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DIFFERENTIAL EQUATIONS INVOLVING CIRCULANT MATRICES

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1. Introduction. This paper develops a theory for the solution of ordinary and partial differential equations whose structure involves the algebra of circulants. Recent interest of circulants is evident in a book by Davis [1]. This paper shows how the algebra of 2×2 circulants relates to the study of the harmonic oscillator, the Cauchy-Riemann equations, Laplace's equation, the Lorentz transformation, and the wave equation. It then uses $n \times n$ circulants to suggest natural generalizations of these equations to higher dimensions.

2. The algebra of circulants. An $n \times n$ circulant is a matrix of the form

$$X = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & \cdots & x_{n-2} & x_{n-1} \\ x_{n-1} & x_0 & x_1 & x_2 & \cdots & x_{n-3} & x_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_2 & x_3 & x_4 & x_5 & \cdots & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_4 & \cdots & x_{n-1} & x_0 \end{bmatrix}.$$

Note that X has arbitrary entries $x_0, x_1, \ldots, x_{n-1}$ in the top row and the entries are moved over one place to the right in each succeeding row. Let K denote the circulant with $x_1 = 1$ and $x_j = 0$ for all $j \neq 1$. Then the arbitrary circulant X equals $\sum_{h=0}^{n-1} x_h K^h$, and $K^n = I$. [$K^0 = I$ also.]

Define complex circulants $E_0, E_1, \ldots, E_{n-1}$ by

$$(1) \qquad E_h = (1/n) \sum_{j=0}^{n-1} \zeta^{-hj} K^j \text{ for } 0 \leqq h \leqq n-1,$$

where $\zeta = e^{2\pi i/n}$. Then $\{E_0, E_1, \ldots, E_{n-1}\}$ is an idempotent basis for complex circulants since

$$(2.1) \qquad E_h^2 = E_h \text{ for } 0 \leqq h \leqq n-1;$$

$$(2.2) \qquad E_h E_j = 0 \text{ if } h \neq j; \text{ and}$$

$$(2.3) \qquad E_0 + E_1 + \cdots + E_{n-1} = I. \text{ (See Davis [1])}.$$

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One can easily express the basis $\{K^0, K^1, \ldots, K^{n-1}\}$ in terms of the basis $\{E_0, \ldots, E_{n-1}\}$ by

$$K^h = \sum_{j=0}^{n-1} \zeta^{hj} E_j \text{ for } 0 \leqq h \leqq n-1. \tag{3}$$

Important properties of circulants are that one can easily express the eigenvalues of a circulant in terms of its entries and that all circulants have the same eigenvectors.

The eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ of a circulant $\sum_{h=0}^{n-1} x_h K^h$ are given by

$$\lambda_h = \sum_{j=0}^{n-1} \zeta^{hj} x_j \text{ for } 0 \leqq h \leqq n-1, \text{ (see Muir [2]), i.e.,} \tag{4}$$

$\boldsymbol{\lambda} = V\mathbf{x}$ where V is the Vandermonde matrix:

$$V = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \cdots & \zeta^{n-1} \\ 1 & \zeta^2 & \zeta^4 & \zeta^6 & \cdots & \zeta^{2(n-1)} \\ 1 & \zeta^3 & \zeta^6 & \zeta^9 & \cdots & \zeta^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \zeta^{n-1} & \zeta^{2(n-1)} & \zeta^{3(n-1)} & & \zeta^{(n-1)^2} \end{bmatrix}$$

It follows that the rows of V are the eigenvectors of X, with row h the eigenvector for eigenvalue λ_h. One can also invert (4) and express the entries of X as linear combinations of its eigenvalues:

$$x_h = (1/n) \sum_{j=0}^{n-1} \zeta^{-hj} \lambda_j \text{ for } 0 \leqq h \leqq n-1. \tag{5}$$

Combining (1), (3), (4), and (5) yields

$$\sum_{h=0}^{n-1} x_h K^h = \sum_{h=0}^{n-1} \lambda_h E_h. \tag{6}$$

There is a natural extension from entire functions on $\mathbf{C}$ to entire functions on matrices defined as follows: if f is an entire function with Taylor series $\sum_{h=0}^{+\infty} a_h X^h$, then $f(A)$ is defined to be the matrix $\sum_{h=0}^{+\infty} a_h A^h$. With circulant matrices, one can avoid the use of infinite series; the following formula holds:

$$f\Big(\sum_{h=0}^{n-1} \lambda_h E_h\Big) = \sum_{h=0}^{n-1} f(\lambda_h) E_h, \text{ (see Davis, [1]).} \tag{7}$$

