ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 1, Winter 1974

AN APPLICATION OF MULTIVARIATE B-SPLINES TO COMPUTER-AIDED GEOMETRIC DESIGN

PETER KOCHEVAR

ABSTRACT. A multidimensional analogue of Schoenberg's Spline Approximation Method is presented within a Computer-Aided Design context. The construction uses multivariate *B*-splines to approximate real valued functions on arbitrary polyhedrons in Euclidean spaces of any dimension.

1. Introduction. Univariate *B*-spline approximation techniques have enjoyed an immense popularity as a tool in the field of Computer-Aided Geometric Design (CAGD). Most applications utilize the Spline Approximation Method of Schoenberg and Greville [15] in which given a sequence of knot points $t_0 < \cdots < t_{N+k+1}$ and a real valued function f defined on $[t_0, t_{N+k+1}]$ the approximation to f is defined by

$$(1.1) \qquad \mathscr{M}_k(f: x) = \sum_{i=0}^{N} f(u_i)\, N(x: t_i, \ldots, t_{i+k+1}).$$

In this expression the $u_i = (t_{i+1} + \cdots + t_{i+k})/k$ are called the nodes and the $N(x: t_i, \ldots, t_{i+k+1})$ are the normalized *B*-spline basis functions defined on the knots $t_i, \ldots, t_{i+k+1}$ of degree k. In CAGD the coefficients $f(u_i)$ are frequently supplied a priori and are the "graphical handles" used to interact with a design. Within this context the function f is chosen to be a piece-wise linear function parametrized so that the coefficients are the breakpoints at the nodes (see Figure 1.1).

This approximation method possesses a number of properties which make it particularly attractive for use in CAGD. First, the method is a local one in that for any evaluation there are at most m nonzero basis functions meaning that small perturbations to the function f result in localized changes to $\mathscr{M}_k(f)$. A second important feature of this scheme is that the basis functions form a partition of unity which has the desirable consequences that the approximation lies within the convex hull of the coefficients of each basis function as well as invariance under rigid motion transformations. The last feature (1.1) possesses is that the approximation oscillates no more than the function f itself. Note, in particular, that this constrains $\mathscr{M}_k(f)$ to reproduce linear functions and in fact the choice of nodes in (1.1) was made so that it would do just that.

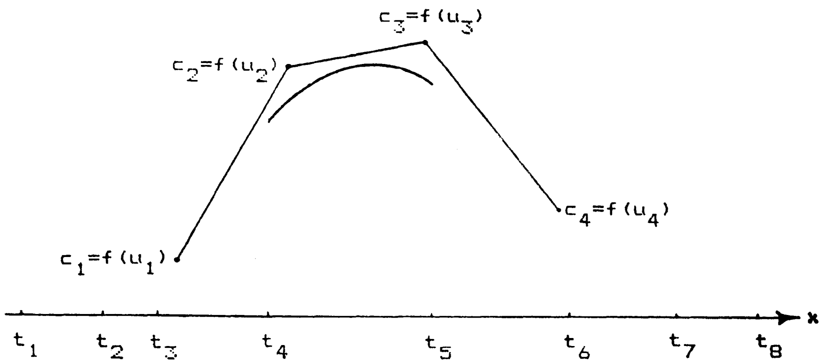

FIGURE 1.1. A cubic B-spline design curve determined by the coefficients c_i.

Extensions to higher dimensions have usually involved tensor products of univariate methods and have generally shown only limited promise. Given knot sequences $s_0 < \cdots < s_{N+p+1}$ and $t_0 < \cdots < t_{M+q+1}$, and a real valued function f defined on $[s_0, s_{N+p+1}] \times [t_0, t_{M+q+1}]$ a bivariate analogue of (1.1) would be

$$\mathcal{M}_{pq}(f\colon x) = \sum_{i=0}^{N} \sum_{j=0}^{M} f(u_i, u_j) \times N(x_1\colon s_i, \ldots, s_{i+p+1})\, N(x_2\colon t_j, \ldots, t_{j+q+1}) \tag{1.2}$$

where u_i and v_j are the nodes and $N(x_1\colon s_i, \ldots, s_{i+p+1})$ and $N(x_2\colon t_j, \ldots, t_{j+q+1})$ are the pth and qth degree univariate B-spline basis functions. As in the one dimensional case the function f is usually chosen to be piece-wise linear with predefined breakpoints occuring at the nodes.

The approximation method given by (1.2) and similar higher dimensional analogues have most of the desirable features of the univariate schemes; specifically local behavior, positive basis functions summing to one, and some shape preserving qualities in that they reproduce linear functions. Unfortunately tensor product schemes also possess some severe limitations with perhaps their most restrictive shortcoming being their dependency on rectilinear knot distributions.

A knot topology of this configuration does not easily lend itself to modelling a wide variety of objects as can be seen, for example, in Figure 1.2 which depicts the wing-body fairing of an airplane. The region where the leading edge of the wing joins the fuselage cannot be satisfactorily modelled unless some special construction is attempted there such as coalescing knot points or substituting some non-B-spline surface. Either

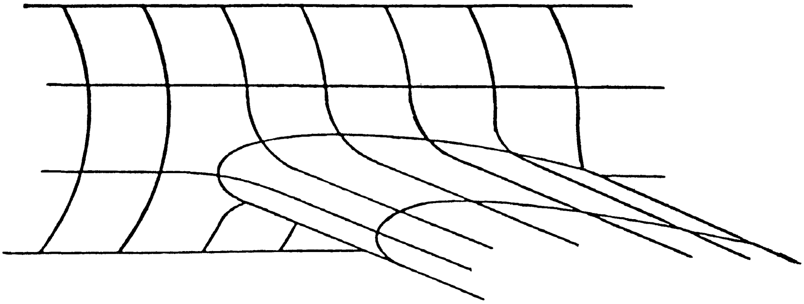

FIGURE 1.2. A surface not easily modelled by tensor product B-splines.

attempt, aside from the added complexity of the construction, has a tendency to circumvent the desirable qualities which prompted the use of B-splines initially. What is truly needed therefore is an alternate generalization of (1.1) that admits arbitrary knot configurations while still preserving its desirable features. The purpose of this paper is to describe such a generalization which uses multivariate B-splines that can be defined for almost any knot configuration giving approximations on any compact polyhedral region in a Euclidean space of any dimension.

