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ARENS REGULARITY OF CONJUGATE BANACH ALGEBRAS WITH DENSE SOCLE

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ABSTRACT. Let A be a semi-simple Banach algebra which is isometrically isomorphic to the conjugate space of a Banach space V. Suppose that A is weakly completely continuous (w.c.c.). We first show that Arens regularity of A can be obtained by imposing certain conditions on V. If, moreover, A has dense socle, we show that these conditions on V can be obtained in turn by demanding that the maximal modular left (right) ideals and minimal idempotents of A have certain properties.

Introduction. Let A be a Banach algebra which is isometrically isomorphic to the conjugate space of a Banach space V. Identify V as a subspace of A^*. By a theorem of Dixmier [4], $A^{**} = \pi(A) \oplus V^\perp$. (See notation in §2.) Hence for A to be Arens regular the two Arens products must agree on $V^\perp$. For w.c.c. Banach algebras this is connected with the A-invariance of V. We obtain conditions for V to be A-invariant. Our conditions involve the concept of an HB-subspace used by Hennefeld [8]. We apply this concept also to the space V and come up with the notion of a VHB-subspace. The subspaces of A which we want to be VHB-subspaces are the maximal modular left (right) ideals. Examples of Banach algebras which have maximal modular left (right) ideals that are HB-subspaces or VHB-subspaces are discussed in §3. In §4 we show a connection between A-invariance of V and VHB-subspaces and present several results on Arens regularity of conjugate Banach algebras.

2. Preliminaries. Let A be a Banach algebra and let A^* and A^{**} be its first and second conjugate spaces. The two Arens products on A are defined in stages as follows [1]. Let $x, y \in A$, $f \in A^*$ and $F, G \in A^{**}$. Define $f \circ x \in A^*$ by $(f \circ x)(y) = f(xy)$, $F \circ f \in A^*$ by $(F \circ f)(x) = F(f \circ x)$, $F \circ G \in A^{**}$ by $(F \circ G)(f) = F(G \circ f)$, $x \circ' f \in A^*$ by $(x \circ' f)(y) = f(yx)$, $f \circ' F \in A^*$ by $(f \circ' F)(x) = F(x \circ' f)$, and $F \circ' G \in A^{**}$ by $(F \circ' G)(f) = G(f \circ' F)$. Then A^{**} is a Banach algebra under the Arens products $F \circ G$ and $F \circ' G$. Both of these products extend the original multiplication on

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A when A is canonically embedded in A^{**}. A Banach algebra A is called *Arens regular* if the two Arens products agree on A^{**}.

Let π denote the canonical embedding of A into A^{**}. We observe that for all $x \in A$, $F \in A^{**}$, $\pi(x) \circ F = \pi(x) \circ' F$ and $F \circ \pi(x) = F \circ' \pi(x)$ [7, Lemma 1.5, p. 116]. A subset V of A^* is called *A-invariant* if $f \circ a$ and $a \circ' f$ are in V for all $a \in A$ and $f \in V$. Similarly V is *A^{**}-invariant* if $F \circ f$ and $f \circ' F$ are in V for all $f \in V$ and $F \in A^{**}$. If A is isometrically isomorphic to the conjugate space of a Banach space V, we will identify A with V^* and say that A is a conjugate Banach algebra. Clearly we may then also identify V as a subspace of A^*, and we will do so in what follows.

Let A be a Banach algebra. Then S_A will denote the socle of A and E_A the set of all minimal idempotents in A. By an ideal in A we will always mean a two-sided ideal unless stated otherwise. For $a \in A$, let L_a and R_a be, respectively, the left and right multiplication operators determined by a. A is called *weakly completely continuous* (w.c.c.) if for every $a \in A$, L_a and R_a are weakly completely continuous operators on A. It follows from the definitions of Arens products and [5, VI. 4.2, p. 482] that A is w.c.c. if and only if $\pi(A)$ is an ideal of A^{**} for either Arens product. Let X be a Banach space and X^* its conjugate space. If S is a subspace of X, then $S^{\perp} = \{f \in X^* : f(s) = 0 \text{ for all } s \in S\}$. If $f \in X^*$, then $f|S$ will denote the restriction of f to S.