In terms of the more direct basis $\{K^h\}$, (7) becomes

$$f\Big(\sum_{h=0}^{n-1} x_h K^h\Big) = \sum_{h=0}^{n-1} \Big[(1/n) \sum_{\ell=0}^{n-1} \zeta^{-h\ell} f\Big(\sum_{j=0}^{n-1} \zeta^{\ell j} x_j\Big)\Big] K^h. \tag{8}$$

EXAMPLE FOR $n = 2$.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus $K^2 = I$. In this case,

$$E_0 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \text{and } E_1 = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} I \\ K \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0 \\ E_1 \end{bmatrix}.$$

Also, if $x_0, x_1, \lambda_0, \lambda_1, \varepsilon$ **C** such that

$$\begin{bmatrix} \lambda_0 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix},$$

then $x_0 I + x_1 K = \lambda_0 E_0 + \lambda_1 E_1$. These formulas are developed and used in Leisenring [3].

3. First-order, linear systems involving circulants. One can easily solve a system of first-order, linear differential equations when the matrix has constant entries. In general, a system with variable entries in the matrix is not so easily solved. However, if the matrix is an $n \times n$ circulant with variable entries, the solution can be written explicitly.

We first investigate the case for 2×2 circulant matrices and solve the system:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} f(t) & g(t) \\ g(t) & f(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{9}$$

First, note that this equation is equivalent to the following:

$$\begin{bmatrix} \dot{x} & \dot{y} \\ \dot{y} & \dot{x} \end{bmatrix} = \begin{bmatrix} f & g \\ g & f \end{bmatrix} \begin{bmatrix} x & y \\ y & x \end{bmatrix}.$$

Using the information of Section 2, we can rewrite this in the form $\dot{x}I + \dot{y}K = (fI + gK)(xI + yK)$. Changing to eigenvalues and idempotents, one finds

$$\begin{aligned} &(\dot{x} + \dot{y})E_0 + (\dot{x} - \dot{y})E_1 \\ &\quad = [(f + g)E_0 + (f - g)E_1][(x + y)E_0 + (x - y)E_1] \\ &\quad = (f + g)(x + y)E_0 + (f - g)(x - y)E_1. \end{aligned}$$

When we equate components, we derive first $\dot{x} + \dot{y} = (f + g)(x + y)$, which has as its solution $x + y = c_0 e^{F+G}$ where c_0 is a constant, $\dot{F} = f$, and $\dot{G} = g$. The other equation is $\dot{x} - \dot{y} = (f - g)(x - y)$ which has as its solution $x - y = c_1 e^{F-G}$ where c_1 is a constant. Thus we get

$$x = (1/2)(c_0e^{F+G} + c_1e^{F-G}),$$
$$y = (1/2)(c_0e^{F+G} - c_1e^{F-G}). \tag{10}$$

The solution vectors

$$\begin{bmatrix} e^{F+G} \\ e^{F+G} \end{bmatrix} \text{and} \begin{bmatrix} e^{F-G} \\ -e^{F-G} \end{bmatrix}$$

are linearly independent for all t, and so they are a fundamental solution set for equation (9).

The same method can be employed to solve the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} f(t) & g(t) \\ g(t) & f(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}. \tag{11}$$

Equation (10) is the general solution of the homogeneous system (9), and a particular solution of the system (11) is

$$x = (1/2)\left[e^{(F+G)} \int (u + v)e^{(-F-G)}\, dt + e^{(F-G)} \int (u - v)e^{(-F+G)}\, dt\right],$$
$$y = (1/2)\left[e^{(F+G)} \int (u + v)e^{(-F-G)}\, dt - e^{(F-G)} \int (u - v)e^{(-F+G)}\, dt\right]. \tag{12}$$

In an analogous manner, using the formulae in the previous section, one can solve explicitly linear $n \times n$ differential systems whose matrix is a circulant.