The following introduces some notation and terminology that will be used throughout the paper. Elements of $\mathbf{R}^n$ will be of the form x and will always be assumed to be column vectors. Superscripts will always denote indices for elements of $\mathbf{R}^n$ and subscripts will in general give the component of a particular vector so that x_q^p references the qth-component of the pth-element from a collection of vectors. The point e_p^q will equal $(\delta_{1p}, \delta_{2p}, \ldots, \delta_{qp})^t$ for q in Z_+ and S_q will be the set of all such e_p^q with p varying between zero and q.

Notation of the form $z = (x, y)^t$ with x in $\mathbf{R}^n$ and y in $\mathbf{R}^k$ will refer to a column vector in $\mathbf{R}^{n+k}$ in which the first n components of z are x and the last k are y. Similarly

$$(x^1 \cdots x^s) \quad \text{and} \quad \begin{pmatrix} x^1 & \cdots & x^s \\ y^1 & \cdots & y^s \end{pmatrix}$$

would be $n \times s$ and $(n + k) \times s$ matrices respectively. If P is a collection of vectors then $[P]$ will denote the convex hull of P and if $t^0, \ldots, t^n$ is a set of points in $\mathbf{R}^n$ then S equal to $[t^0, \ldots, t^n]$ will be referred to as an

n-simplex which has the points $t^0, \ldots, t^n$ as its vertices. The following notations are equivalent

$$\operatorname{vol}(S) = \operatorname{vol}(t^0, \ldots, t^n) = \begin{vmatrix} 1 & \ldots & 1 \\ t^0 & \ldots & t^n \end{vmatrix}$$

all equaling $n!$ times the signed volume of S.

The integer n will always denote the dimension of the space in which an approximation is to take place, k will denote the degree of the multivariate B-splines used in the approximation, and m will always equal $n + k$. The map $\mathscr{P}$ will be defined to be the orthogonal projection of $\mathbf{R}^m$ onto $\mathbf{R}^n$.

2. Multivariate B-splines. The following geometric definition of multivariate B-splines appeared in DeBoor [7] and will prove invaluable in the approximation constructions that are to follow. Suppose $\mathscr{K}$ is any knotset having $m + 1$ knots in $\mathbf{R}^n$ and S is a proper simplex in $\mathbf{R}^m$ whose vertices project orthogonally onto the knots in $\mathscr{K}$. The unnormalized nth-variate B-spline of degree k is defined by

$$M(x\colon \mathscr{K}) = \begin{cases} X_S(x) \,/\, \operatorname{Volume}(S) & m = n, \\ \operatorname{Volume}(\{y \in S\colon \mathscr{P}(y) = x\}) \,/\, \operatorname{Volume}(S) & m > n \end{cases} \tag{2.1}$$

where X_S is the characteristic function on S. As an example suppose $M(x\colon 0, 1, 2, 3)$ is to be calculated for some x_0 in the interval $[1, 2]$. Lift 0, 1, 2, and 3 to $\mathbf{R}^3$ by choosing $s^0 = (0, 1, 0)^t$, $s^1 = (1, 0, 0)^t$, $s^2 = (2, 0, 1)^t$, and $s^3 = (3, 1, 0)^t$ then $M(x_0\colon 0, 1, 2, 3)$ just equals A/V where A is the area of the intersection of the plane $x = x_0$ with the simplex $S =$

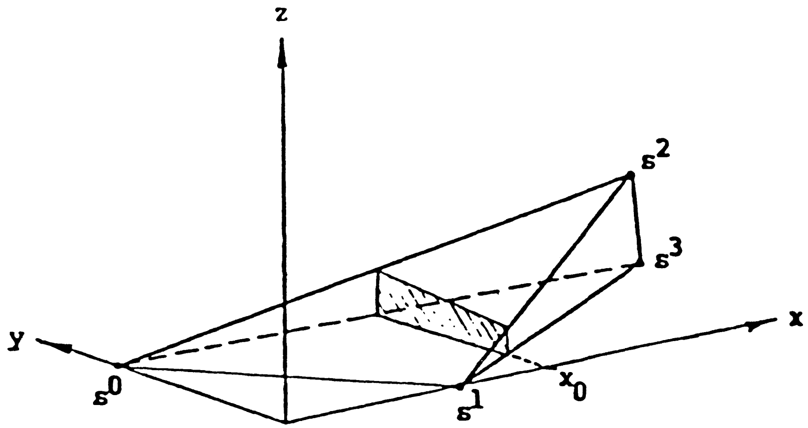

FIGURE 2.1. Geometric construction of $M(x\colon 0, 1, 2, 3)$.

$[s^0, s^1, s^2, s^3]$ and V is the volume of S. Figure 2.1 illustrates the construction.

Although this geometric definition is quite nice conceptually it is horrendous computationally. Fortunately, a recursive definition was discovered independently by both Micchelli **[12]** and Dahmen **[5]**: Given a knotset $\mathscr{K}$ in $\mathbf{R}^n$ of cardinality $m + 1$ and a point x contained in the proper simplex $[t^0, \ldots, t^n]$ with vertices in $\mathscr{K}$ then

$$M(x: \mathscr{K}) = \begin{cases} X_{[\mathscr{K}]}(x)/\mathrm{Volume}([\mathscr{K}]) & m = n \\ (m/k) \sum_{i=0}^{n} b_i(x)\, M(x: \mathscr{K} - \{t^i\}) & m > n \end{cases} \tag{2.2}$$

where $b_i(x)$ is the barycentric coordinate function on $\mathscr{K}$ equal to $\mathrm{vol}(t^0, \ldots, t^{i-1}, x, t^{i+1}, \ldots, t^n)/\mathrm{vol}(t^0, \ldots, t^n)$.

A proper simplex can always be found so that it contains x except in the case where all the knots in $\mathscr{K}$ lie in a hyperplane of dimension less than n. In this degenerate case $M(x: \mathscr{K})$ will be artificially set to zero. The only time a term with a degenerate B-spline could possibly make a contribution in the recurrence relation (2.2) would be when x lies in the knotset over which the B-spline is defined. However, then the barycentric coordinate function associated with the B-spline would be zero annihilating the term. It can be shown in general that the effect a degenerate knotset has on the definition of a B-spline is to reduce its continuity across the degeneracy. Micchelli **[13]** provides a more detailed discussion on using the recurrence relation (2.2) for calculating B-splines on arbitrary knotsets.