All algebras and vector spaces considered here are over the complex field. We follow the terminology of [10]. We gather together some useful results in the following lemma.

LEMMA 2.1. (1) *Let A be a semi-simple Banach algebra with dense socle. Then A is* w.c.c. *if and only if every minimal left* (*right*) *ideal of A is a reflexive Banach space. In particular, a semisimple annihilator Banach algebra is* w.c.c. *with dense socle.*

(2) *Let A be a semi-simple Banach algebra with dense socle. Then every maximal modular left* (*right*) *ideal M is of the form*

$$M = A(1-e) = \{x - xe : x \in A\} \qquad (M = (1-e)A = \{x - ex : x \in A\})$$

for some $e \in E_A$.

PROOF. (1) This follows easily from [15, Theorem 6.2, p. 269]. If A is a semi-simple annihilator algebra, then A has dense socle [2, Theorem 4, p. 157] and each minimal left (right) ideal of A is reflexive [2, Theorem 13, p. 161]. Hence A is w.c.c.

(2) This follows from [17, Lemma 3.3, p. 38] and the last paragraph of [17, p. 41].

We observe that if e is a minimal idempotent in a Banach algebra A, the $A(1-e)$ $((1-e)A)$ is a maximal modular left (right) ideal of A [10].

3. HB-subspaces and VHB-subspaces.

DEFINITION. Let X be a Banach space. A closed subspace M of X is called an HB-subspace if $M^{\perp}$ has a complement M_* in X^* such that for each $f \in X^*$, $\|f\| > \|f_*\|$ and $\|f\| \geqq \|f_{\perp}\|$ whenever $f = f_* + f_{\perp}$ with $f_* \in M_*$ and non-zero $f_{\perp} \in M^{\perp}$. If, moreover, for a closed subspace $V \subset X^*$, $f_*, f_{\perp}$ belong to V whenever $f \in V$, we will say that M is a VHB-subspace.

HB-subspaces enjoy the following properties [**8**].

LEMMA 3.1. *If M is an* HB-*subspace of X, then every $f \in M^*$ has a unique norm preserving extension to X.*

LEMMA 3.2. *Let M be an* HB-*subspace. Then $f \in M_*$ if and only if $\|f|M\| = \|f\|$.*

LEMMA 3.3. *Let M be an* HB-*subspace. If M has a complement N such that for $x \in X$, $\|n\| \leqq \|x\|$ whenever $x = m + n$ with $m \in M$ and $n \in N$, then for every $f \in X^*$, $f_*|N = 0$.*

PROOF. Let $g = f_*|M$ and let g_* be the extension of g to X such that $g_*(n) = 0$ for all $n \in N$. Then by Lemma 3.2

$$\frac{|g_*(x)|}{\|x\|} = \frac{|g_*(m)|}{\|x\|} \leqq \frac{|g_*(m)|}{\|m\|} \leqq \|g\| = \|f_*\|.$$

As $\|g_*\| \geqq \|g\|$, we have $\|g_*\| = \|f_*\|$, and therefore by Lemma 3.1 we have $g_* = f_*$.

In the following two theorems we will discuss examples of Banach algebras which contain maximal modular left (right) ideals that are HB-subspaces or VHB-subspaces.

THEOREM 3.4. *In the Banach algebra c_0 every maximal closed ideal is an* HB-*subspace.*

PROOF. First, c_0 is a commutative Banach algebra with the supremum norm and $(c_0)^* = \ell_1$ [**16**, p. 91]. Second, c_0 is a semi-simple annihilator Banach algebra. Therefore if M is a maximal closed ideal, then $M = A(1 - e)$, where e is a minimal idempotent [**2**, p. 155]. Clearly $e = \{0, ..0, 1, 0, \dots\}$ with one in some *n-th* place. Let $f \in \ell_1$. Then $f = f \circ e + (f - f \circ e)$ and $f \circ e$ vanishes on M. If f is given by the sequence $\{a_k\} \in \ell_1$ then $f - f \circ e$ is given by the sequence $\{b_k\} \in \ell_1$, where $b_k = a_k$ for $k \neq n$ and $b_n = 0$. Hence if $f \circ e \neq 0$, then

$$\|f - f \circ e\| = \sum_{k=1}^{\infty} |b_k| < \sum_{k=1}^{\infty} |a_k| = \|f\|.$$

Thus $\|f - f \circ e\| < \|f\|$ whenever $f \circ e \neq 0$. We have $M^{\perp} = \{f \circ e : f \in A^*\}$ and if we put $M_* = \{f - f \circ e : f \in A^*\}$, the proof is complete.

