symmetric matrix C such that $C^2 = P^{-1}$. So if $w \in [C_0^2(\overline{\Omega})]^r$,

$$\begin{aligned} Q(w) &= \langle w, P^{-1}Lw\rangle \\ &= \langle w, c^2 Lw\rangle \\ &= \langle Cw, LCw\rangle. \end{aligned}$$

The hypotheses on L guarantee that $Q(w) > 0$ unless $Cw \equiv 0$ on $\overline{\Omega}$. But if such is the case $\langle w, P^{-1}w\rangle = \langle Cw, Cw\rangle = 0$. Consequently, since P^{-1} is positive definite, $w \equiv 0$, and (iv) is verified.

3. Uniqueness results.

Let us now assume that $P = \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{-r} \end{pmatrix}$, with $\lambda_k > 0$ fixed for $k = 1, 2, \ldots, r$, that L and M are as in Corollary 2.2, and that

$$m_{kk}(x_0) > 0 \tag{3.1}$$

for some $x_0 \in \Omega$ and some $k \in \{1, 2, \ldots, r\}$. We may now obtain the following

THEOREM 3.1. *Suppose that $\Lambda, L,$ and M are as above. Then there is a unique $s_0 > 0$ such that*

$$Lu = s_0 \Lambda M u \quad \text{in } \Omega \tag{3.2}$$

$$u = 0 \quad \text{on } \partial\Omega$$

has a nontrivial solution u_0 with $u_0^k \geq 0$, for $k = 1, 2, \ldots, r$.

REMARK. The proof of Theorem 3.1 is a special case of the proof of our Theorem 3.8 in [**1**], and, consequently, a fully detailed exposition of the proof is unnecessary. However, in order that this present article be somewhat self-contained, we will give a brief sketch of the main ideas of the proof. A reader seeking further details is referred to [**1**].

PROOF OF THEOREM 3.1. First of all, there is no loss of generality in the additional assumption that

$$-\frac{1}{2r} < m_{kk}(x) < \frac{1}{2r} \tag{3.3}$$

for $x \in \overline{\Omega}$, $k = 1, 2, \ldots, r$, and that

$$0 \leq m_{k\ell}(x) \leq \frac{1}{r} \tag{3.4}$$

for $x \in \overline{\Omega}, k \neq \ell, k, \ell = 1, 2, \ldots, r$. Now consider the family of problems

$$\begin{aligned} Lu &= s\Lambda(M - t)u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.5}$$

which contains (3.2). It is easy to see that (3.5) is equivalent to

$$u = s\Lambda(L + s\Lambda)^{-1}(M - t + 1)u. \tag{3.6}$$

Notice that if $t < 1 - \frac{1}{2r}, M - t + 1$ is nonnegative and, for some $\overline{x} \in \Omega$, irreducible as well. Consequently, the right hand side of (3.6) may be viewed as a compact positive operator on $[C_0^0(\overline{\Omega})]^r$ if $s > 0$ and $t < 1 - \frac{1}{2r}$. It follows as in §3 of [**1**] that, for such s and t, that the existence of a positive solution to (3.6) is equivalent to $r(s\Lambda(L + s\Lambda)^{-1}(M - t + 1)) = 1$, where $r(A)$ is the spectral radius of A. Moreover, if $(\overline{s}, \overline{t})$ is such a point there is a smooth function $t(s) : (\overline{s} - \delta, \overline{s} + \delta) \to (-\infty, 1 - \frac{1}{2r})$, where $\delta > 0$ is sufficiently small, such that $t(\overline{s}) = \overline{t}$ and such that $r(s\Lambda(L+s\Lambda)^{-1}(M-t+1)) = 1$ exactly when $t = t(s)$ if $|(s,t) - (\overline{s}, \overline{t})|$ is sufficiently small.

It follows from (3.3)-(3,4) and [**4**, pp. 188-192] that there is a $t_0 \in (0, 1 - \frac{1}{2r})$ such that (3.6) has no positive solution with $s > 0$ and $t \geq t_0$. Let $t_0^* = \inf\{t : t < 1 - \frac{1}{2r}$ and (3.6) has no positive solution with $s > 0$ at $t\}$. Then it follows from (3.1) that $0 < t_0^* \leq t_0$. We may define a function $f : (-\infty, t_0^*] \to [0, \infty)$ by $f(t) = 1/s$ where s is the smallest positive number for which (3.6) has a positive solution at t provided $t < t_0^*$ and 0 if $t = t_0^*$. That $M - t + 1$ is monotonic in t will imply that f is a decreasing function. Now if $t < t_0^*$ and $s = 1/f(t)$, Corollary 2.2 implies that

$$\dim(N([I - s\Lambda L^{-1}(M - t)]^2)) = \dim(N(I - s\Lambda L^{-1}(M - t))) = 1.$$

A degree theoretic argument, as in [**1**; §3], may now be made to show that f is continuous.

Now suppose there is $\tilde{s} > s_0$ such that (3.6) has a positive solution. Since $0 < 1/\tilde{s} < 1/s_0 = f(0)$, there is a $\tilde{t} \in (0, t_0)$ such that $f(\tilde{t}) = 1/\tilde{s}$. Consequently, $r(\tilde{s}\Lambda(L+\tilde{s}\Lambda)^{-1}(M-0)) = 1 = r(\tilde{s}\Lambda(L+\tilde{s}\Lambda)^{-1}(M-\tilde{t}))$. So $r(\tilde{s}\Lambda(L+\tilde{s}\Lambda)^{-1}(M-t)) = 1$ for $t \in [0, \tilde{t}]$, a contradiction to the solvability of t in terms of s at $(\tilde{s}, \tilde{t})$.

Theorem 3.1 has an immediate consequence which is of substantial interest in the geometric study of generalized spectra of systems of second order elliptic partial differential equations [**5**].

COROLLARY 3.2. *Suppose that L and M satisfy the hypotheses of* Corollary 2.2 *and in addition that* (3.1) *holds. Then the set* $\{(\lambda_1, \lambda_2, \ldots, \lambda_r) : \lambda_k > 0$, *for* $k = 1, 2, \ldots, r$, *and* $Lu = \Lambda Mu$ *has a positive solution in* Ω *with* $u = 0$ *on* $\partial\Omega\}$ *is homeomorphic to* $S = \{(\lambda_1, \lambda_2, \ldots, \lambda_r) : \lambda_k > 0$ *and* $\sum_{k=1}^{r} \lambda_k^2 = 1\}$. *In particular, if* $(\lambda_1^0, \lambda_2^0, \ldots, \lambda_r^0) \in S$ *and* ψ *denotes the homeomorphism,* $\psi((\lambda_1^0, \lambda_2^0, \ldots \lambda_r^0)) = \alpha(\lambda_1^0, \lambda_2^0, \ldots, \lambda_r^0)$, *where* $\alpha > 0$.

REFERENCES

1. R.S. Cantrell and K. Schmitt, *On the eigenvalue problem for coupled elliptic systems,* SIAM J. Math. Anal. **17** (1986), 850-862.

2. P. Hess, *On the eigenvalue problem for weakly-coupled elliptic systems,* Arch. Rat. Mech. Anal. **81** (1983), 151-159.

3. A. Manes and A.M. Micheletti, *Un'estensione della teoria variationale classica degli autovalori per operatori ellitici del secondo ordine,* Boll, U.M.I. **7** (1973), 185-301.

4. M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations,* Prentice-Hall, Englewood Cliffs, N.J., 1967.

5. ———, *The generalized spectrum of second order elliptic systems,* Rocky Mountain J. Math. **9** (1979), 503-518.

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