As an example, suppose $t^0 = (0, 0)^t$, $t^1 = (2, 0)^t$, $t^2 = (1, 1)^t$, and $t^3 = (0, 2)^t$ then for x in $[t^0, t^1, t^2, t^3]$

$$M(x: t^0, t^1, t^2, t^3)$$
$$= 3(b_0(x)\, M(x: t^1, t^2, t^3) + b_1(x)\, M(x: t^0, t^2, t^3) + b_2(x)\, M(x: t^0, t^1, t^3)).$$

In the above, $M(x: t^1, t^2, t^3)$ is a degenerate B-spline so that its value strictly by expression (2.2) is infinite for any x lying on the line segment from t^1 to t^3 but at the same time its coefficient there is zero, signifying that the first term makes no contribution to the sum. Visually the effect of the degeneracy on $M(x: t^0, t^1, t^2, t^3)$ is to introduce a discontinuity across the line segment from t^1 and t^3 as depicted in Figure 2.2. Figures 2.3 and 2.4 show the graphs of two additional bivariate B-spline basis functions along with their associated knotsets.

3. Multivariate B-spline approximation on $[S_n]$. For any degree k let E be the set of knots in $\mathbf{R}^n$ which consists of the elements of S_n each replicated $k + 1$ times, that is,

$$E = \{\underbrace{e_n^0, \ldots, e_n^0}_{k+1}, \ldots, \underbrace{e_n^n, \ldots, e_n^n}_{k+1}\} \tag{3.1}$$

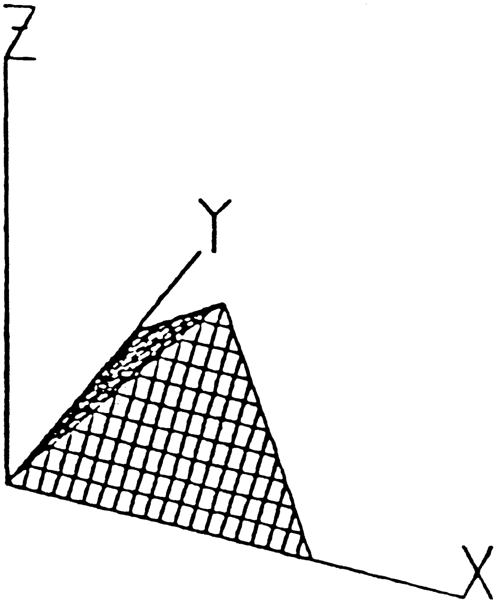

FIGURE 2.2. The effect of degenerate knotsets.

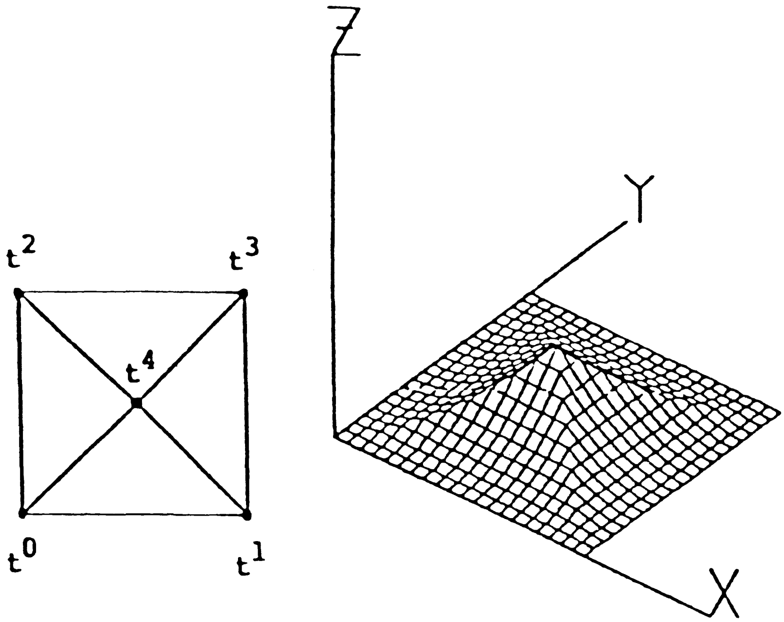

FIGURE 2.3. The quadratic B-spline $M(x\colon t^0, \ldots, t^4)$.

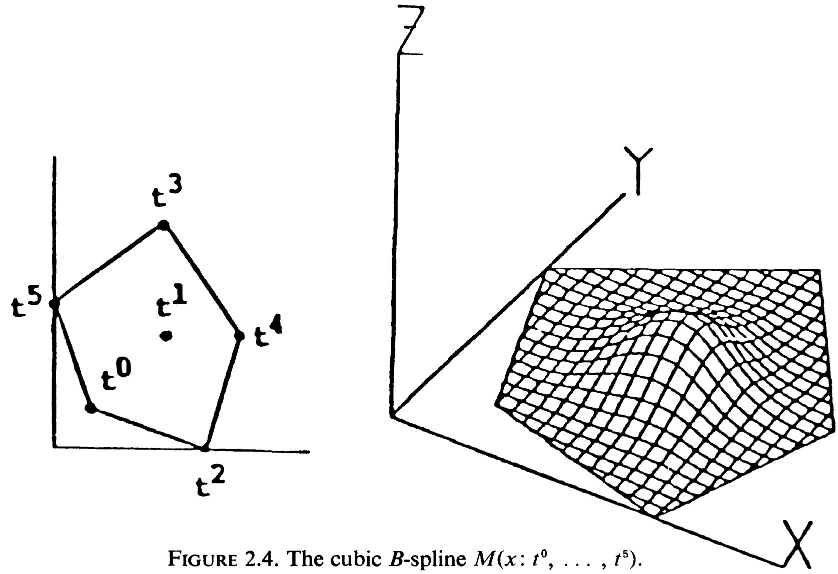

FIGURE 2.4. The cubic B-spline $M(x: t^0, \ldots, t^5)$.

and let S be a collection of knotesets contained in E each having cardinality $m + 1$ where $m = n + k$. Given a real valued function f defined on $[S_n]$ a kth degree approximation will be constructed having the form

$$\mathcal{M}_{n,k}(E, f: x) = \sum_{\mathcal{K} \in S} f(u^{\mathcal{K}})\, N(x: \mathcal{K}) \tag{3.2}$$

where $N(x: \mathcal{K})$ is a normalized multivariate B-spline basis function defined on the knotset $\mathcal{K}$ (which differs from (2.2) by a constant to be defined subsequently) and where $u^{\mathcal{K}}$ in $\mathbf{R}^n$ is a node which depends on the knots in $\mathcal{K}$.