THEOREM 3.5. *Let* $A = m$ *and* $V = \ell_1$. *Then*

(1) *A is isometrically isomorphic to* V^*, *and*

(2) *if e is a minimal idempotent in A, the maximal modular ideal* $M = A(1 - e)$ *is a* VHB-*subspace.*

PROOF. We recall that A is a semi-simple commutative Banach algebra under pointwise multiplication and supremum norm.

(1) This is well known. (See [**16**, p. 91].)

(2) Let $\Pi = \{E(1), \ldots, E(n)\}$ be a partition of $H = \{1, 2, \ldots\}$ and let $f \in A^*$. (See [**16**, p. 93].) For any set $E \subset H$, define $\mu(E) = f(\chi_E)$, where χ_E is the characteristic function of E. It follows that $\|f\| = \sup\{\sum_{E \in \Pi} |\mu(E)| : \Pi \text{ is a partition of } H\}$. We have $f = f \circ e + (f - f \circ e)$ and $f \circ e$ vanishes on M. Suppose that $f \circ e \neq 0$. Then there exists a subset E of H such that $f(e\chi_E) \neq 0$. Since $e = \{0, .. 0, 1, 0, \ldots\}$, one in some k-th place, we see that $e\chi_E = e$. Hence $f(e) \neq 0$. Therefore if Π_0 is a partition that contains E_0 where $\chi_{E_0} = e$, then for any partition $\Pi \geqq \Pi_0$,

$$\sum_{E \in \Pi} |\mu(E) - \nu(E)| < \sum_{E \in \Pi} |\mu(E)|,$$

where $\nu(E) = f(e\chi_E)$. Hence $\|f - f \circ e\| < \|f\|$. We have $M^\perp = \{f \circ e: f \in A^*\}$ and if we take $M_* = \{f - f \circ e: f \in A^*\}$, we obtain that M is an HB-subspace. Moreover if $f \in V$, then clearly $f \circ e$ and $f - f \circ e$ are in V so that M is also a VHB-subspace.

4. VHB-subspaces and Arens regularity.

LEMMA 4.1. *Let A be a Banach algebra which is the conjugate space of a Banach space V. Then the following statements are equivalent.*

(1) *$V^\perp$ is an ideal of A^{**} for either Arens product.*

(2) *V is A-invariant.*

(3) *V is A^{**}-invariant.*

PROOF. (1) $\Leftrightarrow$ (3). This is [**9**, Theorem 3.2, p. 658].

(3) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Suppose that V is A-invariant. We have $A^{**} = \pi(A) \oplus V^\perp$. We show first that $\pi(A) \circ V^\perp \subset V^\perp$ and $V^\perp \circ' \pi(A) \subset V^\perp$. Let $x \in A$ and $F \in V^\perp$. Then for any $f \in V \subset A^*$, $F \circ f = 0$ since $(F \circ f)(x) = F(f \circ x)$ and $f \circ x \in V$ by hypothesis. But $(\pi(x) \circ F)(f) = \pi(x)(F \circ f) = (F \circ f)(x) = 0$ so that $\pi(x) \circ F \in V^\perp$. Hence $\pi(A) \circ V^\perp \subset V^\perp$. Similarly $V^\perp \circ' \pi(A) \subset V^\perp$. Now let $F, G \in V^\perp$. Then for $f \in V$, $(F \circ G)(f) = F(G \circ f) = 0$ since $G \circ f = 0$. Therefore $F \circ G \in V^\perp$. Similarly $F \circ' G \in V^\perp$. Next let $F \in A^{**}$ and $G \in V^\perp$, and write $F = F_1 + F_2$ with $F_1 \in \pi(A)$, and $F_2 \in V^\perp$. Then $G \circ F = G \circ F_1 + G \circ F_2$ and $F \circ G = F_1 \circ G + F_2 \circ G$. But $G \circ F_2$, $F_2 \circ G$, $F_1 \circ G$ and $G \circ' F_1$ are all in $V^\perp$. As $G \circ F_1 = G \circ' F_1$, $G \circ F_1$ is also in $V^\perp$. Hence $F \circ G$ and $G \circ F$ are in $V^\perp$. Similarly $G \circ' F$ and $F \circ' G$ are in $V^\perp$. Thus $V^\perp$ is a closed ideal of A^{**} for either Arens product.

