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ON A THEOREM OF CELLINA FOR SET VALUED FUNCTIONS

GERALD BEER

1. Introduction. By a *multifunction* Γ from a metric space X to a metric space Y, we mean a function that assigns to each $x \in X$ a nonempty subset $\Gamma(x)$ of Y. It is natural to identify Γ with its *graph* in $X \times Y$, the set $\{(x,y) : x \in X \text{ and } y \in \Gamma(x)\}$. In so doing, we can consider a variety of approximation problems with respect to Hausdorff distance in $X \times Y$. Two fundamental questions have been these: (1) when can we approximate a multifunction from above by a decreasing sequence of "continuous" multifunctions? (2) when can we approximate a multifunction by continuous single valued functions? All of the positive results with respect to these questions assume that the multifunctions be convex valued, for they ultimately depend on paracompactness arguments. With respect to the first question, the fundamental result is Hukuhara's Theorem [**9**], recently extended and sharpened by De Blasi [**4**] and De Blasi and Myjak [**5**]. Here, we are interested in the second question, where the fundamental result is due to Cellina [**3**]. Specifically, we show that Cellina's Theorem admits a converse precisely when X is locally compact and Y is complete, and we extend his result to continuous starshaped valued multifunctions.

2. Background material. We first recall the notion of Hausdorff distance between nonempty subsets of a metric space. For this purpose, we denote the union of all open ε-balls whose centers run over a subset E of a metric space $\langle X, d\rangle$ by $S_\varepsilon[E]$ (abbreviating $S_\varepsilon[\{x\}]$ by $S_\varepsilon[x]$). If F_1 and F_2 are nonempty subsets of X and for some $\varepsilon > 0$ both $S_\varepsilon[F_1] \supset F_2$ and $S_\varepsilon[F_2] \supset F_1$, then the *Hausdorff distance* h_d between them is given by the formula

$$h_d(F_1, F_2) = \inf\{\varepsilon : S_\varepsilon[F_1] \supset F_2 \text{ and } S_\varepsilon[F_2] \supset F_1\}.$$

Otherwise, we write $h_d(F_1, F_2) = \infty$. If we restrict h_d to the nonempty closed subsets of X, then h_d defines an infinite valued metric. Basic

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facts about this metric can be found in Castaing and Valadier [**2**] or Klein and Thompson [**11**]. Of course, we are interested in Hausdorff distance in a product of metric spaces $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$. We first need a metric on $X \times Y$ to induce Hausdorff distance on its subsets. Since any two metrics compatible with the product uniformity induce equivalent Hausdorff metrics, for computational simplicity, we choose the *box metric* ρ:

$$\rho[(x_1, y_1), (x_2, y_2)] = \max\{dX(x_1, x_2), d_Y(y_1, y_2)\}.$$

We next turn our attention to multifunctions. If Γ is a multifunction from X to Y and $E \subset X$, we write $\Gamma(E)$ for $\cup\{\Gamma(x) : x \in E\}$. The fundamental notion of upper semicontinuity is that of Kuratowski [**13**]: Γ is *upper semicontinuous* (u.s.c.) at a point p of X if for each neighborhood W of $\Gamma(p)$, there exists a neighborhood V of p such that $\Gamma(V) \subset W$. A multifunction Γ is globally upper semicontinuous if for each open set W in Y, the set $\{x : \Gamma(x) \subset W\}$ is open in X. Evidently, upper semicontinuity so defined is a stronger notion than so-called *Hausdorff upper semicontinuity:* for each $\varepsilon > 0$ there exists a neighborhood V of p such that $\Gamma(V) \subset S_\varepsilon[\Gamma(p)]$. However, these notions do coincide provided Γ is compact valued [**11**] (see also Theorem 3.1 below). We call Γ *continuous* if it is continuous with respect to the topology of Hausdorff distance on the subsets of the target space.