To define (3.2) fully a procedure to construct the collection of knotsets, S and a means of determining the nodes must both be established. These constructions will be motivated by the features of the univariate approximation scheme (1.1) which are responsible for most of its desirable characteristics. Specifically, for x contained in $[S_n]$, S and the $u^{\mathcal{K}}$ will be defined so that

$$\sum_{\mathcal{K} \in S} N(x: \mathcal{K}) = 1 \quad \text{and} \quad \sum_{\mathcal{K} \in S} u^{\mathcal{K}} N(x: \mathcal{K}) = x \tag{3.3}$$

or in other words chosen so that $\mathcal{M}_{n,k}(E, f)$ reproduces linear functions.

Establishing the knotsets. In [7] deBoor suggests a way to establish such a partition of unity on $[U]$ for some set of knots U in $\mathbf{R}^n$. His idea was to select some suitable convex set C in $\mathbf{R}^k$ having unit volume to form the set $[U] \times C$, then lift each knot in U to $[U] \times C$ by appending to it a vector in $\mathbf{R}^k$, and finally triangulating the lifted knots to form a collection of m-simplices, $\mathscr{T}$, such that the union of all simplices in $\mathscr{T}$ is precisely $[U] \times C$ and the intersection of any two is either empty or consists of a common face having dimension less than m. In light of the geometric formulation of multivariate B-splines (2.1), this process then gives for any x in $[U]$

$$\begin{aligned}\sum_{S \in \mathscr{T}} \text{Volume}(\{y \in S: \mathscr{P}(y) = x\}) &= \\ \sum_{S \in \mathscr{T}} \text{Volume}(S)\, M(x: \mathscr{P}(S)) &= \text{Volume}(C) = 1.\end{aligned} \tag{3.4}$$

Using this idea for the problem at hand, take $C = [S_k]$ with the set of knots $U = E$ then $[U] \times C = [S_n] \times [S_k]$. If $\mathscr{T}$ is a proper triangulation of $[S_n] \times [S_k]$ then for any x in $[S_n]$

$$\begin{aligned}k! \sum_{S \in \mathscr{T}} \text{Volume}(\{y \in S: \mathscr{P}(y) = x\}) &= \\ (k!/m!) \sum_{S \in \mathscr{T}} \text{vol}(S)\, M(x: \mathscr{P}(S)) &= \text{vol}(S_k) = 1\end{aligned}$$

as depicted, for example, in Figure 3.1 which shows the construction for n and k both equal to one. On setting

$$N(x: \mathscr{P}(S)) = (k!/m!)\, \text{vol}(S)\, M(x: \mathscr{P}(S)) \tag{3.5}$$

for any S in $\mathscr{T}$ then $\{\mathscr{P}(v): v$ a set of vectors such that $[v] \in \mathscr{T}\}$ is a collection of knotsets which determine a partition of unity for the collection of B-spline basis functions defined on them. The problem of determining the knotsets now reduces to finding an acceptable triangulation of $[S_n] \times [S_k]$.

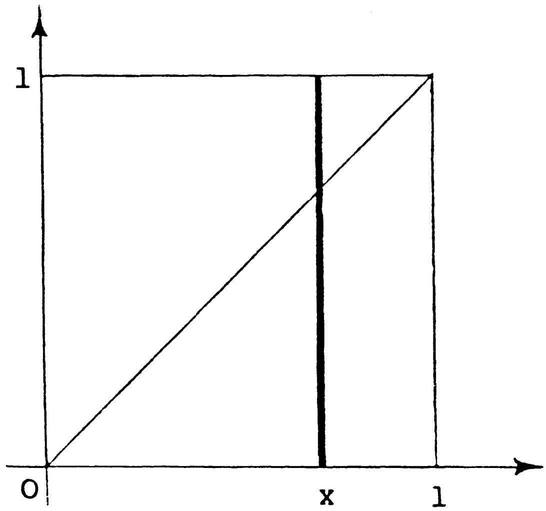

FIGURE 3.1. A partition of unity on $[S_1] \times [S_1]$.

If Per_m is the group of permutations on the integers one through m then for any p in Per_m as m-simplex, S_p, equal to $[v_p^0, \ldots, v_p^m]$ can be defined by setting $v_p^0 = 0$ and $v_p^i = v_p^{i-1} + e_m^{p(i)}$. The collection $\{S_p: p \in \mathrm{Per}_m\}$ is defined as Kuhn's triangulation of the hypercube $[0, 1]^m$. By appropriately mapping $[S_n] \times [S_k]$ to $[0, 1]^m$ with some function F, one can take as the triangulation of $[S_n] \times [S_k]$ the collection of simplices $\{F^{-1}(S): S \in \mathscr{U}\}$ where $\mathscr{U}$ is the set of simplices in the triangulation of $F([S_n] \times [S_k])$ induced by Kuhn's triangulation.

Define $\varepsilon_m^i = (1, 1, \ldots, 1, 0, 0, \ldots 0)^t$ to be a vector in $\mathbf{R}^m$ in which the first i components are one and the remaining $m - i$ components are zero and let $U_m = \{\varepsilon_m^i: 0 \leqq i \leqq m\}$. Let J be a set of multi-indices in $\mathbf{Z}_+^2$ of the form (j_1, j_2) with $0 \leqq j_1 \leqq n$ and $0 \leqq j_2 \leqq k$. For any j in J define

$$a^j = (e_n^{j_1}\ e_k^{j_2})^t \quad \text{and} \quad b^j = (\varepsilon_n^{j_1}\ \varepsilon_k^{j_2})^t \tag{3.6}$$

and let F be the function from $S_n \times S_k$ onto $U_n \times U_k$ which maps a^j to b^j.

The set $U_n \times U_k$ is a subset of the set of vertices of the hypercube $[0, 1]^m$ and therefore Kuhn's triangulation induces a triangulation on $[U_n] \times [U_k]$. Since ε_r^i can never have a component of zero preceding a component of one for any nonnegative i less than r, the subset of Per_m which leads to a triangulation of $[U_n] \times [U_k]$ is just the set P equal to

$$\{p \in \mathrm{Per}_m: p^{-1}(i) < p^{-1}(r) \text{ when } 1 \leqq i < r \leqq n \text{ or } n + 1 \leqq i < r \leqq m\}. \tag{3.7}$$

Define a function G_p from $S_p = [v_p^0, \ldots, v_p^m]$ to $[F^{-1}(v_p^0), \ldots, F^{-1}(v_p^m)]$ for each p in P such that

$$G_p(x) = \sum_{i=0}^{m} b_i(x)\, F^{-1}(v_p^i),$$

the $b_i(x)$ being the barycentric coordinate functions on S_p, and define G so that $G(x) = G_p(x)$ when x is in S_p. The collection of simplices $\mathscr{T} = \{G(S_p): p \in P\}$ then triangulates $[S_n] \times [S_k]$ and a collection of knotsets which results in a partition of unity on $[S_n]$ is just the collection of all orthogonal projections of the vertices of each simplex in $\mathscr{T}$.

