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BOUNDS ON THE DIMENSION OF SPACES OF MULTIVARIATE PIECEWISE POLYNOMIALS

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1. Introduction. Spaces of piecewise polynomials defined over a partition of a planar set are of considerable interest in approximation theory. In addition to their usefulness in a variety of data fitting problems, they also play a central role in the finite element method. Clearly, they are a natural generalization of the classical one-dimensional polynomial spline functions.

Despite their obvious importance, until recently there has been relatively little work on general spaces of piecewise polynomials in two variables. In the last few years, however, the literature has grown considerably—see **[1–25]** and references therein.

Some years ago in **[19]**, I gave a lower bound on the dimension of spaces of piecewise polynomials defined on a triangulation. The purpose of this paper is to present both lower and upper bounds for general rectilinear partitions. The plan of the paper is as follows. In the remainder of this section we introduce the spline spaces of interest and establish some notation. In §2 and §3 we establish our upper and lower bounds, respectively. §4 of the paper contains a variety of applications to special partitions. We conclude the paper with remarks.

Suppose Ω is a closed subset of $\mathbf{R}^2$, and suppose that $\Delta = \{\Omega_i\}$ is a collection of open subsets such that

$$1) \qquad \Omega = \bigcup_1^n \bar{\Omega}_i$$

$$2) \qquad \Omega_i \cap \Omega_j = \emptyset\,, \text{ all } i, j = 1, 2, \ldots, n.$$

We call Δ a partition of Ω. If each Ω_i is a polygon, then we call Δ a rectilinear partition. If each Ω_i is a triangle and if no vertex of any triangle lies in the middle of an edge of another triangle, then we call Δ a triangulation of Ω.

Given a positive integer d we define the space of polynomials of order d (in two variables) by

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$$P_d = \{p(x, y) = \sum_{i=0}^{d} \sum_{j=0}^{d-i} a_{ij}\, x^i y^j,\ a_{ij} \in \mathbf{R}\}.$$

DEFINITION 1.1. Let $0 < r < d$, and set

$$S_d^r(\Delta) = \{s \in C^r(\Omega) : s|_{\Omega_i} \in P_d,\ i = 1, \ldots, n\} \tag{1.1}$$

We call S the space of polynomial splines of order d and smoothness r associated with the partition Δ.

It is clear that S is a linear space. In this paper we are interested in computing its dimension. It turns out that it is not possible to give a general formula—there are some cases where the dimension depends on the exact geometry of the partition — see **[17, 19]**. In general, we must be satisfied with upper and lower bounds for it.

2. An upper bound on dimension. In order to state our main result, we need some additional notation. Throughout the remainder of this section we shall suppose that Δ is a rectilinear partition of a set Ω. Given such a partition, we call the straight line segments making up the partition the edges of the partition, and refer to the points where these edges join together as the vertices of the partition. We denote the number of edges and the number of vertices in the interior of Ω by E and V, respectively.

Associated with the integers d and r, we define

$$\alpha = (d + 1)(d + 2)/2, \qquad \beta = (d - r)(d - r + 1)/2 \tag{2.1}$$

and

$$\gamma = [(d + 1)(d + 2) - (r + 1)(r + 2)]/2. \tag{2.2}$$

We are now ready for the main result of the paper.

THEOREM 2.1. *Suppose that the vertices of the partition Δ are numbered in such a way that each pair of consecutive vertices in the list are corners of a common subset in Δ. For each $i = 1, 2, \ldots, V$, let*

(2.3) $\tilde{e}_i$ = *number of edges with different slopes attached to the i-th vertex but not attached to any of the first $i - 1$ vertices in the list,*

and let

$$\tilde{\sigma}_i = \sum_{j=1}^{d-r} (r + j + 1 - j \cdot \tilde{e}_i)_+ . \tag{2.4}$$

Then

$$\dim S_d^r(\Delta) \leqq \alpha + \beta E - \gamma V + \sum_{i=1}^{V} \tilde{\sigma}_i . \tag{2.5}$$

PROOF. Let N be the number on the right-hand side of (2.5). By an elementary lemma of linear algebra (cf. [19, Lemma 3.3]), it suffices to construct linear functionals $\lambda_1, \ldots, \lambda_N$ such that

(2.6) if $s \in S$ and $\lambda_i s = 0$, $i = 1, \ldots, N$, then $s \equiv 0$.