Salient characteristics of u.s.c. multifunctions with values in a complete target space were revealed by Dolecki and Rolewicz [**7**] and Dolecki and Lechicki [**6**] using measures of noncompactness; we shall find their approach indispensable. Simply put, a measure of noncompactness is an extended real valued functional defined on the power set of a metric space X that measures the degree to which subsets of X fail to be totally bounded. Such measures have a well established place in fixed point theory (see, e.g., [**10**] and the references therein). Here, we work with the *Hausdorff measure of noncompactness functional* χ: we set $\chi(\emptyset) = 0$, and if A is a nonempty subset of X, then

$$\chi(A) = \inf\{\varepsilon : A \text{ has a finite } \varepsilon \text{ - dense subset}\}.$$

Evidently, the functional has these properties:

(i) $\chi(A) = \chi(\mathrm{cl}A)$:

(ii) if $A \subset B$, then $\chi(A) \leq 2\chi(B)$;

(iii) $\chi(S_\varepsilon[A]{\le}\chi(A)+\varepsilon$.

The fundamental result regarding χ is an analog of Cantor's Theorem, due to Kuratowski [**12**].

KURATOWSKI'S THEOREM. *A metric space $\langle X, d\rangle$ is complete if and only if whenever $\langle F_n\rangle$ is a decreasing sequence of nonempty closed subsets of X with $\lim_{n\to\infty}\chi(F_n)=0$, then $\cap_{n=1}^{\infty}F_n$ is nonempty.*

3. A converse to Cellina's theorem. We choose to state Cellina's Theorem in the context of normed linear spaces, rather than in the more general setting of a locally convex metric linear space in which it was proved.

CELLINA'S THEOREM. *Let X and Y be normed linear spaces, and let C be a convex subset of X. Suppose ρ denotes the box metric in $X\times Y$. Let Γ be a convex valued multifunction from C to Y. If Γ is Hausdorff upper semicontinuous and has totally bounded values, then for each $\varepsilon>0$ there exists a continuous function $f: C\to Y$ for which $h_\rho(\Gamma, f)<\varepsilon$.*

Actually, if $\Gamma(C)$ is contained in some closed convex set, then f can be chosen such that $f(C)$ lies in the same set, from which the Kakutani Fixed Point Theorem easily follows. In [**1**] this author showed that C could be replaced by an arbitrary metric space, provided Γ maps isolated points to singletons. Obviously, that Γ have convex values is in no way necessary for the existence of such approximations (see Section 4 below). But does this approximation property force Γ to have totally bounded values, or Γ to be Hausdorff upper semicontinuous? Note that two multifunctions admit the same approximations provided their graphs have the same closure. Thus, in a sense, there is no loss of generality in restricting our attention to multifunctions with closed graph, and without this restriction, there is no hope of obtaining Hausdorff upper semicontinuity as a necessary condition. Our main result falls neatly out of Theorem 3.1 below, which can also be derived from the general results of Dolecki, Rolewicz, and Lechicki.

THEOREM 3.1. *Let X be a metric space and let Y be a complete metric space. Suppose Γ is a multifunction from X to Y with closed graph. The following are equivalent:*

(1) Γ *is upper semicontinuous and compact valued;*

(2) Γ *is Hausdorff upper semicontinuous and has totally bounded values;*

(3) *For each p in X and $\varepsilon > 0$ there exists a neighborhood V of p for which $\chi(\Gamma(V)) \leq \varepsilon$.*

PROOF. (1)→(2). Trivial.

(2)→(3). Fix $\varepsilon > 0$ and $p \in X$. Since Γ is Hausdorff u.s.c., there exists a neighborhood V of p for which $\Gamma(V) \subset S_{\varepsilon/2}[\Gamma(p)]$. Since $\Gamma(p)$ is totally bounded, we have

$$\chi(\Gamma(V)) \leq 2\chi(S_{\varepsilon/2}[\Gamma(p)]) \leq 2(\chi(\Gamma(p)) + \frac{\varepsilon}{2}) = \varepsilon.$$

(3)→(1). By the essential monotonicity of χ ($A \subset B$ implies $\chi(A) \leq 2\chi(B)$), it is clear that Γ has totally bounded values. Since Γ has a closed graph, Γ has closed values, whence by the completeness of the target space, Γ has compact values. Suppose Γ failed to be u.s.c. at some point p in X. We can then find an open neighborhood W of $\Gamma(p)$ such that for each n, $\Gamma(S_{1/n}[p])$ meets W^c. For each n, choose x_n in $\Gamma(S_{1/n}[p]) \cap W^c$, and for each n set $F_n = \mathrm{cl}(\{x_j : j \geq n\})$. We have

$$\chi(F_n) \leq 2\chi(\mathrm{cl}(\Gamma(S_{1/n}[p])) = 2\chi(\Gamma(S_{1/n}[p])).$$

By condition (3), we have $\lim_{n\to\infty} \chi(\Gamma(S_{1/n}[p])) = 0$; so, Kuratowski's Theorem yields a point y in $\bigcap_{n=1}^{\infty} F_n$. Clearly, y is a cluster point of $\langle x_n \rangle$ which lies in W^c. Thus, viewing Γ as a subset of $X \times Y$,the point (p, y) is in the closure of Γ but not in Γ itself, a contradiction.

