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AN INEQUALITY FOR NON-DECREASING SEQUENCES

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1. The following two inequalities are well known [**1**]. If $\{a_i\}$ is a sequence of non-negative numbers and $0 < r < s$, then

$$(1.1) \qquad \left(\sum_{i=1}^{n} a_i^r\right)^{1/r} \geqq \left(\sum_{i=1}^{n} a_i^s\right)^{1/s}.$$

If $\{p_i\}$ is a sequence of non-negative weights and $\sum_{i=1}^{n} p_i \leqq 1$, then

$$(1.2) \qquad \left(\sum_{i=1}^{n} p_i a_i^r\right)^{1/r} \leqq \left(\sum_{i=1}^{n} p_i a_i^s\right)^{1/s}.$$

In a recent paper, Klamkin and Newman [**2**] established an inequality, which may be regarded as a modified version of (1.1) pertaining to non-dcreasing sequences. If $0 = a_0 \leqq a_1 \leqq \cdots \leqq a_n$ satisfies $a_i - a_{i-1} \leqq 1$ and if $r \geqq 1$, $s + 1 = 2(r + 1)$, then

$$(1.3) \qquad \left((r + 1) \sum_{i=1}^{n} a_i^r\right)^{1/(r+1)} \geqq \left((s + 1) \sum_{i=1}^{n} a_i^s\right)^{1/(s+1)}.$$

Our aim here is to prove a "weighted" version of (1.3). The result is, in a certain sense, a converse of (1.2) for non-dcreasing sequences.

THEOREM 1. *Let* $0 \leqq p_0 \leqq p_1 \leqq \cdots \leqq p_n$ *and* $0 = a_0 \leqq a_1 \leqq \cdots \leqq a_n$ *satisfying*

$$(1.4) \qquad a_i - a_{i-1} \leqq (p_i + p_{i-1})/2, \ (i = 1, 2, \ldots, n).$$

If $r \geqq 1$ *and* $s + 1 \geqq 2(r + 1)$, *then*

$$(1.5) \qquad \left((r + 1) \sum_{i=1}^{n} p_i a_i^r\right)^{1/(r+1)} \geqq \left((s + 1) \sum_{i=1}^{n} p_i a_i^s\right)^{1/(s+1)}.$$

REMARKS. (i) The condition that $\{p_i\}$ is a non-decreasing sequence cannot be dispensed with, in general. If $r = 1$, $s = 3$, $p_1 = 3$, $p_2 = 1$, $a_1 = 3$, $a_2 = 5$, then (1.5) does not hold.

(ii) The condition $s + 1 \geqq 2(r + 1)$ is, in general, not dispensable If $r = 1$, $s = 2$, $p_i = 1$, $a_i = i$ $(i = 1, 2, \ldots, n)$, then (1.5) does not hold.

(iii) In order to compare (1.5) with (1.2) observe that setting $\sum_{i=1}^{n} p_i = \lambda$, $p_i/\lambda = q_i$, $a_i/\lambda = b_i$, we have $\sum q_i = 1$ and (1.5) is equivalent to

$$\left((r+1)\sum_{i=1}^{n} q_i b_i^r\right)^{1/(r+1)} \geqq \left((s+1)\sum_{i=1}^{n} q_i b_i^s\right)^{1/(s+1)},$$

whenever $b_i - b_{i-1} \leqq (q_i + q_{i-1})/2$.