Calculation of the nodes. In [11] Marsden and Schoenberg state a univariate B-spline identity which they later use to find the nodes for the approximation scheme (1.1). This identity states that if z is any real number and $t^0 < \cdots < t^N$ are any knots in $\mathbf{R}$ then

$$(z - x)^{m-1} = \sum_{i=0}^{N} (z - t^{i+1}) \cdots (z - t^{i+m-1}) N(x: t^i, \ldots, t^{i+m}) \tag{3.8}$$

for any x in $[t^{m-1}, t^{N-m+1}]$. By expanding both sides into powers of z and setting like terms equal then $\sum_{i=0}^{N} u^i N(x: t^i, \ldots, t^{i+m}) = x$ where $u^i = (t^{i+1} + \cdots + t^{i+m-1})/(m - 1)$. It will be shown that by exploiting this strategy in a multidimensional setting the nodes $u^{\mathscr{K}}$ of the approximation (3.3) can be calculated so that they reproduce linear functions on $[S_n]$.

Let T be any set of knots in $\mathbf{R}^n$ (for the construction to follow T need not be restricted to elements of S_n) and suppose each knot t in T has been lifted to $\mathbf{R}^m$ by appending an element of S_k. If U is the set of all such lifted knots with the volume of $[U]$ greater than zero, then for i from one to n define

$$U_i = \left\{\begin{pmatrix} t \\ (z - t_i)\, a \end{pmatrix} : \begin{pmatrix} t \\ a \end{pmatrix} \in U\right\} \quad \text{for any } z > \max_{t \in T}(t_i) \text{ in } \mathbf{R}. \tag{3.9}$$

Suppose further that $[U]$ has been triangulated into a collection of m-simplices $\mathscr{T}$. For each S in $\mathscr{T}$ define

$$S_i = \left[\left\{\begin{pmatrix} t \\ (z - t_i)\, a \end{pmatrix} : \begin{pmatrix} t \\ a \end{pmatrix} \in S\right\}\right]$$

and let $\mathscr{T}_i = \{S_i : S \in \mathscr{T}\}$. Again, recalling the geometric expression (2.1),

$$\begin{aligned} &k! \sum \text{Volume}(\{y \in S_i : \mathscr{P}(y) = x\}) \\ &\quad = (k!/m!) \sum \text{vol}(S_i)\, M(x : \mathscr{P}(S_i)) = (z - x_i)^k. \end{aligned} \tag{3.10}$$

Figure 3.2 ilustrates this construction in the linear univeriate case.

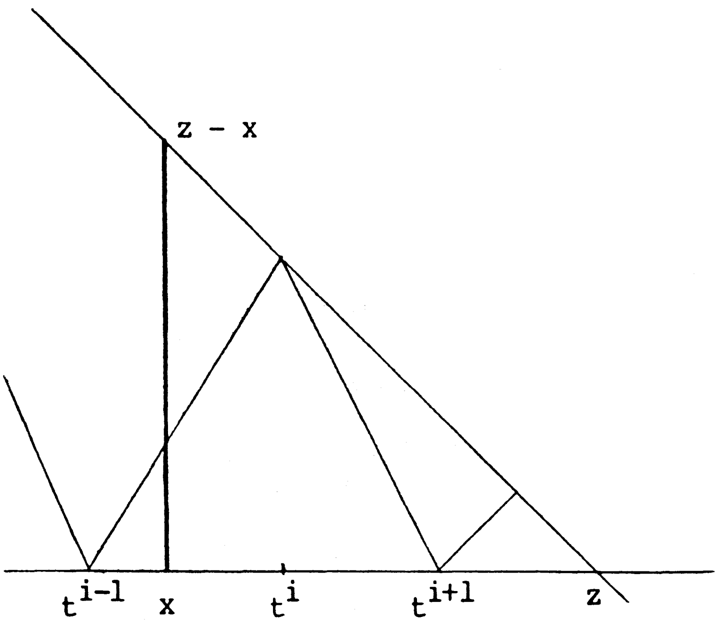

FIGURE 3.2. A geometric interpretation of the multivariate analogue of Marsden's identity.

Note that $\mathscr{P}(S_i) = \mathscr{P}(S)$ for all S_i so that $M(x\colon \mathscr{P}(S_i)) = M(x\colon \mathscr{P}(S))$ and since $M(x\colon \mathscr{P}(S)) = m!\, N(x\colon \mathscr{P}(S)) / (k!\ \mathrm{vol}(S))$ the expression finally gives rise to the identity

$$(3.11) \qquad \sum \mathrm{vol}(S_i)/\mathrm{vol}(S)\ N(x\colon \mathscr{P}(S)) = (z - x_i)^k.$$

Let S be the collection of knotsets determined by the projections of the vertices of each simplex S in $\mathscr{T}$. Suppose $\mathscr{K}$ is the knotset in S consisting of the knots $t^0, \ldots, t^m$ which are the orthogonal projections of the vertices of the simplex S so that

$$\mathrm{vol}(S_i) = \mathrm{vol}\begin{pmatrix} t^0 & \cdots & t^m \\ (z - t_i^0)a^0 & & (z - t_i^m)\, a^m \end{pmatrix}$$

each a^j being an element of S_k. After repeated applications of the Additive Law of Determinants one then finds that $\mathrm{vol}(S_i)$ is equal to

$$(3.12) \qquad \mathrm{vol}(S)\ z^k - \sum_{j=1}^{k} \mathrm{vol}(B_i^{0j}, \ldots, B_i^{mj})\ z^{k-1} + \mathrm{O}(z^{k-2})$$

where B_i^{pq} equals $(t^p a^p)^t$ but with the qth component of a^p multiplied by t_i^p. The right side of (3.11) expands to the form $(z - x_i)^k = z^k - k\ x_i\ z^{k-1} + \mathrm{O}(z^{k-2})$ and on equating like powers of z in the expression (3.12), $u_i^{\mathscr{K}}$, the ith node component, equals

$$1 \,/\, (k\ \mathrm{vol}(S)) \sum_{j=1}^{k} \mathrm{vol}(B_i^{0j}, \ldots, B_i^{mj})$$

where i varies between one and n.