THEOREM 3.2. *Let X be a locally compact and Y complete. Suppose Γ is a multifunction from X to Y with closed graph, and for each $\varepsilon > 0$, there exists a continuous function $f : X \to Y$ for which $h_\rho(f, \Gamma) < \varepsilon$. Then Γ is Hausdorff upper semicontinuous and has totally bounded values (equivalently, Γ is upper semicontinuous and has compact values).*

PROOF. We show condition (3) of Theorem 3.1 holds. Fix p in X and $\varepsilon > 0$. Choose $\delta \in (0, \varepsilon)$ for which $\mathrm{cl}(S_\delta[p])$ is compact. Next, choose a continuous function f for which $h_\rho(f, \Gamma) < \delta/2$. As a result, $\Gamma(S_{\delta/2}[p]) \subset S_{\delta/2}[f(S_\delta[p])]$. Since $f(\mathrm{cl}(S_\delta[p]))$ is compact, it measure of noncompactness is zero; so,

$$\chi(S_{\delta/2}[f(S_\delta[p])]) \leq \chi(f(S_\delta[p])) + \frac{\delta}{2} = \frac{\delta}{2}.$$

Again, by the essential monotonicity of χ, we have $\chi(\Gamma(S_{\delta/2}[p])) \leq \delta < \varepsilon$.

It turns out that Theorem 3.2 is quite sharp, i.e., the necessary conditions fail to be necessary if either X is not locally compact or Y is not complete, even if we restrict our attention to convex valued multifunctions.

EXAMPLE 3.3. We show that if Y is any normed linear space and X is a metric space that is not locally compact, then there exists a convex valued multifunction with closed graph from X to Y which admits h_ρ-approximations by continuous functions from X to Y, but which does not have totally bounded values. Let C be a separable closed convex subset of Y containing the origin θ with $\chi(C) > 0$, e.g., C could be a finite dimensional subspace. Suppose $p \in X$ has no compact neighborhood. Let $\varepsilon_1 = 1$, and choose a countably infinite subset E_1 of $\{x : 0 < d(x,p) < \varepsilon_1\}$ with no limit point. Choose $\varepsilon_2 > 0$ for which $E_1 \cap S_{2\varepsilon_2}[p] = \emptyset$. Now let E_2 be a countably infinite subset of $\{x : 0 < d(x,p) < \varepsilon_2\}$ with no limit point. Choose $\varepsilon_3 > 0$ for which $S_{2\varepsilon_3}[p] \cap E_2 = \emptyset$. Continuing, we produce for each $n \in Z^+$ a countably infinite set E_n with no limit points and a real sequence $\langle \varepsilon_n \rangle$ such that for each n, $0 < \varepsilon_{n+1} < \frac{1}{2}\varepsilon_n$ and $E_n \subset \{x : \varepsilon_{n+1} < d(x,p) < \varepsilon_n\}$. By the Dugundji Extension Theorem [8], for each n we can find a continuous function $g_n : X \to C$ such that $g_n(E_n)$ is dense in C, and $g_n(x) = \theta$ if either $d(x,p) \leq \varepsilon_{n+1}$ or $d(x,p) \geq \varepsilon_n$. Set $g = \sum_{n=1}^{\infty} g_n$, and define Γ by

$$\Gamma(x) = \begin{cases} C & \text{if } x = p \\ \{g(x)\} & \text{if } x \neq p. \end{cases}$$

The set $\Gamma(p)$ is not totally bounded, but for each n, $\sum_{k=1}^{n} g_k : X \to Y$ is continuous, and $h_\rho(\Gamma, \sum_{k=1}^{n} g_k) \leq 2^{2-n}$.