Suppose the vertices of the partition are $\xi_1, \ldots, \xi_V$, and let Ω^0 be a set in Δ with a corner at ξ_1 (cf. Figure 1). It is well-known (cf. [16]) that we can find a set Λ^0 of α point functionals in Ω^0 which annihilate P_d. Let E_1 be the number of edges attached to the vertex ξ_1. We claim that we can find an additional $\beta E_1 - \gamma + \tilde{\sigma}_1$ functionals to obtain a set Λ^1 which annihilates any function in $s|_{\Omega^1}$, where $\Omega^1 = \bigcup\{\Omega_i\colon \Omega_i$ has a vertex at $\xi_1\}$. To show this, we may suppose that ξ_1 is at the origin, and that the figure is rotated so that none of the edges lie on the x or y axes. Suppose we number the edges counter clockwise, starting from Ω^0. Each of these edges is described by an angle θ_j or equivalently by an equation $y + \alpha_j x = 0$, $j = 1, \ldots, E_1$.

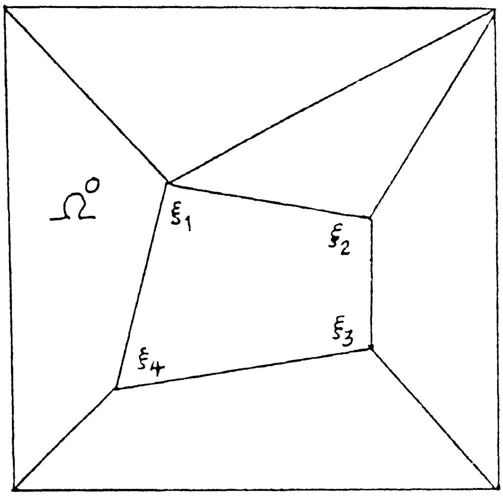

FIGURE 1. A rectilinear partition.

Now suppose $s \equiv 0$ on Ω^0. Then after crossing the first edge, S must have the form

$$s(x, y) = \sum_{j=1}^{d-r} \sum_{k=1}^{j} c_{j,k}\, \phi^1_{j,k}(x. y), \tag{2.7}$$

where in general $\phi^i_{j,k}(x, y) = x^{j-k}(y + \alpha_i x)^{r+k}$. Define

$$\phi^i_{j,k}(x, y)_+ = \begin{cases} \phi^i_{j,k}(x, y) & \text{if } \arctan(y/x) \geqq \theta_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows that the set of functions in $S|_{\Omega^1}$ Which vanish on Ω^0 are precisely the functions of the form

$$s(x, y) = \sum_{\nu=1}^{n} \sum_{j=1}^{d-r} \sum_{k=1}^{j} c_{\nu jk} \phi^\nu_{jk}(x, y)_+, \qquad (n = E_1)$$

where the coefficients satisfy the equations

$$\sum_{\nu=1}^{n} \sum_{j=1}^{d-r} \sum_{k=1}^{j} c_{\nu jk} \phi^\nu_{jk}(x, y) = 0$$

for all $x, y \in \Omega^0$. By equating the coefficients of the various powers of $x^\nu y^\mu$ to zero, we can rewrite this as a homogeneous system of linear equations

(2.8) $$AC = 0,$$

where $c = (c_1, \ldots, c_{d-r})^T$, $c_j = (c_{1jj}, \ldots, c_{1j1}, \ldots, c_{njj}, \ldots, c_{nj1})^T$, and

$$A = \begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & A_{d-r} \end{bmatrix},$$

where for each $j = 1, \ldots, d-r$, A_j is an $r + j = 1$ by $n \cdot j$ matrix of the form $A_j = [A_{j1}, \ldots, A_{jn}]$ with A_{ji} given by

$$\begin{bmatrix} 1 & & & \\ \binom{r+j}{1}\alpha_i & 1 & & \\ \binom{r+j}{2}\alpha_i^2 & \binom{r+j-1}{1}\alpha_i & & 1 \\ \cdots & & & \\ \binom{r+j}{r+j}\alpha_i^{r+j} & \binom{r+j-1}{r+j-1}\alpha_i^{r+j-1} & \cdots & \binom{r+1}{r+1}\alpha_i^{r+1} \end{bmatrix}.$$

Clearly (2.8) is a system of γ equations in $n\beta$ unknowns. It is shown in **[19]** that the rank of A is $\gamma - \bar{\sigma}_1$. We conclude that we can add $n\beta - \gamma + \bar{\sigma}_1$ requations to force the coefficients to be zero. Since $n = E_1$, this is equivalent to adding $\beta E_1 - \gamma + \bar{\sigma}_1$ linear functionals to Λ^0 to get Λ^1.