4. Approximation on arbitrary polyhedrons. The special approximation $\mathscr{M}_{n,k}(E, f)$ given by (3.2) was shown in [9] to be nothing more than a reformulation of the Generalized Bezier Approximation to real valued functions defined on $[S_n]$. In this section $\mathscr{M}_{n,k}(E, f)$ will be extended to a general scheme defined on arbitrary polyhedrons by relying on two interconnected constructions. One construction is that of knot refinement where new knots are added to some initial set in such a way as to extend the original approximation to a new one defined on the larger set. Unfortunately when the initial set of knots is just the vertices of $[S_n]$ any refinement will extend to an approximation which may not have the maximum attainable continuity. In this case not every set of $n + 1$ knots in some knotset will be of dimension n which by a result of Micchelli [13, p. 14] means that the degree of continuity of the basis function defined on that knotset may be less than $k - 1$. Although some applications in CAGD could capitalize on these discontinuities to form creases in a design surface (e.g., the trailing edge of an airplane wing) a means will be developed which can guarantee the maximum degree of continuity.

This process will turn out to be nothing more than a multidimensional analogue of the end conditions that must be specified in ordinary univariate B-spline approximation.

Boundary conditions. Each knot in the initial set E defined by (3.1) will be termed an "exterior" knot while any appended to E through knot refinement will be termed an "interior" knot. If the exterior knots can be displaced appropriately before knot insertion occurs then the continuity class of the resulting approximation can be improved. In [5] Dahmen performs such an alteration by lifting each knot to $\mathbf{R}^m$, applying a certain affine transformation to them, and finally projecting them back to $\mathbf{R}^n$. The goal in what follows is to also construct an affine transformation which will move the knots in E into general positions while at the same time preserving the partition of unity on $[S_n]$.

Following Dahmen, define a transformation, $\mathscr{A}$, acting on any u in $\mathbf{R}^m$ to equal

$$\begin{pmatrix} 1 & & A \\ & \ddots & \\ & & 1 \end{pmatrix} \operatorname{diag}_m((d_1, \ldots, d_n, 1, \ldots, 1)^t)u + (v, 0)^t \tag{4.1}$$

where $A = (a_{i,j})$ is some arbitrary $n \times k$ "boundary condition" matrix, $\operatorname{diag}_m(x)$ is an $m \times m$ diagonal matrix having x as the diagonal, and v is an element of $\mathbf{R}^n$. Note that $\mathscr{A}$ is the product of skew and a scale transformation followed by a translation. Define U_i to be the set $S_n \times \{e_k^i\}$ and let V_i be the collection $\{\mathscr{P}(\mathscr{A}(u)): u \in U_i\}$ with i varying between zero and k. Each $[V_i]$ is a simplex in $\mathbf{R}^n$ which is just a scaled translate of $[S_n]$ and therefore there exists a natural correspondence between the vertices in S_n and those in each V_i. Let U be the intersection of all $[V_i]$ and note that U can either be empty, a j-simplex where $j < n$, or a scaled translate of $[S_n]$ which equals the set

$$W = \{x \in \mathbf{R}^n : k! \operatorname{Volume}(\{y \in \mathscr{A}([S_n] \times [S_k]) : \mathscr{P}(y) = x\}) = 1\}.$$

Assuming for simplicity that $d_1 = \cdots = d_n = d$, the scale factor d and a translate v will need to be determined for the transformation $\mathscr{A}$ so that U equals $[S_n]$.

Let

$$c^i = \mathscr{P}(\mathscr{A}(0\ e_k^i)^t) = \begin{cases} 0 & \text{if } i = 0 \\ (a_{1i} \cdots a_{ni})^t & \text{if } 1 \leqq i \leqq n \end{cases}$$

be the point in each set V_i which corresponds to the origin in S_n. Suppose U is a scaled translate of $[S_n]$ and V is the set of vertices of the simplex U, then the point c, where

$$c_j = \max_{0 \leq i \leq n}(c_j^i) = \max\{0, a_{j1}, \ldots, a_{jk}\},\ 1 \leq j \leq n,$$

is the point in V which corresponds to the origin in S_n (see Figure 4.1). In [9] it was demonstrated that the translate v in (4.1) must equal $-c$ and the scale factor d must be

$$1 + v_1 + \cdots v_n - \min_{0 \leq i \leq n}\{c_1^i + \cdots + c_n^i\}.$$

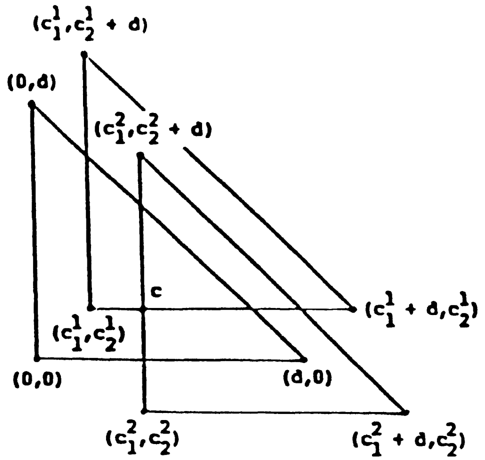

FIGURE 4.1. Displacement of the exterior knots.

If $\mathscr{T}$ is the collection of simplices which triangulate $[S_n] \times [S_k]$ then $\mathscr{T}^* = \{\mathscr{A}(S)\colon S \in \mathscr{T}\}$ triangulates $\mathscr{A}([S_n] \times [S_k])$ and thus determines a collection of knotsets which are the projections of the vertices of each simplex in $\mathscr{T}^*$. For any x in the simplex $[S_n]$

$$k! \sum_{S^* \in \mathscr{T}^*} \mathrm{Volume}(\{y \in S^*\colon \mathscr{P}(y) = x\})$$
$$= (k!/m!) \sum_{S^* \in T^*} \mathrm{vol}(S^*) M(x\colon \mathscr{P}(S^*)) = \mathrm{vol}(S_k) = 1$$

which suggests that each basis function be normalized by setting $N(x\colon \mathscr{P}(S^*)) = (k!/m!)\,\mathrm{vol}(S^*) M(x\colon \mathscr{P}(S^*))$. In addition, for any simplex S^* a node u^{S^*} can be determined by the node calculation procedure outlined earlier. Thus a new approximation to real valued functions f defined on the convex hull of T^* equal to $[\mathscr{P}(\mathscr{A}(S_n \times S_k))]$ is

$$\mathscr{M}_{n,k}(T^*, f\colon x) = \sum_{S^* \in \mathscr{T}^*} f(u^{S^*}) N(x\colon \mathscr{P}(S^*)) \tag{4.2}$$

having the property that for any x in $[S_n]$ linear functions are reproduced.

