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ON MORE RESTRICTED PARTITIONS

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1. **Introduction.** Recently Gupta [2] studied the function $g(n, m, h, k)$ which enumerates the number of partitions of n into exactly k summands each less than or equal to m and in which the number of distinct summands is exactly h. Earlier Cheema and Haskell [1] studied $p(n, r, m, k)$, the number of partitions of n into r summands such that each summand is less than or equal to k and greater than or equal to m. In this article it is shown that the above results can be generalized to study $g(n, l, m, h, k)$ (and other related functions) which enumerates the number of partitions of n into exactly k summands each less than or equal to m and greater than or equal to l in which the number of distinct summands is exactly h.

2. $g(n, l, m, h, k)$ is the coefficient of $x^n z^k t^h$ in

$$(2.1)\quad \prod_{i=l}^{m}\left(1+\frac{x^i z}{1-x^i z}t\right) \quad \text{i.e., in } \prod_{i=l}^{m}\left(\frac{1+x^i z(t-1)}{1-x^i z}\right)=\frac{f(t)}{f(0)}$$

where

$$f(t)=\prod_{i=l}^{m}(1+x^i z(t-1))$$

$$(2.2)\quad \begin{aligned} &= 1+\begin{bmatrix} m-l+1 \\ 1 \end{bmatrix} x^{l-1}z(t-1) \\ &\quad +\begin{bmatrix} m-l+1 \\ 2 \end{bmatrix} x^{2(l-1)}z^2(t-1)^2 \\ &\quad +\cdots+\begin{bmatrix} m-l+1 \\ m-l+1 \end{bmatrix} x^{(m-l+1)(l-1)}z^{m-l+1}(t-1)^{m-l+1}, \end{aligned}$$

where

$$(2.3)\quad \begin{bmatrix} m \\ i \end{bmatrix}=\frac{(1-x^m)(1-x^{m-1})\cdots(1-x^{m-i+1})}{(1-x)(1-x^2)\cdots(1-x^i)}x^{i(i+1)/2}.$$

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Thus the coefficient of t^h in $f(t)$ is

$$
\begin{aligned}
&x^{h(l-1)}\begin{bmatrix} m-l+1 \\ h \end{bmatrix} z^h \\
(2.4)\qquad &-\binom{h+1}{1}\begin{bmatrix} m-l+1 \\ h+1 \end{bmatrix} x^{(h+1)(l-1)}z^{h+1} \\
&+\cdots+(-1)^{m-h}\binom{m-l+1}{m-l+1-h}\begin{bmatrix} m-l+1 \\ m-l+1 \end{bmatrix} \\
&\qquad\cdot x^{(m-l+1)(l-1)}z^{m-l+1} \\
&= z^h A_h(x,z).
\end{aligned}
$$

Using partial fractions we have

$$
(2.5)\qquad A_h(x,z)\prod_{i=l}^{m}(1-x^i z)^{-1} = \sum_{i=l}^{m} B_i^{(h)}(x)/(1-x^i z)
$$

where

$$
(2.6)\qquad
\begin{aligned}
&B_i^{(h)}(x) \\
&= \frac{A_h(x,x^{-i})}{(1-x^{-1})(1-x^{-2})\cdots(1-x^{l-i})(1-x)(1-x^2)\cdots(1-x^{m-i})}, \\
&B_l^{(h)}(x) = \frac{A_h(x,x^{-l})}{(1-x)(1-x^2)\cdots(1-x^{m-l})}.
\end{aligned}
$$

The coefficient of z^{k-h} in (2.5) is given by $F(x,l,m,h,k) = \sum_{i=l}^{m} B_i^{(h)}(x)x^{i(k-h)}$ and $g(n,l,m,h,k)$ is obtained as the coefficient of x^n in $F(x,l,m,h,k)$.