We now continue this process one vertex at a time. In particular, if E_i denotes the number of edges attached to the vertex ξ_i (but not to any of the vertices $\xi_1, \ldots, \xi_{i-1}$), then we can add a set of $\beta E_i - \gamma + \bar{\sigma}_i$ functionals to Λ^{i-1} to get a set Λ^i which annihilates splines on

$$\Omega^i = \Omega^{i-1} \bigcup \{\Omega_j\colon \Omega_j \text{ has a corner at } \xi_i\}.$$

After proceeding through all vertices and adding β functionals associated with each remaining uncounted edge, we end up with a set of N linear functionals which annihilates all of S. This completes the proof.

For convenience, we give values of α, β, and γ in Table 1 for several choices of d and r.

d	r	α	β	γ
2	1	6	1	3
3	1	10	3	7
4	1	15	6	12
5	1	21	10	18
3	2	10	1	4
4	2	15	3	9
5	2	21	6	15
4	3	15	1	5
5	3	21	3	11

TABLE 1. The coefficients (2.1)–(2.2) for some choices of d and r.

It is clear that the upper bound given in Theorem 2.1 is numerically computable. The following example shows that its value depends on the ordering of the vertices.

EXAMPLE 2.2. Let Ω and Δ be as shown in Figure 2, and let $d = 2$ and $r = 1$. Compute an upper bound for the associated spline space.

DISCUSSION. Here $\alpha = 6$, $\beta = 1$, and $\gamma = 3$. If we order the vertices so that the lower one comes first, then we have $\tilde{\sigma}_1 = (3 - \tilde{e}_1)_+ = 0$ and $\tilde{\sigma}_2 = (3 - \tilde{e}_2)_+ = 1$, and hence $\dim S_2^1(\Delta) \leqq 6 + 7 - 2 \cdot 3 + 1 = 8$.

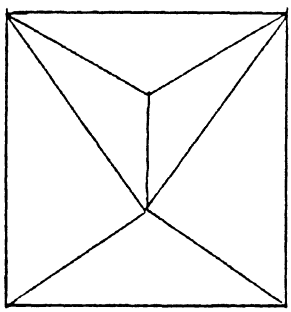

FIGURE 2. The partition for Example 2.2.

On the other hand, if we order the vertices so that the upper one comes first, then $\tilde{\sigma}_1 = \tilde{\sigma}_2 = 0$, snd now dim $S_2^1(\Delta) \leqq 6 + 7 - 2 \cdot 3 = 7$.

3. A lower bound on dimension. In order to be able to use the upper bound of Theorem 2.1 to determine the exact dimension of a space of splines, we need to have a lower bound to combine it with. We begin by presenting a lower bound which applies to arbitrary rectilinear partitions.

THEOREM 3.1. *Let Δ be a rectilinear partition of a set Ω, and let α, β and γ be as in* (2.1)-(2.2). *Given any ordering of the vertices, let*

(3.1) e_i = *number of edges with different slopes attached to the i-th vertex,*

$$\sigma_i = \sum_{j=1}^{d-r} (r + j + 1 - j \cdot e_i)_+, \quad i = 1, \ldots, V. \tag{3.2}$$

Then

$$\dim S_d^r(\Delta) \geqq \alpha + \beta E - \gamma V + \sum_{i=1}^{V} \sigma_i. \tag{3.3}$$

PROOF. The proof follows along the same lines as the proof of Theorem 3.1 of [**19**]. In particular, if Δ is a partition of a set Ω with only one interior vertex, then the argument proceeds exactly as before. To get the result for a general partition, we use a merging procedure. Indeed, given any vertex on the boundary of Ω connected to an interior vertex ξ^* by an edge, we may remove one polygon having these vertices to get a new set Ω_1 with one less interior vertex (cf. Figure 3). Assuming the result for partitions with $V - 1$ interior vertices, we can then merge this space with a spline space over the cell Ω_2 with interior vertex ξ^* to get the result — cf. the argument in [**19**].