THEOREM 1. *The system of equations*

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \end{bmatrix} = \left(\sum_{h=0}^{n-1} a_h(t)K^h\right) \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} b_0(t) \\ b_1(t) \\ \vdots \\ b_{n-1}(t) \end{bmatrix} \tag{13}$$

has as its general solution

$$x_h = (1/n)\sum_{j=0}^{n-1} \zeta^{hj}e^{F_j}\left(c_j + \int g_je^{-F_j}dt\right) \tag{14}$$

for $0 \leqq h \leqq n-1$ *where* $\zeta = e^{2\pi i/n}$, *and*

$$f_h = \sum_{j=0}^{n-1} \zeta^{hj}a_j \; (0 \leqq h \leqq n-1);$$

$$g_h = \sum_{j=0}^{n-1} \zeta^{-hj}b_j \; (0 \leqq h \leqq n-1);$$

$\dot{F}_j = f_j \; (0 \leqq j \leqq n-1)$; *and* $c_0, c_1, \ldots, c_{n-1}$ *are constants. The expression*

$$x_h = (1/n) \sum_{j=0}^{n-1} \zeta^{hj}c_je^{F_j} \; (0 \leqq h \leqq n-1) \tag{15}$$

is the general solution of the homogeneous system, and

$$x_h = (1/n)\sum_{j=0}^{n-1} \zeta^{hj} e^{F_j} \int g_j e^{-F_j} dt \tag{16}$$

$(0 \leqq h \leqq n-1)$ *is a particular solution of the nonhomogeneous system.*

4. The equation $d^n x/dt^n - x = 0$. As a special case of equation (13), note that the companion matrix form of the equation $x^{(n)} - x = 0$ is

$$\begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ \vdots \\ \\ x^{(n)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \\ \vdots \\ \\ x^{(n-1)} \end{bmatrix}$$

i.e., it contains the matrix K. Therefore, $\exp tK = \sum_{h=0}^{n-1} (\exp_h t) K^h$ is a Wronskian of solution of the equation $x^{(n)} - x = 0$, where

$$\exp_h t = (1/n)\sum_{l=0}^{n-1} \zeta^{-hl} \exp \zeta^l t = \sum_{j=0}^{+\infty} t^{h+nj}/(h+nj)!$$

for $0 \leqq h \leqq n-1$. (See Rubel and Stolarsky [4].)

5. Generalization of the Lorenz transformation and of the wave equation. In his study of the geometry of relativity, Leisenring [3] used the equation

$$\exp\begin{bmatrix} 0 & \phi \\ \phi & 0 \end{bmatrix} = \begin{bmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{bmatrix} \tag{17}$$

and showed that the matrix on the right is a Lorentz transformation, i.e., it preserves the quadratic form

$$x^2 - y^2 = \det\begin{bmatrix} x & y \\ y & x \end{bmatrix}.$$

Indeed, in the physical Lorentz transformations

$$x = (x' + ut')/\sqrt{1-u^2/c^2}, \quad t = (t' + ux'/c^2)/\sqrt{1-u^2/c^2}$$

let $u = c \tanh \phi$ and $c = 1$. Then, the equations become

$$\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} \operatorname{ch} \phi & \operatorname{sh} \phi \\ \operatorname{sh} \phi & \operatorname{ch} \phi \end{bmatrix} \begin{bmatrix} x' \\ t' \end{bmatrix},$$

and it follows that $x^2 - t^2 = x'^2 - t'^2$. The latter fact can also be shown using circulants. If

$$\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} \operatorname{ch}\phi & \operatorname{sh}\phi \\ \operatorname{sh}\phi & \operatorname{ch}\phi \end{bmatrix} \begin{bmatrix} x' \\ t' \end{bmatrix},$$

then

$$\begin{bmatrix} x & t \\ t & x \end{bmatrix} = \begin{bmatrix} \operatorname{ch}\phi & \operatorname{sh}\phi \\ \operatorname{sh}\phi & \operatorname{ch}\phi \end{bmatrix} \begin{bmatrix} x' & t' \\ t' & x' \end{bmatrix},$$

so

$$\det \begin{bmatrix} x & t \\ t & x \end{bmatrix} = \det \begin{bmatrix} \operatorname{ch}\phi & \operatorname{sh}\phi \\ \operatorname{sh}\phi & \operatorname{ch}\phi \end{bmatrix} \det \begin{bmatrix} x' & t' \\ t' & x' \end{bmatrix},$$

or $x^2 - t^2 = 1 \cdot (x'^2 - t'^2)$.

By the chain rule, it also follows that

$$\begin{bmatrix} \partial/\partial x' \\ \partial/\partial t' \end{bmatrix} = \begin{bmatrix} \operatorname{ch}\phi & \operatorname{sh}\phi \\ \operatorname{sh}\phi & \operatorname{ch}\phi \end{bmatrix} \begin{bmatrix} \partial/\partial x \\ \partial/\partial t \end{bmatrix};$$

so the above circulant linear transformation also preserves the wave operator

$$\partial^2/\partial x^2 - \partial^2/\partial t^2 = \det \begin{bmatrix} \partial/\partial x & \partial/\partial t \\ \partial/\partial t & \partial/\partial x \end{bmatrix}.$$