EXAMPLE 3.4. Let Y be a normed linear space that is not a Banach space. We produce a compact convex valued multifunction Γ with closed graph from $[0,1]$ to Y that admits continuous approximations in the above sense, yet is not upper semicontinuous. Let $\langle y_n \rangle$ be a nonconvergent Cauchy sequence in Y. Let $g : (0,1] \to Y$ map $[1/(n+1), 1/n]$ linearly onto the line segment joining y_{n+1} to y_n for each $n \in Z^+$. Since $\{[(1/n+1),(1/n)] : n \in Z^+\}$ is a locally finite closed cover of $(0,1]$, g is continuous. Thus, $\Gamma : [0,1] \to Y$ defined by

$$\Gamma(x) = \begin{cases} \{\theta\} & \text{if } x = 0 \\ \operatorname{conv}\{\theta, g(x)\} & \text{otherwise} \end{cases}$$

is u.s.c. at each x in $(0,1]$. Thus, to show Γ has a closed graph, we need only look at $x = 0$. If not, there exists a sequence $\langle x_n \rangle$ in $(0,1]$ convergent to 0 and a sequence $\langle \alpha_n \rangle$ in $(0,1]$ such that $\langle \alpha_n g(x_n) \rangle$ converges to a nonzero vector z. Since $\{y_n : n \in Z^+\}$ is bounded, $g((0,1])$ is bounded, and we conclude that $\liminf_{n\to\infty} \alpha_n > 0$. By passing to a subsequence, we may assume $\langle \alpha_n \rangle$ converges to a positive number α, so that $\langle g(x_n) \rangle$ converges to $\frac{1}{\alpha} z$. It follows that $\langle y_n \rangle$ has the same limit, an impossibility. Thus, Γ is closed. Obviously, Γ is not u.s.c. at $x = 0$

Now we turn to the approximation of Γ by continuous functions. Let $\varepsilon > 0$ be arbitrary, and choose $n > 2/\varepsilon$ is large that $||y_k - y_n|| < \varepsilon/2$ for each $k > n$. By Cellina's Theorem, there exists a continuous $g : [1/n, 1] \to Y$ such that $h_\rho(g, \Gamma|[1/n, 1]) < \varepsilon/2$. Define $f : [0,1] \to Y$ by

$$f(x) = \begin{cases} n \times g, (1/n) & \text{if } 0 \le x < 1/n \\ g(x) & \text{if } 1/n \le x. \end{cases}$$

Continuity of f is obvious, and it is a routine matter to show $h_\rho(f, \Gamma) \le \varepsilon$.

The completeness of the target space Y plays no role whatsoever in the necessary condition of totally bounded values. On the other hand, local compactness of the domain is not superfluous for the necessary condition of Hausdorff upper semicontinuity.

EXAMPLE 3.5. We construct a single valued discontinuous function f from the unit ball B in ℓ_2 to itself with a closed graph that can be approximated in Hausdorff distance by continuous functions. The function, viewed as a multifunction, will be compact convex valued, but will not be upper semicontinuous. To this end, let $\{e_n : n \in Z^+\}$ be the standard orthonormal basis for ℓ_2. To describe f, we first introduce an auxiliary function $\psi : [0,1] \to [0,1]$ defined by

$$\psi(t) = \begin{cases} 2t & \text{if } 0 \le t < 1/2 \\ 2 - 2t & \text{if } 1/2 \le t \le 1. \end{cases}$$

For each $n \in Z^+$, let $\alpha_n = \frac{1}{n} - \frac{1}{n+1}$, and define f by

$$f(x) = \begin{cases} \theta & \text{if } x = \theta \\ \psi\left[\frac{1}{\alpha_n}\left(\frac{1}{n} - ||x||\right)\right] \cdot e_n & \text{if } \frac{1}{n+1} < ||x|| \leq \frac{1}{n} \text{ for some } n. \end{cases}$$

If we restrict f to any ray emanating from the origin, then its graph looks like a series of spikes rotating about an axis in infinite dimensional space. Evidently, f is continuous at all points except $x = \theta$, and if $\langle (x_n, f(x_n)) \rangle$ converges to (θ, y), we must have $y = \theta$. Thus, f has a closed graph. To show that f can be approximated by continuous functions, let ε be positive, and choose $n \in Z^+$ for which $2/n < \varepsilon$. Choose $\delta \in (0, \frac{1}{2n})$ such that $\{S_\delta[\frac{1}{2n}e_j] : j \in Z^+\}$ is a disjoint family of balls. Define $g : B \to B$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } ||x|| \geq 1/n \\ (1 - \frac{1}{\delta}||x - \frac{1}{2n}e_j||) \cdot e_{n+j-1} & \text{if } ||x - \frac{1}{2n}e_j|| < \delta \text{ for some } j \\ \theta & \text{otherwise.} \end{cases}$$