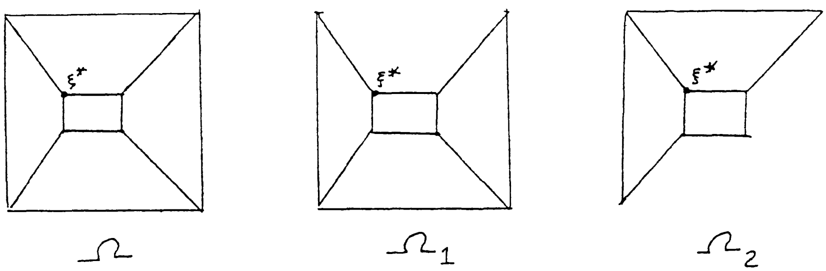

FIGURE 3. Merging cells of a partition.

We note that the lower bound in (3.3) has exactly the same form as the upper bound in (2.5), the only difference being that here σ_i replaces $\tilde{\sigma}_i$.

By the definitions (2.4) and (3.2), it is clear that for each $i = 1, \ldots, n$, $\sigma_i \leqq \tilde{\sigma}_i$, and thus the upper bound is greater than or equal to the lower bound. We have the following immediate corollary.

COROLLARY 3.2. *Suppose that for some ordering of the vertices of a triangulation Δ we have*

$$\sigma_i = \tilde{\sigma}_i, \; i = 1, \ldots, V. \tag{3.4}$$

Then the expressions in (2.5) *and* (3.3) *agree and give the dimension of the spline space* (1.1).

The following example (cf. [**17, 19**)] shows that even for relatively simple triangulations, it may happen that our lower and upper bounds do not agree.

EXAMPLE 3.3. Let Δ be the triangulation show in Figure 4, and let $d = 2$ and $r = 1$. Compute the dimension of the corresponding spline space.

DISCUSSION. It is easy to see that $\sigma_1 = \sigma_2 = \sigma_3 = 0$ while no matter how we order the veitices, we will always have $\tilde{\sigma}_3 = 1$. It follows that $6 \leqq \dim S_2^1(\Delta) \leqq 7$. Indeed, it is known (cf. [**17, 19**]) that the exact dimension of this spline space depends on the location of the vertices. If the figure is symmetric, the dimension is seven; otherwise it is six.

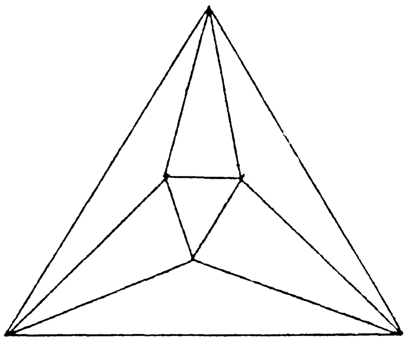

FIGURE 4. The partition in Example 3.3.

We now give an example of a rectangular partition where the upper and lower bounds do agree. Other examples can be found in §5.

EXAMPLE 3.4. Let Δ be the rectilinear partition shown in Figure 5. Let $d = 2$ and $r = 1$. Compute the dimension of the associated spline space.

DISCUSSION. It is easily seen that if we number the vertices as shown in

Figure 5, then $\tilde{\sigma}_i = (3 - \tilde{e}_i)_+ = 0$, $i = 1, \ldots, 4$, It follows that $\dim S_2^1(\Delta) = 6 + 17 - 4 \cdot 3 = 11$.

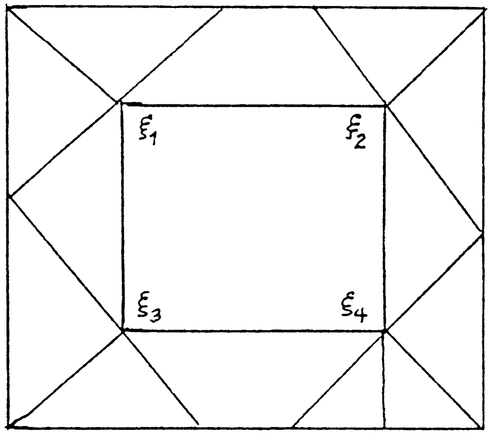

FIGURE 5. The partition in Example 3.4.

