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PRIME GRAPHS, MATCHINGS AND THE CASTELNUOVO-MUMFORD REGULARITY

TÜRKER BIYIKOĞLU AND YUSUF CIVAN

ABSTRACT. We demonstrate the effectiveness of prime graphs for the calculation of the (Castelnuovo-Mumford) regularity of graphs. Such a notion allows us to reformulate the regularity as a generalized induced matching problem and perform regularity calculations in specific graph classes, including (C_3, P_5)-free graphs, P_6-free bipartite graphs and all Cohen-Macaulay graphs of girth at least five. In particular, we verify that the five cycle graph C_5 is the unique connected graph satisfying the inequality $\mathrm{im}(G) < \mathrm{reg}(G) = \mathrm{m}(G)$. In addition, we prove that, for each integer $n \geq 1$, there exists a vertex decomposable perfect prime graph G_n with $\mathrm{reg}(G_n) = n$.

1. Introduction. The Castelnuovo-Mumford regularity (or, merely, the regularity) is something of a two-way study in the sense that it is a fundamental invariant both in commutative algebra [**5**] and discrete geometry [**9**]. The regularity is a type of universal bound for measuring the complexity of an object (a module, a sheaf or a simplicial complex). We recall that, when $G = (V, E)$ is a (simple) graph, its *edge ideal* I_G is defined to be the ideal in the polynomial ring $R = \Bbbk[V]$ with a finite set $V = \{x_1, \ldots, x_n\}$ of indeterminates over a field $\Bbbk$, generated by the quadratic monomials $x_i x_j$ corresponding to edges of G. Most of the recent work in the area has been devoted to the existence of applicable bounds on the regularity $\mathrm{reg}(G) := \mathrm{reg}(R/I_G)$ of the edge ring of a given graph [**2, 3, 5, 13, 17**]. One way of attacking such a problem goes by translating the underlying algebraic or topological language

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to the language of graphs. Such an approach may enable us to bound the regularity of a graph via other graph invariants. The most likely candidates involve the matching parameters of graphs.

In **[3]**, we introduced the notion of a *prime graph* that brings a new strategy for the calculation of the regularity. We show here that such a notion allows us to reformulate the regularity of any graph as a generalized induced matching problem and perform the regularity calculations in specific graph classes including (C_3, P_5)-free graphs (Theorem 3.7), bipartite P_6-free graphs (Theorem 3.9) and well-covered block cactus graphs (Theorem 4.6) that, in turn, contain all Cohen-Macaulay graphs of girth at least five.

We prove that a 3-path (an *ear*) *addition* to any end vertices of an edge of a prime graph gives rise to a new prime graph under which the regularity increases exactly by one. By way of application, we prove that, for each integer $n \geq 1$, there exists a vertex decomposable perfect prime graph G_n with $\mathrm{reg}(G_n) = n$ that, in a sense, reveals the difficulty behind the calculation of the regularity of vertex decomposable graphs. Moreover, the existence of such graphs allows us to construct a vertex decomposable prime graph H_s for each $s \geq 1$ satisfying $\mathrm{reg}(H_s) - \mathrm{im}(H_s) = s$.

It is already known **[10, 6]** that the inequality $\mathrm{im}(G) \leq \mathrm{reg}(G) \leq \mathrm{m}(G)$ holds for any graph G. The existence of graphs realizing the invariants in the above inequality by possible integers was the subject of a recent paper by Hibi et al. **[7]**. Apart from the exceptional case $\mathrm{im}(G) < \mathrm{reg}(G) = \mathrm{m}(G)$, they showed that there exist infinite families of connected graphs for which any inequality derived from the above inequality holds (see **[7,** Theorem 1.9]). In particular, they observed that the only graph up to seven vertices satisfying the inequality $\mathrm{im}(G) < \mathrm{reg}(G) = \mathrm{m}(G)$ is the five-cycle graph C_5, and noted that such graphs might be rare. Indeed, we confirm their observation by showing that the graph C_5 with these properties is unique.

2. Preliminaries. By a (simple) graph G, we will mean a finite undirected graph without loops or multiple edges. If G is a graph, $V(G)$ and $E(G)$ (or simply V and E) denote its vertex and edge sets. If $U \subset V$, the graph induced on U is written $G[U]$, which is the graph on the set U of vertices together with any edges whose end vertices are both in U, and in particular, we abbreviate $G[V \setminus U]$ to $G - U$, and write $G - x$ whenever $U = \{x\}$. For a given subset $U \subseteq V$, the (open) neighborhood of U is defined by $N_G(U) := \cup_{u \in U} N_G(u)$, where $N_G(u) := \{v \in V \colon uv \in E\}$, and similarly, $N_G[U] := N_G(U) \cup U$ is the closed neighborhood of U. In particular, the degree $\deg_G(x)$ of a vertex x in G is the cardinality of $N_G(x)$. A subgraph H of a graph G is said to be a *dominating subgraph* if $N_G[V(H)] = V(G)$. The *distance* $d_G(x, y)$ between vertices x and y is the smallest number of edges in a path joining x and y.