The function g so constructed is well defined and continuous. For each $j \in Z^+, g$ maps $S_\delta[(1/2n)e_j]$ onto the line segment joining θ to e_{n+j-1}. As a result, the image of $S_{1/n}[\theta]$ under g is the union of the line segments joining θ to e_j for $j = n, n+1, n+2, \ldots$. From this observation, it is easy to check that $h_\rho(g, f) \leq 2/n < \varepsilon$.

4. Some extensions of Cellina's theorem. In this section we look at some extensions of Cellina's Theorem to multifunctions that do not have convex values. At the heart of these extensions, as well as Cellina's Theorem itself, is the following paracompactness theorem (see pages 414-415 of [**3**] and page 181 of [**1**]).

LEMMA 4.1. *Let $\langle X, d \rangle$ be a metric space with no isolated points and let Ω be an open cover of X. Then there exists a locally finite open refinement $\{V_i : i \in I\}$ of Ω and closed balls $\{B_i : i \in I\}$ such that $B_i \subset V_i$ for each $i \in I$, and $B_i \cap V_j = \theta$ whenever $i \neq j$.*

LEMMA 4.2. *Let p be a limit point of metric space X and let E be a path connected totally bounded subset of a metric space Y. If $\varepsilon > 0$, $\delta > 0$ and $y \in E$ are arbitrary, then there exists a continuous function f from X to E such that $f(S_\delta[p])$ is ε-dense in E and $f(x) = y$ whenever $d_X(x, p) \geq \delta$.*

PROOF. Let $\{y_1, y_2, \ldots, y_n\}$ be an ε-dense subset of E; w.l.o.g., we may assume that $y_n = y$. Let $\{x_1, x_2, \ldots, x_n\}$ be n points in

$S_\delta[p]$ such that $d_X(p,x_i) < d_X(p,x_j)$ whenever $i < j$. For each $i \in \{1,2,\ldots,n-1\}$, let $\lambda_i = d_X(p,x_{i+1}) - d_X(p,x_i)$. For each such i, let ϕ_i be a path from $[0,1]$ to E such that $\phi_i(0) = y_i$ and $\phi_i(1) = y_{i+1}$. The desired function f is given by

$$f(x) = \begin{cases} y_1 & \text{if } d_X(p,x) < d_X(p,x_1) \\ \phi_i[\frac{1}{\lambda_i}(d_X(p,x) - d_X(p,x_i))] & \text{if } d_X(p,x_i) \le d_X(p,x) < d_X(p,x_{i+1}) \\ y_n & \text{if } d_X(p,x_n) \le d_X(p,x). \end{cases}$$

THEOREM 4.3. *Let $\langle X,d\rangle$ be a metric space and let Y be a normed linear space. Suppose Γ is Hausdorff upper semicontinuous multifunction from X to Y with totally bounded path connected values, and Γ contains some Hausdorff upper semicontinuous convex valued multifunction $\sum$. If Γ maps isolated points of X to singletons, then for each $\varepsilon > 0$ there exists a continuous function $f : X \to Y$ for which $h_\rho(f,\Gamma) \le \varepsilon$.*

PROOF. For simplicity, we assume that X has no isolated points (the general case can be handled as in Theorem 1 of [**1**], using Lemma 5 of [**1**]). By the Hausdorff upper semicontinuity of Γ and $\sum$, there exists for each $x \in X$ a positive number $\lambda_x < \varepsilon$ such that if $z \in S_{\lambda_x}[x]$, then both $\Gamma(z) \subset S_{\varepsilon/2}[\Gamma(x)]$ and $\sum(z) \subset S_\varepsilon[\sum(x)]$. Let $\{V_i : i \in I\}$ and $\{B_i : i \in I\}$ satisfy the conditions of Lemma 4.1 with respect to the cover $\Omega = \{S_{\lambda_x/2}[x] : x \in X\}$ of X. For each $i \in I$, choose $x(i)$ in X such that $V_i \subset S_{\lambda_{x(i)}/2}[x_i]$, and let $f_i : X \to \Gamma(x(i))$ have these properties:

(i) $f_i(B_i)$ is $\varepsilon/2$-dense in $\Gamma(x(i))$;

(ii) $f_i(B_i^c)$ is a singelton subset of $\sum(x(i))$.