Knot refinement. The final construction required before fully defining the general approximation on arbitrary polyhedrons is an iterative procedure for refining the approximation $\mathcal{M}_{n,k}(T^*, f)$. Suppose T is a set of knots either equalling T^* or containing T^* as well as a finite number of additional knots lying in $[S_n]$. Suppose further that K is a collection of knotsets in T so that $\mathcal{M}_{n,k}(T, f)$ is a refined approximation derived from $\mathcal{M}_{n,k}(T^*, f)$ having the property that it reproduces all linear functions in $[S_n]$. Given any y in $[S_n]$ the iterative step which extends $\mathcal{M}_{n,k}(T, f)$ to an approximation on the set of knots $T \cup \{y\}$ will now be outlined.

A theorem due to Micchelli [13] states that if $y = \sum_{t \in \mathcal{K}} b_t t$, b_t in $\mathbf{R}$, for some knotset $\mathcal{K}$ in K then

$$M(x: \mathcal{K}) = \sum_{t \in \mathcal{K}} b_t M(x: (\mathcal{K} - \{t\}) \cup \{y\}). \tag{4.3}$$

For each $\mathcal{K}$ in the collection $U = \{\mathcal{K} \in K: y \in [\mathcal{K}]\}$, a set of knots, $t^0, \ldots, t^n$ can be chosen which lie on the boundary of $[\mathcal{K}]$ such that y is in the n-simplex, S, composed of those knots. If the volume of $[S]$ is greater than zero then $y = \sum_{i=0}^{n} b_i(y) t^i$ and relation (4.3) becomes

$$N(x: \mathcal{K}) = c_{\mathcal{K}} \sum_{i=0}^{n} b_i(y) M(x: \mathcal{K}^i) \tag{4.4}$$

where $\mathcal{K}^i = (\mathcal{K} \cup \{y\}) - \{t^i\}$, $c_{\mathcal{K}}$ is the normalization constant such that $N(x: \mathcal{K})$ equals $c_{\mathcal{K}} M(x: \mathcal{K})$, and the $b_i(x)$ are the barycentric coordinate functions on S. Note that by choosing only simplices whose vertices lie on the boundary of $[\mathcal{K}]$ it is guaranteed no other knots in the set $T - \mathcal{K}$ lie in the interior of $[\mathcal{K}^i]$ provided this was the case for all $\mathcal{K}$ in K initially.

It will turn out in general that not all knotsets ν in the collection $V = \{\mathcal{K}^i: 0 \leq i \leq n, \mathcal{K} \in U\}$ are distinct. To rectify this situation let W be the collection of all distinct knotsets in V and for all μ in W define

$$N(x: u) = \sum_{\substack{\nu \in V \\ \nu = \mu}} c_\nu b_\nu M(x: \mu) \tag{4.5}$$

where c_ν are normalization constants and b_ν are barycentric coordinate functions evaluated at y corresponding to a knotset $\mathcal{K}^i$ in the expression (4.4). For all x in $[V]$ the refined approximation defined on $T \cup \{y\}$ is then given by

$$\mathcal{M}_{n,k}(T \cup \{y\}, f: x) = \sum_{\mathcal{K} \in K - U} f(u^{\mathcal{K}}) N(x: \mathcal{K}) + \sum_{\mu \in W} f(u^\mu) N(x: \mu) \tag{4.6}$$

with the property that the basis functions form a partition of unity.

All that remains to fully specify the scheme (4.6) is to calculate the new nodes u^μ so that

$$\sum_{\mathscr{K}\in K-U} u^{\mathscr{K}} N(x:\mathscr{K}) + \sum_{\mu\in W} u^{\mu} N(x:\mu) = x.$$

For any μ in W, relations (4.4) and (4.5) signify that the node u^μ must satisfy the expression

$$u^{\mu} N(x:\mu) = u^{\mu}\Bigl(\sum_{\substack{\nu\in V\\ \nu=\mu}} c_\nu b_\nu\Bigr) M(x:\mu) = \Bigl(\sum_{\substack{\nu\in V\\ \nu=\mu}} u^{\nu} c_\nu b_\nu\Bigr) M(x:\mu) \tag{4.7}$$

where u^ν is the old node associated with the basis function defined on the knotset ν in (4.4). Solving for u^μ in (4.7)

$$u^{\mu} = \sum_{\substack{\nu\in V\\ \nu=\mu}} u^{\nu} c_\nu b_\nu \Big/ \sum_{\substack{\nu\in V\\ \nu=\mu}} c_\nu b_\nu$$

and since $c_\nu b_\nu > 0$ for all ν, u^μ is a convex combination of old nodes which insures that the new nodes lie in $[T]$.

Approximation on polyhedrons. A polyhedron in $\mathbf{R}^n$ is a region which can be partitioned into a finite collection of proper n-simplices in which the intersection of any two is either empty or is a common face having dimension less than n. It is easy to see how the approximation (4.2) on $[S_n]$ can be used as a basis for an approximation on any such polyhedron.

Without loss of generality, assume P to be an arbritrary polyhedron lying in $[S_n]$ and choose a boundary condition matrix A so that $\mathscr{M}_{n,k}(T^*, f)$ is the approximation given by (4.2) with the set of knots T^* equalling $\mathscr{P}(\mathscr{A}(S_n \times S_k))$, $\mathscr{A}$ being defined by the transformation (4.1). If T is the set of knots comprised of those in T^* and any other finite set

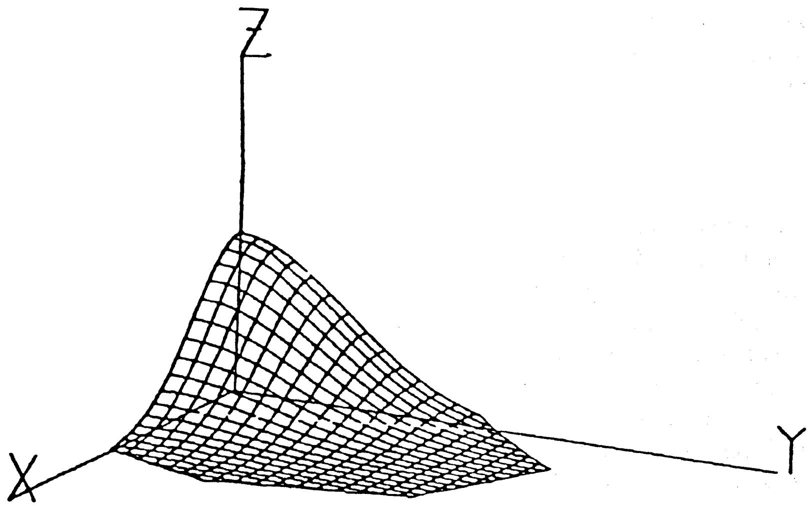

FIGURE 4.2. The function $f(x, y) = \exp(-16(x^2 + y^2))/4$.