There are natural generalizations of these properties of 2-dimensional relativity to n dimensions. If $s_1, s_2, \ldots, s_{n-1} \in \mathbf{C}$, then by equation (8)

$$\exp\left(\sum_{h=1}^{n-1} s_h K^h\right) = \sum_{h=0}^{n-1} \left[(1/n) \sum_{\ell=0}^{n-1} \zeta^{-h\ell} \exp\left(\sum_{j=1}^{n-1} \zeta^{\ell j} s_j\right)\right] K^h; \tag{18}$$

and $\det \exp(\sum_{h=1}^{n-1} s_h K^h) = 1$. So the linear transformation $\exp(\sum_{h=1}^{n-1} s_h K^h)$ preserves the nth order form

$$\begin{aligned} \det\left(\sum_{h=0}^{n-1} x_h K^h\right) &= \prod_{h=0}^{n-1} \left(\sum_{j=0}^{n-1} \zeta^{hj} x_j\right) \\ &= \prod_{h=0}^{n-1} (x_0 + \zeta^h x_1 + \zeta^{2h} x_2 + \cdots + \zeta^{(n-1)h} x_{x_{n-1}}). \end{aligned} \tag{19}$$

By the above method, it also leaves invariant the linear partial differential operator

$$\begin{aligned} \det\left(\sum_{h=0}^{n-1} (\partial/\partial x_h) K^h\right) &= \prod_{h=0}^{n-1} \left(\sum_{j=0}^{n-1} \zeta^{hj} (\partial/\partial x_j)\right) \\ &= \prod_{h=0}^{n-1} \left(\frac{\partial}{\partial x_0} + \zeta^h \frac{\partial}{\partial x_1} + \zeta^{2h} \frac{\partial}{\partial x_2} + \cdots \right. \\ &\qquad \left. + \zeta^{(n-1)h} \frac{\partial}{\partial x_{n-1}}\right), \end{aligned} \tag{20}$$

where the product denotes composition.

6. Solutions of a homogeneous, partial differential equation. The method of circulants which led to the formation of the partial differential equation

$$\left[\prod_{h=0}^{n-1}\left(\sum_{j=0}^{n-1}\zeta^{hj}(\partial/\partial x_j)\right)\right](u) = 0, \text{ where } \zeta = e^{2\pi i/n}, \tag{21}$$

also leads to solutions of this equation. This is a natural generalization of the homogeneous wave equation, since for $n = 2$ (21) becomes $\partial^2 u/\partial x_0^2 - \partial^2 u/\partial x_1^2 = 0$. Let $z_0, z_1, \ldots, z_{n-1}$ be new variables given by $z_h = (1/n)\sum_{j=0}^{n-1}\zeta^{-hj}x_j$ $(0 \leqq h \leqq n-1)$. [These formulas are like equation (5) in Section 2.] Then $x_h = \sum_{j=0}^{n-1}\zeta^{hj}z_j$ $(0 \leqq h \leqq n-1)$; and, by chain rule, $\partial/\partial z_h = \sum_{j=0}^{n-1}\zeta^{hj}(\partial/\partial x_j)$ $(0 \leqq h \leqq n-1)$. Thus equation (21) takes the form $(\prod_{h=0}^{n-1}(\partial/\partial z_h))(u) = 0$. Now let $F_0, F_1, \ldots, F_{n-1}$ be n C^∞ functions: $\mathbf{C}^{n-1} \to \mathbf{C}$ and let

$$\begin{aligned} u(z_0, z_1, \ldots, z_{n-1}) &= F_0(z_1, z_2, \ldots, z_{n-1}) \\ &\quad + \sum_{h=1}^{n-2} F_h(z_0, z_1, \ldots, z_{h-1}, z_{h+1}, \ldots, z_{n-1}) \\ &\quad + F_{n-1}(z_0, z_1, \ldots, z_{n-2}) \end{aligned} \tag{22}$$

i.e., for each h, F_h is independent of z_h. It is easy to verify that u given by (22) is a solution of equation (21). It is a reasonable conjecture that this method constructs all solutions, as it does for $n = 2$.

If in equation (22), the conditions

$$\overline{F_0(\overline{\alpha_{n-1}}, \overline{\alpha_{n-2}}, \ldots, \overline{\alpha_1})} = F_0(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$$

and

$$\overline{F_h(\bar\alpha_0, \bar\alpha_{n-2}, \bar\alpha_{n-3}, \ldots, \bar\alpha_1)} = F_{n-h}(\alpha_0, \alpha_1, \ldots, \alpha_{n-2}) \quad (1 \leqq h \leqq n-1)$$

hold for all complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, then u as a function of $x_0, x_1, \ldots, x_{n-1}$ maps $\mathbf{R}^n$ into $\mathbf{R}$.