Now let $\{p_i : i \in I\}$ be a partition of unity subordinated to $\{V_i : i \in I\}$. We claim that $f = \sum p_i f_i$ is the desired continuous function. First,we show that $\Gamma \subset S_\varepsilon[f]$. If $z \in X$ is arbitrary, choose $i \in I$ such that $z \in V_i \subset S_{\lambda_{x(i)}/2}[x_i]$. By construction, we have

$$\Gamma(z) \subset S_{\varepsilon/2}[\Gamma(x(i))] \subset S_\varepsilon[f_i(B_i)].$$

Now since $f|B_i = f$, for each $y \in \Gamma(z)$ there exists x in B_i such that $\|y - f(x)\| < \varepsilon$. Also, since x and z both lie in V_i, we obtain $d_X(x,z) < \lambda_{x(i)} < \varepsilon$. This proves $\Gamma \subset S_\varepsilon[f]$. We finally show that $f \subset S_\varepsilon[\Gamma]$. Again, let $z \in X$ be arbitrary. If there exists an index i for which $z \in B_i$, then $f(z)$ lies in $\Gamma(x(i))$, whence $(z, f(z)) \in$

$S_{\lambda_{x(i)}}[\{x(i)\} \times \Gamma(x(i))] \subset S_\varepsilon[\Gamma]$. Otherwise, let $\{i_1, i_2, \ldots, i_n\}$ be those indices for which $p_i(z)$ is nonzero. Now $f_{i_j}(z)$ is actually a point in $\sum(x(i_j))$ for each $j \in \{1, 2, \ldots, n\}$, and there exists some fixed index j such that $\{x(i_1), x(i_2), \ldots, x(i_n)\}$ is contained in the ball of radius $\lambda_{x(i_j)}$ with center $x(i_j)$. As a result, $f(z)$ is a convex combination of points of the convex set $S_\varepsilon[\sum(x(i_j))]$. Since $\sum(x(i_j)) \subset \Gamma(x(i_j))$, the point $f(z)$ has distance less than ε from some point in $\Gamma(x(i_j))$, and it easily follows that $(z, f(z)) \in S_\varepsilon[\Gamma]$.

A continuous single valued function, when viewed as a multifunction, is upper semicontinuous. Thus, a Hausdorff upper semicontinuous multifunction with path connected totally bounded values can be approximated in Hausdorff distance by continuous functions, provided it admits a continuous selection. But we have a completely different application in mind. Recall that if x and y are points of a set E in a linear space, then we say *x sees y via E* if the line segment $\operatorname{conv}\{x, y\}$ joining x to y lies in E. The set E is called *starshaped* if there exists some point p of E that sees each other point of E via E. The set of points with this property is convex [**16**], and is called the *kernel* of E. We denote this set by $\ker E$.

LEMMA 4.4. *Let X be a metric space and let Y be a normed linear space. Let Γ be a continuous multifunction from X to Y with compact starshaped values. Then the kernel multifunction $\sum$ associated with Γ defined by $\sum(x) = \ker \Gamma(x)$ is Hausdorff upper semicontinuous and convex valued.*

PROOF. Fix p in X. If $\sum$ is not Hausdorff upper semicontinuous at p, then there exists $\varepsilon > 0$, a sequence $\langle x_n \rangle$ convergent to p, and for each n $y_n \in \sum(x_n)$, such that $y_n \notin S_\varepsilon[\sum(p)]$. Since $\Gamma(p)$ is compact and Γ is u.s.c. at p, by passing to a subsequence, we may assume that $\langle y_n \rangle$ is convergent to some point $y \in \Gamma(p) \cap (S_\varepsilon[\sum(p)])^c$. Now let $z \in \Gamma(p)$ be arbitrary. Since $\langle \Gamma(x_n) \rangle$ converges to $\Gamma(p)$ in Hausdorff distance, we have $\Gamma(p) = \lim \Gamma(x_n)$ in the sense of Kuratowski, i.e., $\operatorname{Li}\Gamma(x_n) = \operatorname{Ls}\Gamma(x_n) = \Gamma(p)$ (see e.g., [**13**], [**14**], or [**15**]). Since $\Gamma(p) \subset \operatorname{Li}\Gamma(x_n)$, there is a sequence $\langle z_n \rangle$ convergent to z such that for each n, $z_n \in \Gamma(x_n)$. Since y_n sees z_n via $\Gamma(x_n)$ and $\operatorname{Ls}\Gamma(x_n) \subset \Gamma(p)$, we conclude that y sees z via $\Gamma(p)$, an impossibility, for y does not lie in the kernel of $\Gamma(p)$.