These results are easily extended to the study of $f(n,m,h,k)$ which enumerates the number of partitions of n into summands greater than or equal to m such that the total number of summands is equal to k and the number of distinct summands is h. In this case $f(n,m,h,k)$ is the coefficient of $x^n z^k t^h$ in $\prod_{i=m}^{\infty}(1+z(t-1)x^i)/(1-zx^i)$. Using the identity

$$
(2.7)\qquad \prod_{i=m}^{\infty}(1-zx^i)^{-1} = \sum_{r=0}^{\infty}\frac{z^r x^{rm}}{(1-x)(1-x^2)\cdots(1-x^r)}
$$

and

$$\prod_{i=1}^{r} (1 - x^i)^{-1} = \sum_{r=0}^{\infty} q_r(n) x^n \tag{2.8}$$

reduces the problem to that of finding the coefficient $z^k x^n$ in

$$\left\{ \sum_{s=0}^{\infty} (-1)^s \binom{h+s}{s} \frac{z^{s+h} x^{(h+s)m+(h+s)(h+s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^{h+s})} \right\} \times \left\{ \sum_{r=0}^{\infty} \frac{z^r x^{rm}}{(1-x)(1-x^2)\cdots(1-x^r)} \right\}$$

i.e., in

$$\left[\sum_{s=0}^{\infty} (-1)^s \binom{h+s}{s} z^{s+h} \sum_{i=(h+s)m+\binom{h+s}{2}} x^u \, q_{h+s}^{\left(u-\left\{(h+s)m+\binom{h+s}{2}\right\}\right)} \right] \cdot \left[\sum_{r=0}^{\infty} z^r \sum_{t=rm}^{\infty} q_r^{(t-rm)} x^t \right]. \tag{2.9}$$

Thus

$$f(n, m, h, k) = \sum_{u+t=n} \sum_{r+s=k-h} (-1)^s \times \binom{h+s}{s} q_{h+s}^{\left(u-\left\{(h+s)m+\binom{h+s}{2}\right\}\right)} q_r^{(t-rm)} \tag{2.10}$$

summations being over all nonnegative integers u, t, r, s under the above conditions.

If another extra restriction is imposed that the parts should be odd, the corresponding generating functions are

$$\prod_{i=m}^{\infty} \left(1 + \frac{x^{2i+1}z}{1-x^{2i+1}z} t \right) = \sum f'(n, m, h, k) x^n z^k t^h, \tag{2.11}$$

$$\prod_{i=0}^{m} \left(1 + \frac{x^{2i+1}z}{1-x^{2i+1}z} t \right) = \sum g'(n, m, h, k) x^n z^k t^h. \tag{2.12}$$

Again one can use the following identities

$$\prod_{m=0}^{i-1} (1 + ax^{2m+1}) = 1 + a\frac{(1-x^{2i})x}{1-x^2} + a^2 \frac{(1-x^{2i-2})(1-x^{2i})}{(1-x^2)(1-x^4)} x^4 + \cdots + a^m \frac{(1-x^{2i-2m+2})(1-x^{2i-2m+4})\cdots(1-x^{2i})}{(1-x^2)(1-x^4)\cdots(1-x^{2m})} x^{m^2} + \cdots + a^i x^{i^2}, \tag{2.13}$$

$$
\begin{aligned}
(2.14) \qquad & (1 + ax^{2m+1})(1 + ax^{2m+3})(1 + ax^{2m+5}) \cdots \\
&= 1 + \frac{ax^{2m+1}}{1 - x^2} + \frac{a^2x^{4+4m}}{(1 - x^2)(1 - x^4)} \\
&\quad + \cdots + \frac{a^r x^{r^2+2rm}}{(1 - x^2)(1 - x^4) \cdots (1 - x^{2r})} + \cdots.
\end{aligned}
$$

The details are similar.

3. **Concluding remarks.** Let $g^*(n, l, m, h, k)$ denote the number of those g-type partitions of n in which the greatest part used is m and let $G(n, l, m, h, k)$ be the number of those in which the number of summands is at most k ($\geqq h$). Thus

$$g^*(n, l, m, h, k) = g(n, l, m, h, k) - g(n, l, m - 1, h, k),$$

$$G(n, l, m, h, k) = \sum_{j=1}^{k} g(n, l, m, h, j).$$

It also follows from their generating functions that $g(n, l, m, h, k) = g(n - k(l - 1), m, h, k)$. Thus the properties of $g(n, l, m, h, k)$ reduce to those of $g(n, m, h, k)$ studied by Gupta in [2] and it is easy to show that $F(x, l, m, h, k)$ is a polynomial in x of degree exactly $mk + h(2l - h - 1)/2$. The reduction formulas derived by Gupta can also be extended.

References

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2. H. Gupta, *Highly restricted partitions*, J. Res. Nat. Bur. Standards Sect. B **73B** (1969), 329-350. MR **40** #4135.

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