7. A generalization of the Cauchy-Riemann conditions. Recall that if the Jacobian matrix of functions $u(x, y)$ and $v(x, y)$, namely

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

is of the form

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

(the usual representation of $\mathbf{C}$), that is $u_x = v_y$ and $u_y = -v_x$, then u and v satisfy Laplace's equation $(\partial^2/\partial x^2 + \partial^2/\partial y^2)(f) = 0$. Note too that

$$\det\begin{bmatrix} \partial/\partial x & \partial/\partial y \\ -\partial/\partial y & \partial/\partial x \end{bmatrix} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

An analogous property holds for 2×2 circulants. If the Jacobian

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

is a 2×2 circulant, then $u_x = v_y$ and $u_y = v_x$. If two C^∞ functions u and v satisfy these equations, then there exist two analytic functions f and g such that u and v are of the form

$$u = (1/2)[f(x+y) + g(x-y)],$$
$$v = (1/2)[f(x+y) - g(x-y)].$$

Also u and v satisfy the wave equation $(\partial^2/\partial x^2 - \partial^2/\partial y^2)(F) = 0$. We notice here too that

$$\det\begin{bmatrix} \partial/\partial x & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}.$$

These facts can be generalized to $\mathbf{C}^n$ using circulants.

If $u_0, u_1, \ldots, u_{n-1}$ are entire functions mapping $\mathbf{C}^n$ into $\mathbf{C}$, their Jacobian is

$$\begin{bmatrix} \partial u_0/\partial x_0 & \partial u_0/\partial x_1 & \cdots & \partial u_0/\partial x_{n-1} \\ \partial u_1/\partial x_0 & \partial u_1/\partial x_1 & \cdots & \partial u_1/\partial x_{n-1} \\ \vdots & \vdots & & \vdots \\ \partial u_{n-1}/\partial x_0 & \partial u_{n-1}/\partial x_1 & \cdots & \partial u_{n-1}/\partial x_{n-1} \end{bmatrix}.$$

This matrix is a circulant if and only if

(i) $$\partial u_0/\partial x_0 = \partial u_1/\partial x_1 = \cdots = \partial u_{n-1}/\partial x_{n-1}$$

and

(ii) $$\begin{aligned} \partial u_0/\partial x_h &= \partial u_1/\partial x_{h+1} = \cdots = \partial u_{n-h-1}/\partial x_{n-1} \\ &= \partial u_{n-h}/\partial x_0 = \partial u_{n-h+1}/\partial x_1 = \cdots = \partial u_{n-1}/\partial x_{h-1} \end{aligned}$$

for $1 \leqq h \leqq n-1$.

THEOREM 2. *Let $u_0, u_1, \ldots, u_{n-1}$ be entire functions mapping $\mathbf{C}^n$ into $\mathbf{C}$ such that their Jacobian is a circulant. Then*

(1) *$u_0, u_1, \ldots, u_{n-1}$ satisfy the partial differential equation*

$$\left[\prod_{h=0}^{n-1}\left(\sum_{j=0}^{n-1} \zeta^{hj}\frac{\partial}{\partial x_j}\right)\right](u) = 0;$$

(2) *there exist entire functions $g_0, g_1, \ldots, g_{n-1}$ mapping $\mathbf{C}$ into $\mathbf{C}$ such that $u_0, u_1, \ldots, u_{n-1}$ are of the form*

$$u_j = (1/n)\sum_{\ell=0}^{n-1} \zeta^{-j\ell} g_\ell\left(\sum_{h=0}^{n-1} \zeta^{h\ell} x_h\right);$$

(3) *if* $\overline{g_0(\bar{z})} = g_0(z)$ *and* $\overline{g_\ell(\bar{z})} = g_{n-\ell}(z)$ *for* $1 \leqq \ell \leqq n-1$, *then* $u_0, u_1, \ldots, u_{n-1}$ *all map* $\mathbf{R}^n$ *into* $\mathbf{R}$.