LEMMA 4.2. *Let A be a Banach algebra which is the conjugate space of a Banach space V. If $e \in E_A$ is such that $\|L_e\| \leqq 1$ and the maximal modular right ideal $M = (1 - e) A$ is a* VHB-*subspace, then $f \circ e \in V$ for all $f \in V$.*

PROOF. Let $e \in E_A$ be such that $\|L_e\| \leqq 1$ and $M = (1 - e)A$ is a VHB-subspace. Let $f \in V$. Then $f = f_* + f_\perp$ with $f_* \in M_* \cap V$ and non-zero $f_\perp \in M^\perp \cap V$. (See §3.) Since $f_\perp$ vanishes on M, we have $f_\perp(x - ex) = 0$ or $f_\perp(x) = (f_\perp \circ e)(x)$ for all $x \in A$ so that $f_\perp = f_\perp \circ e$. By the Pierce decomposition, $(1 - e)A \oplus eA = A$. Since $\|L_e\| \leqq 1$, by Lemma 3.3 we have $f_*|eA = 0$. Therefore $(f \circ e)(x) = f(ex) = f_\perp(ex) = f_\perp(x)$ for all $x \in A$, i.e., $f \circ e = f_\perp$. Hence $f \circ e \in V$.

In the same way we can show that $e \circ' f \in V$ for all $f \in V$ whenever $\|R_e\| \leqq 1$ and the maximal modular left ideal $M = A(1 - e)$ is a VHB-subspace.

THEOREM 4.3. *Let A be a semi-simple* w.c.c. *Banach algebra with dense socle which is the conjugate space of a Banach space V. Assume that every maximal modular left (right) ideal M of A is a* VHB-*subspace and that for every $e \in E_A$, $\|L_e\| \leqq 1$ and $\|R_e\| \leqq 1$. Then V is A-invariant.*

PROOF. Let R^{**} be the radical of $(A^{**}, \circ)$. Since A is w.c.c., by [**12**, Lemma 6.1, p. 11], $R^{**} = \{G \in A^{**}\colon \pi(A) \circ G = (0)\}$. We claim that $V^\perp = R^{**}$. By Lemmas 2.1 and 4.2, $f \circ e$ and $e \circ' f$ are in V for all $e \in E_A$ and $f \in V$. Hence if $e \in E_A$ and $F \in V^\perp$, then $\pi(e) \circ F \in V^\perp$ since $(\pi(e) \circ F)(f) = F(f \circ e) = 0$ for all $f \in V$. As A is w.c.c., $\pi(e) \circ F \in \pi(A)$ and, as $V^\perp \cap \pi(A) = (0)$, it follows that $\pi(e) \circ F = 0$ for all $e \in E_A$ and $F \in V^\perp$. But then $\pi(\text{ae}) \circ F = \pi(a) \circ (\pi(e) \circ F) = 0$ for all $a \in A$, $e \in E_A$ and $F \in V^\perp$. Hence, since every element in S_A is of the form $a_1 e_1 + \cdots + a_n e_n$, where $e_i \in E_A$ and $a_i \in A$, $i = 1, \ldots, n$, we see that $\pi(S_A) \circ V = (0)$ for all $F \in V^\perp$. As S_A is dense in A, we obtain $\pi(A) \circ F = (0)$ for all $F \in V^\perp$. Therefore $V^\perp \subset R^{**}$. To see that $R^{**} \subset V^\perp$, let $G \in R^{**}$ and write $G = G_1 + G_2$ with $G_1 \in \pi(A)$ and $G_2 \in V^\perp$. As $V^\perp \subset R^{**}$, $G_2 \in R^{**}$. Hence $G_1 = G - G_2 \in R^{**}$. But $\pi(A) \cap R^{**} = (0)$ [**6**, Theorem 4.6, p. 130]. Therefore $G_1 = 0$ and so $G \in V^\perp$, i.e., $R^{**} \subset V^\perp$. Hence $V^\perp = R^{**}$ and therefore $V^\perp$ is an ideal of $(A^{**}, \circ)$.