Throughout, K_n, $K_{l,k}$, P_n and C_m will denote the complete, complete bipartite, path and cycle graphs for any $n, l, k \geq 1$ and $m \geq 3$, respectively. For an integer $n \geq 2$ and a graph G, we denote by nG the disjoint union of n copies of G.

For any family of graphs $\mathcal{H}$, we say that a graph G is *$\mathcal{H}$-free*, if G contains no induced subgraph isomorphic to any graph $H \in \mathcal{H}$. A graph G is called *chordal* if it is C_r-free for any $r > 3$, and a graph G is said to be *cochordal* if its complement $\overline{G}$ is a chordal graph. Moreover, G is said to be a *weakly chordal graph* if G and its complement $\overline{G}$ are C_k-free for any $k \geq 5$. The *girth* of a graph G is the length of a shortest induced cycle in G, and if G is cycle-free, its girth is defined to be ∞.

Recall that a subset $M \subseteq E$ is called a *matching* of G if no two edges in M share a common end, and a maximum matching is a matching that contains the largest possible number of edges. The *matching number* $\mathrm{m}(G)$ of G is the cardinality of a maximum matching. Moreover, a matching M of G is an *induced matching* if it occurs as an induced subgraph of G, and the cardinality of a maximum induced matching is called the *induced matching number* of G and denoted by $\mathrm{im}(G)$.

A graph G is said to be *well-covered* if all maximal independent sets in G are of the same size, and G is a Cohen-Macaulay graph if so is its edge ring R/I_G.

Remark 2.1. In order to simplify the notation, we note that when we mention the homology, homotopy or a suspension of a graph, we mean that of its independence complex, so whenever it is appropriate, we drop $\mathrm{Ind}(-)$ from our notation.

3. Prime graphs. In this section, we first recall the notion of prime graphs and prime factorization **[3]** and then perform the regularity calculations in some hereditary graph classes.

A connected graph G is called a *prime graph* over a field $\Bbbk$, if $\mathrm{reg}_{\Bbbk}(G-x) < \mathrm{reg}_{\Bbbk}(G)$ for any vertex $x \in V(G)$. Furthermore, we call a connected graph G a *perfect prime graph* if it is a prime graph over any field.

The graph K_2, the cycles C_{3k+2} and the complement of cycles $\overline{C}_m$ for any $k \geq 1$ and $m \geq 4$ are examples of perfect prime graphs.

The *null graph* $N = (\emptyset, \emptyset)$ is the degenerate case, where its independence complex satisfies $\mathrm{Ind}(N) = \{\emptyset\}$, in which we count it as the (trivial) perfect prime. This is consistent with the usual conventions that $\widetilde{H}_{-1}(\{\emptyset\}; \Bbbk) \cong \Bbbk$ and $\widetilde{H}_p(\{\emptyset\}; \Bbbk) \cong 0$ for any $p \neq -1$ in that case, where $\widetilde{H}_*(-; \Bbbk)$ denotes the (reduced) singular homology.

The following provides an inductive bound on the regularity of graphs.

Lemma 3.1. **[4, 13]** *Let G be a graph and let $v \in V$ be given. Then*

$$\mathrm{reg}(G) \leq \max\{\mathrm{reg}(G-v),\ \mathrm{reg}(G - N_G[v]) + 1\}.$$

Moreover, $\mathrm{reg}(G)$ *always equals to one of* $\mathrm{reg}(G-v)$ *or* $\mathrm{reg}(G - N_G[v]) + 1$.

Note that, if G is a prime graph, then $\mathrm{reg}(G) = \mathrm{reg}(G - N_G[x]) + 1$ holds for any vertex $x \in V$ as a consequence of Lemma 3.1.

We prove in **[3]** that prime graphs cannot contain any pair of vertices whose open or closed neighborhoods are comparable with respect to the inclusion.