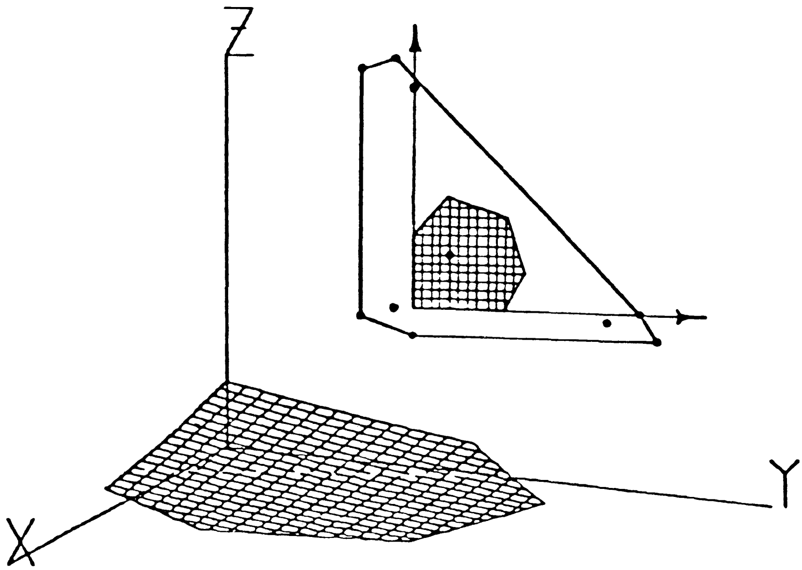

FIGURE 4.3. A quadratic approximation to f.

of points in P then the approximation $\mathcal{M}_{n,k}(T^*, f)$ can be extended to a method $\mathcal{M}_{n,k}(T, f)$ by iterative refinement. The restriction of $\mathcal{M}_{n,k}(T, f)$ to P is then a method which approximates any real valued function f defined on $[T]$ in the polyhedron P having the property that it reproduces linear functions there.

Figure 4.2 shows the graph of the function $f(x, y) = \exp(-16(x^2 + y^2))/4$ which has been approximated in Figure 4.3 by a quadratic method on the depicted polyhedron. The boundary condition matrix used is given by

$$A = \begin{pmatrix} -0.20 & 0.10 \\ -0.05 & -0.15 \end{pmatrix}.$$

5. Conclusion. It has been demonstrated that the mathematical basis used for most CAGD applications is readily extendable to higher dimensions in principle. But, whether or not this extension will turn out to be a viable alternative in practice is still an open question. The gap in complexity between one and higher dimensions is a large one and hopefully the apparent advantages of multivariate B-spline approximation over current multidimensional methods will more than make up for it. Nonetheless, if CAGD is not to benefit directly from multivariate B-spline techniques it will undoubtedly benefit indirectly as the underlying mathematics becomes enriched by further research.

REFERENCES

1. R. E. Barnhill and R. F. Riesenfeld, *Computer Aided Geometric Design*, Academic Press, New York, 1974.

2. A. Bowyer, *Computing Dirichlet tessellations*, The Computer Journal (2) **24** (1981), 162–166.

3. I. Brueckner, *Construction of Bezier points of quadrilaterals from those of triangles*, CAD (1) **12** (1980), 21–24.

4. H. B. Curry and I. J. Schoenberg, *On Polya frequency functions IV, the fundamental spline functions and their limits*, J. d'Analyse Math. **17** (1966), 71–107.

5. W. Dahmen, *On multivariate B-splines*, SIAM J. Numer. Anal. **17** (1980), 179–191.

6. ———, *Approximation by linear combinations of multivariate B-splines*, J. of Approx. Theory **31** (1981), 299–324.

7. C. de Boor, *Splines as linear combinations of B-splines*, in Approx. Theory II, G. G. Lorentz, C. K. Chui, & L. L. Schumaker, eds., Academic Press, New York, 1976.

8. S. Karlin, *Total Positivity*, Stanford Univ. Press, Stanford, Calif., 1968.

9. P. Kochevar, *A multivariate analogue of Schoenberg's spline approximation method*, Master's Thesis, Dept. of Math., Univ. of Utah, 1982.

10. J. Lane, *Shape operators for computer aided geometric design*, Doctoral Dissertation, Dept. of Comp. Science, Univ. of Utah, 1977.

11. M. Marsden and I. J. Schoenberg, *On variation diminishing spline approximation methods*, Mathematica (31) **8** (1966), 61–82.

12. C. A. Micchelli, *A constructive approach to Kergin interpolation in* $\mathbf{R}^k$: *multivariate B-splines and Lagrange interpolation*, Univ. of Wisc., Math. Research Center Report No. 1895, 1978.

13. ———, *On a numerically efficient method for computing multivariate B-splines*, IBM Research Report RC7716 (#33432), June 1979.

14. R. Riesenfeld, *Applications of B-spline approximation to geometric problems of computer-aided design*. Univ. of Utah Technical Report UTEC-CSc-73-126, Univ. of Utah, Salt Lake City, March 1973.

15. I. J. Schoenberg, *On spline functions* (with, supplement by T. N. E. Greville), in Inequalities, O. Shisha, ed., Academic Press, New York, 1967.

16. D. F. Watson, *Computing the n-dimensional Delaunay tessellation with application to Voronoi polytopes*. The Computer Journal (2) **24** (1981), 167–172.

PROGRAM OF COMPUTER GRAPHICS, 120 RAND HALL, CORNELL UNIVERSITY, ITHACA, NY 14853