Using again [**12**, Lemma 6.1, p. 11] for the algebra $(A^{**}, \circ')$ and the fact that $e \circ' f \in V$ for all $e \in E_A$ and $f \in V$, we can show that $V^\perp = R_1^{**} = \{G \in A^{**}\colon G \circ' \pi(A) = (0)\}$, the racial of $(A^{**}, \circ')$. Thus $V^\perp$ is an ideal of $(A^{**}, \circ')$. Hence, by Lemma 4.1, V is A-invariant.

THEOREM 4.4. *Let A be a semi-simple* w.c.c. *Banach algebra which is the conjugate space of a Banach space V. If V is A-invariant, then A is Arens regular.*

PROOF. Suppose V is A-invariant. By Lemma 4.1, $V^\perp$ is an ideal of A^{**} for either Arens product. As A is w.c.c., $\pi(A)$ is an ideal of A^{**} for either

Arens product. Since $\pi(A) \cap V^{\perp} = (0)$, $\pi(A) \circ V^{\perp} = V^{\perp} \circ \pi(A) = (0)$. Let $F, G \in V^{\perp}$ and let $\{a_\alpha\}$ be a net in A such that $\pi(a_\alpha)$ w^*-converges to F. Then $\pi(a_\alpha) \circ G$ w^*-converges to $F \circ G$ [7, Lemma 1.4, p. 116] and, as $\pi(a_\alpha) \circ G = 0$, we get $F \circ G = 0$. Similarly by taking a net $\{b_\beta\}$ in A such that $\pi(b_\beta)$ w^*-converges to G, we can show that $F \circ' \pi(b_\beta)$ w^*-converges to $F \circ' G$ and that $F \circ' G = 0$. Now let $F, G \in A^{**}$ and write $F = F_1 + F_2$ and $G = G_1 + G_2$ with $F_1, G_1 \in \pi(A)$ and $F_2, G_2 \in V^{\perp}$. Then $F \circ G = F_1 \circ G_1 + F_1 \circ G_2 + F_2 \circ G_1 + F_2 \circ G_2 = F_1 \circ G_1$. Likewise $F \circ' G = F_1 \circ' G_1$. Since $F_1 \circ G_1 = F_1 \circ' G_1$, we get $F \circ G = D \circ' G$. Therefore A is Arens regular.

THEOREM 4.5. *Let A be a semi-simple* w.c.c. *Banach algebra with dense socle which is the conjugate space of a Banach space V. If every maximal modular left (right) ideal of A is a* VHB-*subspace and for every $e \in E_A$, $\|L_e\| \leqq 1$ and $\|R_e\| \leqq 1$, then A is Arens regular.*

PROOF. By Theorem 4.3. V is A-invariant and therefore, by Theorem 4.4, A is Arens regular.

We include two simple applications of Theorem 4.4.

COROLLARY 4.6. *The algebra* tc(H) *of trace-class operators on a Hilbert space H is Arens regular. More generally, if $\{H_\lambda\}$ is a family of Hilbert spaces, then the L_1-direct sum $(\sum_\lambda \mathrm{tc}(H_\lambda))_1$ is Arens regular.*

PROOF. tc(H) and $(\sum_\lambda \mathrm{tc}(H_\lambda))_1$ are dual A^*-algebras [**14**, Theorem 9.2, p. 65] and therefore are w.c.c. (with dense socle) by Lemma 2.1. Consider first $A = \mathrm{tc}(H)$. Let $V = \mathrm{LC}(H)$, the algebra of all compact linear operators on H. Then by [**11**, Theorem 1, p. 46] A is isometrically isomorphic to V^*. Also by [**11**, Theorem 2, p. 47] A^* is isometrically isomorphic to $\mathrm{L}(H)$, the algebra of all continuous linear operators on H. It is easy to see that V is A-invariant (in fact it is A^{**}-invariant). Hence, by Theorem 4.4, A is Arens regular.