PROOF OF (1). If the Jacobian of $(u_0, u_1, \ldots, u_{n-1})$ is a circulant, then conditions (i) and (ii) hold. It then follows that

$$\sum_{h=0}^{n-1} (\partial u_j/\partial x_h)K^h = K^j \cdot \left(\sum_{h=0}^{n-1} (\partial u_0/\partial x_h)K^h\right)$$

for $0 \leqq j \leqq n-1$. Let

$$L(u) = \left[\prod_{h=0}^{n-1}\left(\sum_{j=0}^{n-1} \zeta^{hj} \frac{\partial}{\partial x_j}\right)\right](u), \text{ and}$$

let $z_h = (1/n) \sum_{j=0}^{n-1} \zeta^{-hj}x_j$ for $0 \leqq h \leqq n-1$. Then $x_h = \sum_{j=0}^{n-1} \zeta^{hj} z_j$ and $\partial/\partial z_h = \sum_{j=0}^{n-1} \zeta^{hj} (\partial/\partial x_j)$ for $0 \leqq h \leqq n-1$. Thus $L = \prod_{h=0}^{n-1} (\partial/\partial z_h)$. Using the relationship between the entries and eigenvalues of a circulant matrix as shown in §2,

$$\sum_{h=0}^{n-1} \frac{\partial}{\partial x_h} K^h = \sum_{h=0}^{n-1}\left(\sum_{j=0}^{n-1} \zeta^{hj} \frac{\partial}{\partial x_j}\right) E_h = \sum_{h=0}^{n-1} \frac{\partial}{\partial z_h} E_h.$$

So

$$\sum_{h=0}^{n-1} (\partial u_j/\partial x_h) K^h = \sum_{h=0}^{n-1} (\partial u_j/\partial z_h)E_h$$

for $0 \leqq j \leqq n-1$. But

$$\begin{aligned}\sum_{h=0}^{n-1} (\partial u_j/\partial x_h) K^h &= K^j \sum_{h=0}^{n-1} (\partial u_0/\partial x_h)K^h \\ &= \left(\sum_{h=0}^{n-1} \zeta^{hj} E_h\right)\left(\sum_{h=0}^{n-1} (\partial u_0/\partial z_h)E_h\right) \\ &= \sum_{h=0}^{n-1} \zeta^{hj}(\partial u_0/\partial z_h)E_h.\end{aligned}$$

Since the E_h's form a basis, $\partial u_j/\partial z_h = \zeta^{hj}(\partial u_0/\partial z_h)$ for $0 \leqq h \leqq n-1$ and $0 \leqq j \leqq n-1$.

Now let $v_k = \sum_{j=0}^{n-1} \zeta^{-kj}u_j$ for $0 \leqq k \leqq n-1$. Then

$$\begin{aligned}\partial v_k/\partial z_h &= (\partial/\partial z_h)\left(\sum_{j=0}^{n-1} \zeta^{-kj}u_j\right) = \sum_{j=0}^{n-1} \zeta^{-kj} (\partial u_j/\partial z_h) \\ &= \sum_{j=0}^{n-1} \zeta^{-kj} \zeta^{hj}(\partial u_0/\partial z_h) = \sum_{j=0}^{n-1} \zeta^{(-k+h)j}(\partial u_0/\partial z_h).\end{aligned}$$

If $k \neq h$, then $\partial v_k/\partial z_h = [\sum_{j=0}^{n-1} \zeta^{(-k+h)j}] (\partial u_0/\partial z_h) = 0$. Thus $(\prod_{h=0}^{n-1} (\partial/\partial z_h))(v_k) = 0 = L(v_k)$.

So,

$$L(v_k) = L\left(\sum_{j=0}^{n-1} \zeta^{-kj} u_j\right) = \sum_{j=0}^{n-1} \zeta^{-kj} L(u_j) = 0,$$

for $0 \leqq k \leqq n-1$. Since the matrix of coefficients of the $L(u_j)$'s is non-singular, $L(u_0) = L(u_1) = \cdots = L(u_{n-1}) = 0$.

PROOF OF (2). In the proof of (1), we showed that $\partial v_k/\partial z_h = 0$ if $k \neq h$. Thus there exists an entire function f_k mapping **C** into **C** such that $v_k = f_k(z_k)$. Therefore, $u_j = (1/n) \sum_{k=0}^{n-1} \zeta^{kj} f_k(z_k)$ for $0 \leqq j \leqq n-1$. Now write $u_0, u_1, \ldots, u_{n-1}$ as entries of a circulant, i.e., take $\sum_{j=0}^{n-1} u_j K^j$. Then

$$\begin{aligned}\sum_{j=0}^{n-1} u_j K^j &= f_0(z_0)E_0 + \sum_{k=1}^{n-1} f_{n-k}(z_{n-k})\, E_k \\ &= f_0\left((1/n)\sum_{\ell=0}^{n-1} x_\ell\right) E_0 + \sum_{k=1}^{n-1} f_{n-k}\left((1/n)\sum_{\ell=0}^{n-1} \zeta^{k\ell} x_\ell\right) E_k.\end{aligned}$$