Proposition 3.2 (**[3]**). *If $N_G(y) \subseteq N_G(x)$ for vertices x and y, then G cannot be a prime graph. Similarly, if $N_G[u] \subseteq N_G[v]$ holds in G such that $\deg_G(v) \geq 2$, then G cannot be a prime graph.*

Let G be a graph, and let $\mathcal{R} = \{R_1, \ldots, R_r\}$ be a set of pairwise vertex disjoint induced subgraphs of G such that $|V(R_i)| \geq 2$ for each

$1 \le i \le r$. Then, $\mathcal{R}$ is said to be an *induced decomposition* of G if the induced subgraph of G on $\bigcup_{i=1}^{r} V(R_i)$ contains no edge of G that is not contained in any of $E(R_i)$, and $\mathcal{R}$ is maximal with this property. The set of induced decompositions of a graph G is denoted by $\mathcal{ID}(G)$.

Let $\mathcal{R} = \{R_1, \dots, R_r\}$ be an induced decomposition of a graph G. If each R_i is a prime graph, then we call $\mathcal{R}$ a *prime decomposition* of G, and the set of prime decompositions of a graph G is denoted by $\mathcal{PD}(G)$. Obviously, the set $\mathcal{PD}(G)$ is non-empty for any graph G.

Theorem 3.3 (**[3]**). *For any graph G and any field $\Bbbk$, we have*

$$\operatorname{reg}_{\Bbbk}(G) = \max\left\{ \sum_{i=1}^{r} \operatorname{reg}_{\Bbbk}(H_i) \colon \{H_1, \dots, H_r\} \in \mathcal{PD}_{\Bbbk}(G) \right\}.$$

Definition 3.4. A prime decomposition $\mathcal{R}$ of a graph G for which the equality of Theorem 3.3 holds is called a *prime factorization* of G, and the set of prime factorizations of G is denoted by $\mathcal{PF}(G)$.

We may restate Theorem 3.3, which shows that the regularity calculation of graphs exactly corresponds to a generalized induced matching problem.

Definition 3.5. Let G be a graph, $\mathcal{T} = \{T_1, \dots, T_k\}$ a set of connected graphs and $\mathfrak{a} = (a_1, \dots, a_k)$ a sequence of non-negative integers. We then call the integer

$$\operatorname{im}(G; \mathcal{T}; \mathfrak{a}) := \max\{a_1 n_1 + \cdots + a_k n_k \colon \{n_1 T_1, \dots, n_k T_k\} \in \mathcal{ID}(G)\}$$

the *induced matching number* of G with respect to the pair $(\mathcal{T}, \mathfrak{a})$. We make the convention that $\operatorname{im}(G; \mathcal{T}; \mathfrak{a}) := 0$ if no sequence exists of non-negative integers $(n_1, \dots, n_k)$ such that $\{n_1 T_1, \dots, n_k T_k\} \in \mathcal{ID}(G)$.

Whenever it is convenient, we drop the sequence $\mathfrak{a}$ from our notation and simply write $\operatorname{im}(G; \mathcal{T})$ instead of $\operatorname{im}(G; \mathcal{T}; \mathfrak{a})$. In particular, we remark that the integer $\operatorname{im}(G; K_2) := \operatorname{im}(G; K_2; \mathbf{1})$, where $\mathbf{1}$ is the sequence consisting of 1's is exactly the induced matching number of G, that is, $\operatorname{im}(G; K_2) = \operatorname{im}(G)$.

Corollary 3.6. *For any graph G, we have* $\operatorname{reg}(G) \geq \operatorname{im}(G; \mathcal{R}; \mathfrak{a}_{\mathcal{R}})$ *for each prime decomposition* $\mathcal{R} = \{H_1, \dots, H_k\} \in \mathcal{PD}(G)$*, where* $\mathfrak{a}_{\mathcal{R}} = (\operatorname{reg}(H_i) \colon i \in [k])$*. In particular, the equality* $\operatorname{reg}(G) = \operatorname{im}(G; \mathcal{R}; \mathfrak{a}_{\mathcal{R}})$ *holds if* $\mathcal{R} \in \mathcal{PF}(G)$*.*

Corollary 3.6 implies that, once we know the family of induced prime subgraphs, say $\mathcal{P}_G$, of a given graph G (over a fixed field $\Bbbk$), the calculation of the regularity $\operatorname{reg}(G)$ turns into a generalized induced $\mathcal{P}_G$-matching problem that can also be considered as a maximum weighted induced $\mathcal{P}_G$-matching problem in which the weight of any subgraph H in $\mathcal{P}_G$ equals its regularity $\operatorname{reg}(H)$. Therefore, Corollary 3.6 is more useful when we know the set of induced prime subgraphs of a given graph.