4. Examples and applications. In this section we apply our upper and lower bounds to several cases of interest. We begin with a special partition of a rectangle. Let

$$\Omega = [a, b] \times [\tilde{a}, \tilde{b}]$$

$$a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b \tag{4.1}$$

$$\tilde{a} = \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_{\tilde{k}} < \tilde{x}_{\tilde{k}+1} = \tilde{b}$$

If Δ is the partition of Ω which is obtained by drawing grid lines at the point $x_1, \ldots, x_k$ and $\tilde{x}_1, \ldots, \tilde{x}_{\tilde{k}}$ along with the upward sloping diagonals (cf. Figure 6), then we call Δ a type one partition.

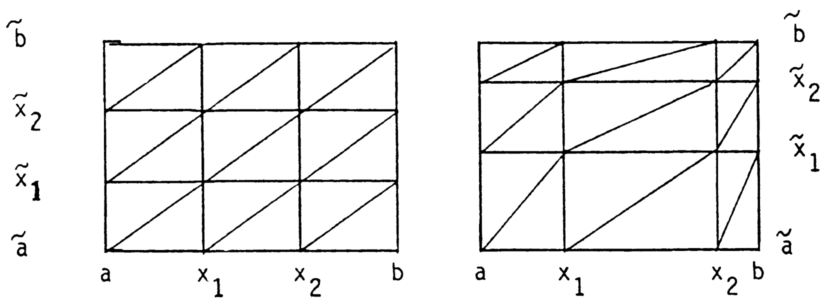

FIGURE 6. Type-1 partitions.

THEOREM 4.1. *Let Δ be an equally spaced type-1 partition of a rectangle Ω. Then for all* $0 \leqq r < d$,

$$\begin{aligned} \dim S_d^r(\Delta) &= k\tilde{k}(d^2 - 3rd + 2r^2 + \sigma) \\ &\quad + (k + \tilde{k})(d^2 - 2rd + d - r + r^2) \\ &\quad + (2d^2 + 4d - 2rd - r + r^2 + 2)/2, \end{aligned} \tag{4.2}$$

where

$$\sigma = \begin{cases} r^2/4 & \textit{if } r \textit{ is even and } 3r + 1 > 2d, \\ (r^2 - 1)/4 & \textit{if } r \textit{ is odd and } 3r + 1 > 2d, \\ (d - r)(2r - d) & \textit{otherwise.} \end{cases} \tag{4.3}$$

PROOF. We easily check that $V = k\tilde{k}$ and $E = 3k\tilde{k} + 2(k + \tilde{k}) + 1$. If we put the vertices of the partition in lexicographical order, then we note that $e_i = \tilde{e}_i = 3$ for all i, and thus

$$\sigma_i = \tilde{\sigma}_i = \sigma := \sum_{j=1}^{d-r} (r + 1 - 2 \cdot j)_+$$

for all $i = 1, \ldots, V$. It follows that Corollary 3.2 can be applied, and after some algebra, we obtain (4.2)

Our next theorem deals with arbitrary type-1 partitions.

THEOREM 4.2. *Let Δ be a general type-1 partition. Then for all* $1 < d$,

$$\dim S_d^1(\Delta) = k\tilde{k}(d^2 - 3d + 2) + (k + \tilde{k})(d^2 - d) + (d^2 + d + 1). \tag{4.4}$$

PROOF. If we put the vertices in lexicographical order, then it is easy to see that $\tilde{e}_i$ is always at least three, and thus $\tilde{\sigma}_i = 0$ for $i = 1, \ldots, V$. Now applying Corollary 3.2, we obtain (4.4).

Theorem 4.2 gives the dimension of C^1 spline spaces on general type-1 partitions. The following example shows that for $r > 1$, our upper and lower bounds do not agree, and in fact the actual dimension can be equal to the upper bound.

EXAMPLE 4.3. Let Δ be the unequally spaced type-1 partition shown in Figure 6 with $k = \tilde{k} = 2$. Let $d = 3$ and $r = 2$.

DISCUSSION. If we put the vertices in lexicographical order, then we see that $\sigma_i = 0$ and $\tilde{\sigma}_i = \delta_{i3}$, $i = 1, \ldots, 4$. It follows that $15 \leqq \dim S_3^2(\Delta) \leqq 16$. The actual dimension of this space is sixteen. Indeed, in addition to the ten linearly independent polynomials in S, the following six splines also belong to the space:

$$(x - x_1)_+^3,\ (x - x_2)_+^3,\ (y - \tilde{x}_1)_+^3,\ (y - \tilde{x}_2)_+^3,$$
$$(y - \tilde{x}_2 - (x - x_0)(\tilde{x}_3 - \tilde{x}_2)/(x_1 - x_0))_+^3,\ (y - x_0 - (x - x_2)(\tilde{x}_1 - \tilde{x}_0)/(x_3 - x_2))_+^3.$$

Theorem 4.1 asserts that this space has dimension nineteen in the case where Δ is an equally spaced type-1 partition.