2. PROOF. The convexity of x^r $(r \geqq 1)$ implies that

$$\int_a^b x^r dx \leqq (b-a)\frac{a^r + b^r}{2}$$

for $0 \leqq a < b$. Hence

$$a_i^{r+1} - a_{i-1}^{r+1} \leqq \frac{r+1}{2}(a_i^r + a_{i-1}^r)(a_i - a_{i-1}).$$

Since $\{a_i\}$ and $\{p_i\}$ are non-decreasing,

$$(a_i^r + a_{i-1}^r)\frac{p_i + p_{i-1}}{2} \leqq a_i^r p_i + a_{i-1}^r p_{i-1}.$$

Combining this, (1.4) and the previous inequality we have

$$a_i^{r+1} - a_{i-1}^{r+1} \leqq \frac{r+1}{2}(a_i^r p_i + a_{i-1}^r p_{i-1}). \tag{2.1}$$

If we set $\sigma_j = \sum_{i=1}^{j} a_i^r p_i$ and sum both sides of (2.1) for $1 \leqq i \leqq j$, we get

$$a_j^{r+1} \leqq \frac{r+1}{2}(\sigma_j + \sigma_{j-1}).$$

Using the notation $k = (s+1)/(r+1)$, the last inequality yields

$$a_j^{s-r} \leqq (r+1)^{k-1}((\sigma_j + \sigma_{j-1})/2)^{k-1}. \tag{2.2}$$

Now, since $k - 1 \geqq 1$, the convexity of x^{k-1} implies that

$$\int_a^b x^{k-1}\, dx \geqq (b-a)\left(\frac{a+b}{2}\right)^{k-1}$$

for $0 \leqq a < b$. Hence

$$k(\sigma_j - \sigma_{j-1})\left(\frac{\sigma_j + \sigma_{j-1}}{2}\right)^{k-1} \leqq \sigma_j^k - \sigma_{j-1}^k. \tag{2.3}$$

From (2.2) and (2.3) we conclude that

$$kp_j a_j^s = ka_j^{s-r}(\sigma_j - \sigma_{j-1}) \leqq (r+1)^{k-1}(\sigma_j^k - \sigma_{j-1}^k).$$

Whence, after summing for $1 \leqq j \leqq n$, we obtain

$$k\sum_{j=1}^{n} p_j a_j^s \leqq (r+1)^{k-1}\left(\sum_{k=1}^{n} p_i a_i^r\right)^k.$$

Replacing k by $(s+1)/(r+1)$, (1.5) follows.

3. If we replace assumption (1.4) by $a_i - a_{i-1} \leqq p_i$, we obtain a slightly different inequality. The proof is analogous to that of Theorem 1, hence will be omitted here.

THEOREM 2. *Let* $0 \leqq p_1 \leqq p_2 \leqq \cdots \leqq p_n$ *and* $0 = a_0 \leqq a_1 \leqq \cdots \leqq a_n$ *satisfying* $a_i - a_{i-1} \leqq p_i$ $(i = 1, 2, \ldots, n)$. *If* $r \geqq 1$ *and* $s + 1 \geqq 2(r + 1)$, *then*

$$\left((r+1)\sum_{i=1}^{n} a_i^r \frac{p_i + p_{i+1}}{2}\right)^{1/(r+1)} \geqq \left((s+1)\sum_{i=1}^{n} a_i^s \frac{P_i + P_{i+1}}{2}\right)^{1/(s+1)}. \tag{3.1}$$

If the sequence $\{a_i\}$ is non-decreasing and convex (i.e., $a_i - a_{i-1} \geqq 0$ and $a_{i+1} + a_{i-1} - 2a_i \geqq 0$), then we may set $p_i = a_i - a_{i-1}$ in Theorem 2. Inequality (3.1) now becomes

$$\left(\frac{r+1}{2}\sum_{i=1}^{n-1} a_i^r(a_{i+1} - a_{i-1})\right)^{1/(r+1)} \geqq \left(\frac{s+1}{2}\sum_{i=1}^{n-1} a_i^s(a_{i+1} - a_{i-1})\right)^{1/(s+1)}.$$

Finally, if we set $p_i = 1$ $(i = 0, 1, \ldots, n)$ in Theorem 1 (or Theorem 2), we obtain the generalization of the Klamkin-Newman inequality (1.3) for the cases $s + 1 \geqq 2(r + 1)$.

REFERENCES

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