EXAMPLE 4.5. A continuous compact starshaped valued multifunction need not have a continuous kernel: consider Γ from $[0,1]$ to R^2 defined by $\Gamma(x) = \operatorname{conv}\{(0,0),(0,1)\} \cup \operatorname{conv}\{(0,1),(x,2)\}$. An u.s.c. compact *starshaped* valued multifunction need not have an u.s.c kernel: consider Γ from $[0,1]$ to R^2 defined by

$$\Gamma(x) = \begin{cases} \operatorname{conv}\{(0,0),(1,0)\} & \text{if } x > 0 \\ \operatorname{conv}\{(0,0),(1,0)\} \cup \operatorname{conv}\{(0,0),(0,1)\} & \text{if } x = 0. \end{cases}$$

THEOREM 4.6. *Let $\langle X, d\rangle$ be a metric space and let Y be a normed linear space. Suppose Γ is continuous multifunction from X to Y with totally bounded starshaped values. If Γ maps isolated points to singletons, then for each $\varepsilon > 0$, there exists a continuous function $f : X \to Y$ for which $h_\rho(f,\Gamma) \le \varepsilon$.*

PROOF. Let W be a complete normed linear space in which Y is linearly and isometrically imbedded, and let Γ^* from X to W assign to each x in X the W-closure of $\Gamma(x)$. Clearly, each set $\Gamma^*(x)$ is both compact and starshaped, and the kernel of $\Gamma^*(x)$ contains the kernel of $\Gamma(x)$. Also, for each x and z in X, we have $h_\rho[\Gamma^*(x),\Gamma^*(z)] = h_\rho[\Gamma(x),\Gamma(z)]$: so, Γ^* is continuous. By Lemma 4.4 and Theorem 4.3, Γ^* can be approximated in Hausdorff distance by continuous functions from X to Z. Now with respect to Γ^*, the functions $\{f_i : i \in I\}$ in the proof of Theorem 4.3 can be constructed so that the range of each f_i lies in $\Gamma(x(i))$ rather than in the larger set $\Gamma^*(x(i))$, because an $\varepsilon/2$-dense subset of $\Gamma^*(x(i))$ can be chosen from $\Gamma(x(i))$, and the kernel of $\Gamma(x(i))$ contains the kernel of $\Gamma^*(x(i))$. Thus, with respect to Γ^*, the values of the amalgamated function f in the proof of Theorem 4.3 can be arranged to be convex combinations of points of Y rather than W. As a result, Γ^*, and hence Γ, can be approximated in Hausdorff distance by continuous functions from X to Y.

EXAMPLE 4.7. Not all continuous continua valued multifunctions can be approximated by continuous functions in Hausdorff distance. Let C denote the unit disc in the plane. Our continua valued multifunction Γ is defined on C as follows: Γ maps the origin to the unit circle, and each point of C with polar coordinates (r,ϕ) to an antipodal arc with center $(1,\phi+\pi)$ and arc length $2\pi(1-r)$. We claim that for some $\varepsilon > 0$ there is no continuous function from C to the plane that ε-approximates Γ in Hausdorff distance. Otherwise, for each $n \in Z^+$ we can find $f_n : C \to R^2$ with $h_\rho(f_n,\Gamma) < 1/n$. If we follow each f_n by the standard retraction of R^2 onto C, we obtain a self-map f_n^* of the disc which still satisfies $h_\rho(f_n^*, \Gamma) < 1/n$. By Brouwer's Theorem, each f_n^* has a fixed point, and we conclude Γ contains some point (x_n, y_n) whose ρ-distance from the diagonal in $C \times C$ is at most $1/n$. Since the diagonal is compact and Γ is closed (in fact, compact), Γ meets the diagonal. This means that for some $x \in C$, we have $x \in \Gamma(x)$, a contradiction.

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DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, LOS ANGELES, LOS ANGELES, CA 90032