We turn now to another special partition of a rectangle. Let Ω and grid points $x_1 \ldots, x_k$ and $\tilde{x}_1, \ldots, \tilde{x}_{\tilde{k}}$ be given as in (4.1). If Δ is a partition of Ω which is obtained by drawing all the grid lines plus both diagonals in each subrectangle, we call Δ a type-2 partition of Ω. Typical type-2 partitions are shown in Figure 7.

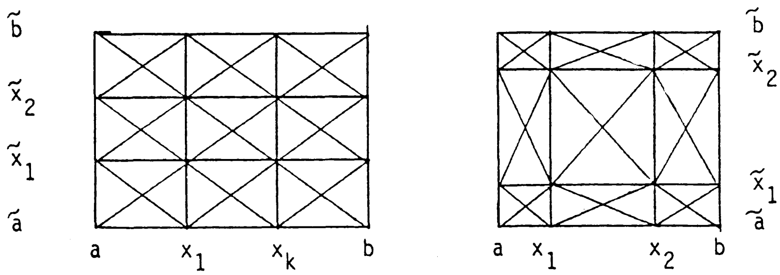

FIGURE 7. Type-2 partitions.

THEOREM 4.4. *Let Δ be an equally spaced type-2 partition of a rectangle Ω. Then for all* $0 \leqq r < d$,

$$\begin{aligned} \dim S_d^r(\Delta) = {} & k\tilde{k}(2d^2 - 6rd + 4r^2 + \sigma_g + \sigma_c) \\ & + (k + \tilde{k})(2d^2 - 5rd + d - r + 3r^2 + \sigma_c) \qquad (4.5) \\ & + (4d^2 + 4d - 8rd - r + 5r^2 + 2 + 2\sigma_c)/2, \end{aligned}$$

where

$$\sigma_g = \sum_{j=1}^{d-r} (r + 1 - 3 \cdot j)_+, \qquad \sigma_c = \sum_{j=1}^{d-r} (r + 1 - j)_+. \qquad (4.6)$$

PROOF. Here there are $k\tilde{k}$ vertices at the corners of the grid and an additional $k\tilde{k} + (k + \tilde{k}) + 1$ vertices where the diagonals cross. The number of edges is given by $E = 6k\tilde{k} + 5(k + \tilde{k}) + 4$. If we put the grid vertices in lexicographical order, followed by the cross vertices, then we note that $\sigma_i = \widetilde{\sigma_i} = \sigma_g$ for $i = 1, \ldots, k\tilde{k}$. For the remaining points we have $\sigma_i = \widetilde{\sigma_i} = \sigma_c$, $i = k\tilde{k} + 1, \ldots, V$. Now Corollary 3.2 applies, and after some algebra, we obtain (4.5).

The theorem above deals with equally-spaced type-2 partitions. In the unequally-spaced case, we have the following result.

THEOREM 4.5. *Let Δ be an arbitrary type-2 partition of a rectangle Ω. Then for all $r < d$ with $r = 0, 1, 2$,*

(4.7) $\dim S_d^r(\Delta)$ *is given by the formula* (3.3).

PROOF. If we order the vertices as in the proof of Theorem 4.4, then since $\tilde{e}_i \geqq 4$, we see that $\tilde{\sigma}_i = 0$, $i = 1, \ldots, k\tilde{k}$. On the other hand, since $e_i = \tilde{e}_i = 2$ and thus $\sigma_i = \tilde{\sigma}_i$ for $i = k\tilde{k} + 1, \ldots, V$, Corollary 3.2 applies to establish the result.

We close this section with a result on a more general kind of partition. Let Ω be the closure of an arbitrary domain, and suppose that a partition Δ is a simple cross cut partition (cf. [2]) obtained by drawing L lines across Ω. (Being simple requires that exactly two lines meet at each vertex in Ω—see Figure 8).