Now let $A = (\sum_\lambda \mathrm{tc}(H_\lambda))_1$ and $V = (\sum_\lambda \mathrm{LC}(H_\lambda))_0$, the $B(\infty)$-sum of $\mathrm{LC}(H_\lambda)$ [**10**, p. 106]. Then A is isometrically isomerphic to V^* [**14**, p. 64] and A^* is isometrically isomorphic to the normed full direct sum of $\mathrm{L}(H_\lambda)$. Clearly V is A-invariant. Hence by Theorem 4.4, A is Arens regular.

COROLLARY 4.7. *As a Banach algebra under pointwise multiplication, ℓ_1 is Arens regular.*

PROOF. ℓ_1 is a semi-simple annihilator Banach algebra and therefore is w.c.c. by Lemma 2.1. Moreover, $\ell_1 = (c_0)^*$. Since c_0 is ℓ_1-invariant, by Theorem 4.4, ℓ_1 is Arens regular. (See also [**3**, p. 863].)

We conclude with the following observations.

THEOREM 4.8. *Let A be a semi-simple* w.c.c. *Banach algebra with dense socle which is the conjugate space of a Banach space V. If V is A-invariant,* *then* $A^*\circ A = \{f\circ a: f \in A^*,\ a \in A\}$ *and* $A\circ' A^* = \{a\circ' f: a \in A, f \in A^*\}$ *are subsets of* V.

PROOF. For $e \in E_A$ let $V_e = \{v\circ e: v \in V\}$. Since V is A-invariant, $V_e \subset V$, and sice $V^* = A$, we have $(V_e)^* = eA$. By Lemma 2.1, V_e is a reflexive Banach space. It is easy to see that $(eA)^*$ may be identified with the subspace $A^*\circ e = \{f\circ e: f \in A^*\}$. Also, for any $a \in A$ and $f \in A^*$, $f\circ ae = (f\circ a)\circ e \in (eA)^*$. Since $V_e^{**} = (eA)^*$ and V_e is reflexive, we see that $f\circ ae \in V_e$. Hence if $a_1, \ldots, a_n \in A$ and $e_1, \ldots, e_n \in E_A$, then $f\circ(a_1e_1 + \cdots + a_ne_n) \in V_{e_1} + \cdots + V_{e_n} \subset V$. Thus $f\circ s \in V$ for all $f \in A^*$ and $s \in S_A$. Let $a \in A$. Since S_A is dense in A, there is a sequence $s_n \in S_A$ such that $s_n \to a$ as $n \to \infty$. Then for $f \in A^*$, $f\circ s_n \to f\circ a$, and since V is closed, it follows that $f\circ a \in V$. Hence $A^*\circ A \subset V$. Similarly, using the subspace $V_e' = \{e\circ' v: v \in V\}$, we can show that $A\circ' A^* \subset V$.

THEOREM 4.9. *Let A be a semi-simple* w.c.c. *Banach algebra with dense socle which is the conjugate space of a Banach space V. Suppose A contains a bounded approximate identity. If V is A-invariant, then A is reflexive.*

PROOF. Suppose that V is A-invariant. Then by Theorem 4.4, A is Arens regular. By [3, Lemma 3.8, p. 855] A^{**} contains an identity. Since A is w.c.c. and Arens regular, by [13, Corollary 4. 3, p. 298] A^{**} is semi-simple. But $A^{**} = \pi(A) \oplus V^\perp$ and $V^\perp$ is contained in the radical of A^{**} (see the proof of Theorem 4.3). Hence $V^\perp = (0)$ and $A^{**} = \pi(A)$ so that A is reflexive.

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