Now let $g_0(z) = f_0((1/n)z)$ and $g_k(z) = f_{n-k}((1/n)z)$ for $1 \leqq k \leqq n-1$. Then

$$\sum_{j=0}^{n-1} u_j K^j = \sum_{k=0}^{n-1} g_k\left(\sum_{\ell=0}^{n-1} \zeta^{k\ell} x_\ell\right) E_k.$$

Note also that

$$u_j = (1/n)\sum_{\ell=0}^{n-1} \zeta^{-j\ell} g_\ell\left(\sum_{h=0}^{n-1} \zeta^{h\ell} x_h\right)$$

for $0 \leqq j \leqq n-1$. Since $g_0, g_1, \ldots, g_{n-1}$ are entire functions mapping **C** into **C**, part (2) follows.

PROOF OF (3). The conditions imply that

$$u_j(\overline{x_0}, \overline{x_1}, \ldots, \overline{x_{n-1}}) = \overline{u_j(x_0, x_1, \ldots, x_{n-1})}.$$

Now we discuss the concept of derivative of a function on $n \times n$ circulants. To repeat what we said earlier, a complex function $f(x + iy) = u(x, y) + iv(x, y)$ has a derivative (i.e., is analytic) if the Cauchy-Riemann conditions, $u_x = v_y$ and $u_y = -v_x$, hold; and these conditions hold if and only if the Jacobian matrix

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

is of the form of a matrix representation of **C**, i.e.,

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}.$$

Similarly, a function on circulants has a derivative which is a function on circulants if the Jacobian matrix of the original function is in the form of a circulant.

If $u = (u_0, u_1, \ldots, u_{n-1})$ is a function from $\mathbf{C}^n$ into $\mathbf{C}^n$, u can be extended to a function $\tilde{u}$ sending complex circulants into complex circulants via

$$\tilde{u}\left(\sum_{h=0}^{n-1} x_h K^h\right) = \sum_{h=0}^{n-1} u_h(x_0, x_1, \ldots, x_{n-1})K^h.$$

The difference quotient of $\tilde{u}$ in the direction of the complex axis K^h is

$$\tilde{u}' = \lim_{\Delta x_h \to 0} \frac{\tilde{u}[x_0 I + \cdots + (x_h + \Delta x_h)K^h + \cdots + x_{n-1}K^{n-1}] - \tilde{u}(x_0 I + \cdots + x_{n-1}K^{n-1})}{\Delta x_h K^h}.$$

[This definition is well-defined because since $K^n = I$, $I/K^h = K^{n-h}$.] u can be called differentiable if $\tilde{u}'$ is the same in the directions of all the axes I, $K, \ldots, K^{n-1}$. Then it follows that

$$\tilde{u}' = \sum_{j=0}^{n-1} (\partial u_j/\partial x_0)K^j = \sum_{j=0}^{n-h-1} (\partial u_{j+h}/\partial x_h)K^j + \sum_{j=n-h}^{n-1} (\partial u_{j-n+h}/\partial x_h)K^h$$

for $0 \leqq h \leqq n-1$. Note that for all these quotients to be equal, conditions (i) and (ii) hold. Also, $\tilde{u}'$ is the transpose of the Jacobian matrix of $u = (u_0, u_1, \ldots.\ u_{n-1})$ and it is a circulant. (The transpose of a circulant is a circulant.)

Theorem 2 shows that $u_0, u_1, \ldots, u_{n-1}$ are entire functions: $\mathbf{C}^n \to \mathbf{C}$ and satisfy conditions (i) and (ii) if and only if there exist entire functions $g_0, g_1, \ldots, g_{n-1}$: $\mathbf{C} \to \mathbf{C}$ such that

$$u_j = (1/n) \sum_{\ell=0}^{n-1} \zeta^{-j\ell} g_\ell\left(\sum_{h=0}^{n-1} \zeta^{h\ell} x_h\right)$$

for $0 \leqq j \leqq n-1$. One can show that by changing basis,

$$\begin{aligned}\tilde{u} &= \sum_{j=0}^{n-1}\left[(1/n)\sum_{\ell=0}^{n-1} \zeta^{-j\ell} g_\ell\left(\sum_{h=0}^{n-1} \zeta^{h\ell} x_h\right)\right]K^j \\ &= \sum_{h=0}^{n-1} g_h\left(\sum_{\ell=0}^{n-1} \zeta^{h\ell} x_\ell\right)E_h,\end{aligned}$$