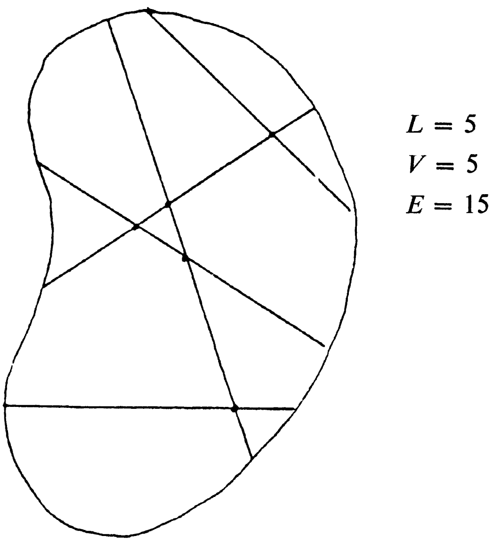

FIGURE 8, A Simple cross-cut partition

THEOREM 4.6. *Let Δ be a simple cross-cut partition of a set Ω, and let α and β be as in* (2.1). *Then for all* $0 \leqq r < d < 2r + 1$,

(4.8) $$\dim S_d^r(\Delta) = \alpha + \beta E - V(d^2 + 3d - 2r^2 - 4r)/2,$$

while if $2r + 1 \leqq d$, *then*

$$\dim S_d^r(\Delta) = \alpha + \beta(E - 2V). \tag{4.9}$$

PROOF. Here $e_i = \tilde{e}_i = 2$ for $i = 1, \ldots, V$, and so

$$\sigma_i = \tilde{\sigma}_i = \sum_{j=1}^{d-r} (r+1-j)_+ = \begin{cases} r(r+1)/2 & \text{if } 2r+1 < d, \\ -3r^2 - r + d + 4dr - d^2 & \text{if } r < d \leqq 2r. \end{cases}$$

The result now follows from Corollary 3.2.

The result (4.9) agrees with the formula obtained in Theorem 5.1 of **[2]** when we take note of the fact (cf. Lemma 5.1 of **[2]**) that $L = E - 2V$.

5. Remarks. 1. In this paper we have confined our attention to rectilinear partitions since it is very difficult to see what the connection would be between polynomials in adjoining regions separated by a curved boundary. The boundary of Ω itself can, of course, be curved.

2. The idea of obtaining an upper bound on dimension by placing linear functionals was used already in several earlier papers—see, e.g., **[17–19]**. Despite using them in a similar way on some special cases in **[19]**, I did not see the general result at the time, however.

3. Generally I have followed the notation of **[19]** throughout this paper. One notable change, however, is that here I am using d for the degree of the polynomials, while in **[19]** I used m for the order. The two are connected by $m = d + 1$.

4. Corollary 3.2 is easiest to apply when the number of edges at each vertex is relatively large compared with the smoothness order r. Unfortunately, it is easy to construct a variety of examples similar to Example 3.3 where the upper and lower bounds do not agree. On the other hand, there are also many examples where they do agree and provide a dimension statement in situations where no other presently available methods apply. Example 4.3 is a case which does not seem to fit any available theory, not even the quasi-cross-cut theory of **[6]**. It is of interest to note that in this example the correct dimension equals the upper bound rather than the lower one.

5. I had hoped that the upper bound presented here would shed some light on why high degree splines with low smoothness do not seem to be subject to the difficulty inherent in Example 3.3. (It is known **[16]** that for all $d \geqq 5$, the C^1 splines on the partition in Figure 4 have dimension given by the lower bound. The cases $d = 3, 4$ remain unclarified).

6. Type-1 and Type-2 partitions have been considered in a variety of papers — see, e.g., **[3–10, 19]**. Dimension statements for the equally spaced case and for several values of d and r can be found in **[19]**. Slightly different looking (but equivalent) formulae were found for general d and r in **[6]**. The results for unequally spaced partitions are new.

7. In comparing the upper and lower bounds given here for a variety of

quasi-cross-cut partitions, I found them to agree with each other and with the dimensionality formulae given in [**6**]. Hence, I conjecture that this holds for general quasi-cross-cut partitions. The problem in constructing a proof is that there does not seem to be a simple relation between vertices, edges, and lines in such a partition.

8. Recently there have appeared a number of results on the dimension of spaces of splines which satisfy boundary conditions — see [**8–10**]. The tools presented here can also be used on these kinds of spline spaces.

9. After identifying the dimension of a space of splines, the next important question is to construct a basis, and if possible a local basis, for the space. Considerable work has been done on this problem for special partitions—see, e.g., [**1–10**] and references therein.

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