i.e., $\tilde{u}$ is decomposed into a sum of entire functions on the idempotent axes. Then too,

$$\begin{aligned}\tilde{u}' &= \sum_{j=0}^{n-1} (\partial u_j/\partial x_0)K^j \\ &= \sum_{j=0}^{n-1}\left[(1/n)\sum_{\ell=0}^{n-1} \zeta^{-j\ell} g_\ell'\left(\sum_{h=0}^{n-1} \zeta^{h\ell} x_h\right)\right]K^j \\ &= \sum_{h=0}^{n-1} g_h'\left(\sum_{\ell=0}^{n-1} \zeta^{h\ell} x_\ell\right)E_h,\end{aligned}$$

i.e., if $\tilde{u}$ satisfies hypothesis of Theorem 2, one can differentiate $\tilde{u}$ by differentiating the above entire functions $g_0, \ldots, g_{n-1}$ on their respective idempotent axes.

In particular, the function

$$\exp\Bigl(\sum_{h=0}^{n-1} x_h K^h\Bigr) = \sum_{j=0}^{n-1}\Bigl[(1/n)\sum_{\ell=0}^{n-1} \zeta^{-j\ell} \exp\Bigl(\sum_{h=0}^{n-1} \zeta^{h\ell} x_h\Bigr)\Bigr] K^j$$

is its own derivative.

Skew-circulants (defined in Davis [1]) also have these differentiation properties. A skew-circulant has entries positioned like those of a circulant except with minus signs below the main diagonal, i.e., for $n = 3$,

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ -x_2 & x_0 & x_1 \\ -x_1 & -x_2 & x_0 \end{bmatrix}.$$

If v is the skew-circulant with $x_1 = 1$ and $x_j = 0$ for $j \neq 1$, then $v^n = -I$ and the skew-circulant can be expressed in the form $\sum_{h=0}^{n-1} x_h v^h$. This algebra is isomorphic over $\mathbf{C}$ to regular circulants via the correspondence $v \mapsto \alpha K$ where $\alpha = e^{\pi i/n}$ $(\alpha^n = -1)$.

If the Jacobian matrix of $u = (u_0, u_1, \ldots, u_{n-1})$ [where $u_0, u_1, \ldots, u_{n-1}$: $\mathbf{C}^n \to \mathbf{C}$] is a skew-circulant matrix, then

(iii) $$\partial u_0/\partial x_0 = \partial u_1/\partial x_1 = \cdots = \partial u_{n-1}/\partial x_{n-1}$$

and

(iv) $$\begin{aligned} \partial u_0/\partial x_h &= \partial u_1/\partial x_{h+1} = \cdots = \partial u_{n-h-1}/\partial x_{n-1} \\ &= -\partial u_{n-h}/\partial x_0 = -\partial u_{n-h+1}/\partial x_1 = \cdots = -\partial u_{n-1}/\partial x_{h-1} \end{aligned}$$

for $1 \leqq h \leqq n-1$.

For $n = 2$, these are exactly the Cauchy-Riemann equations. Using a method similar to the proof of Theorem 2, one can show the following

THEOREM 3. *Let $\zeta = e^{2\pi i/n}$ and $\alpha = e^{\pi i/n}$. Let $u_0, u_1, \ldots, u_{n-1}$ be entire functions: $\mathbf{C}^n \to \mathbf{C}$ such that conditions* (iii) *and* (iv) *hold. Then there exist entire functions $g_0, g_1, \ldots, g_{n-1}$: $\mathbf{C} \to \mathbf{C}$ such that $u_0, u_1, \ldots, u_{n-1}$ are of the form*

$$u_j = \alpha^{-j}(1/n)\sum_{\ell=0}^{n-1} \zeta^{-j\ell} g_\ell\Bigl(\sum_{h=0}^{n-1} \zeta^{h\ell}\alpha^h x_h\Bigr)$$

for $0 \leqq j \leqq n-1$.

Since in the algebra of skew-circulants, $E_h = (1/n)\sum_{j=0}^{n-1} \zeta^{hj} \alpha^{-j} v^j$ for $0 \leqq h \leqq n-1$ is an idempotent basis,

$$\sum_{j=0}^{n-1} u_j v^j = \sum_{h=0}^{n-1} g_h\Bigl(\sum_{\ell=0}^{n-1} \zeta^{h\ell}\alpha^\ell x_\ell\Bigr) E_h.$$

Thus differentiation properties here are similar to those of circulants.

See Leisenring [3] for an application of this type of differentiation to the geometry of the bicomplex plane $\mathbf{C} \times \mathbf{C}$.

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