Theorem 3.7. *If G is a (C_3, P_5)-free prime graph, then G is isomorphic to either K_2 or C_5. In particular, if G is a (C_3, P_5)-free graph, we then have* $\operatorname{reg}(G) = \operatorname{im}(G; K_2, C_5) \leq 2\operatorname{im}(G)$*.*

Proof. Suppose that G is a prime and (C_3, P_5)-free graph. If G is C_5-free, then it is a weakly chordal graph; hence, we have $\operatorname{reg}(G) = \operatorname{im}(G)$ **[17]**. However, since G is connected and (C_3, P_5)-free, we must have $\operatorname{im}(G) = 1$. Indeed, assume otherwise that $\operatorname{im}(G) > 1$, and let M be an induced matching of size $\operatorname{im}(G)$. If $xy,\ uv \in M$ are two edges, since G is connected, there exists a path $P := \{x = x_0, x_1, \dots, x_{l-1}, x_l = u\}$ of minimum length $l \geq 2$ between the vertices x and u in G.

Note that the case $l \geq 3$ is not possible, since G is P_5-free. Therefore, we only need consider the case $l = 2$. Thus, if $P = \{x, z, u\}$, the edges yz and vz cannot be present in G, since G is triangle-free. However, in such a case, the set $\{y, x, z, u, v\}$ induces a P_5 in G, a contradiction. It then follows that $\operatorname{im}(G) = 1$, which, in turn, implies that $G \cong K_2$, since G is prime. Hence, we may assume that G contains at least one induced five cycle C, say on the vertices $x_1, \dots, x_5$, such that $x_i x_{i+1} \in E(G)$ in a cyclic fashion.

Suppose first that any vertex of G not contained in $V(C)$ has exactly two neighbors in C, and let y be such a vertex. We may assume, without loss of generality, that neighbors of y in C are x_1

and x_3. Then, by Proposition 3.2, there exist vertices $u \in N_G(y)$ and $v \in N_G(x_2)$ such that ux_2, $yv \notin E(G)$. Note that we must have $uv \in E(G)$, since otherwise, the set $\{u, x_2, x_1, y, v\}$ induces a P_5. To prevent the existence of induced 5-paths in G, the vertices u and v must have at least one neighbor in C. However, since G is triangle-free, the only possible neighbors would be x_4 and x_5. If $ux_4 \in E(G)$, then the set $\{u, x_4, x_5, x_1, x_2\}$ induces a P_5, while if $ux_5 \in E(G)$, then the set $\{u, x_5, x_4, x_3, x_2\}$ induces a P_5 in G, any of which is impossible.

Assume now that any vertex in $V(G) \setminus V(C)$ has exactly one neighbor in C. If y is such vertex and its neighbor in C is x_1, then it follows from Proposition 3.2 that y has a neighbor z outside of C. However, in such a case, either the set $\{z, y, x_1, x_5, x_4\}$ or the set $\{z, y, x_1, x_2, x_3\}$ induces a P_5 in G, since z can have at most one neighbor in C by our assumption. Therefore, any such graph must be isomorphic to a C_5.

Finally, the inequality $\operatorname{reg}(G) \leq 2\operatorname{im}(G)$ follows from Corollary 3.6 when G is a (C_3, P_5)-free graph, since any induced copy of a C_5, which has regularity two, can contribute one edge to an induced matching. $\square$

The following is a direct consequence of Theorem 3.7:

Corollary 3.8. *If G is a $(2K_2, C_3)$-free graph, then* $\operatorname{reg}(G) \leq 2$.

Observe that, for any P_5-free bipartite graph B, we have $\operatorname{reg}(B) = \operatorname{im}(B)$ by Theorem 3.7. Note that, since bipartite P_5-free graphs are weakly chordal, the equality $\operatorname{reg}(B) = \operatorname{im}(B)$ also follows from a result of **[17]** for such graphs. However, we can extend it further:

Theorem 3.9. *If G is a bipartite P_6-free graph, then* $\operatorname{reg}(G) = \operatorname{im}(G)$.

Proof. Once again, it suffices to prove that the only induced prime in such a graph is isomorphic to a K_2. Thus, let H be a prime and P_6-free bipartite graph. Following the characterization of P_6-free graphs due to van't Hof and Paulusma **[15]**, such a graph must contain either a dominating complete bipartite subgraph or else an induced dominating

C_6. We accordingly divide the proof into two cases, while noting that the methods of the proof in both cases are almost identical.

Suppose that $K := K_{m,n}$ is a dominating complete bipartite subgraph of H, and assume for contradiction that $H \not\cong K_2$. So, let $V(H) = U \cup V$ and $V(K) = U' \cup V'$ such that $U' \subseteq U$ and $V' \subseteq V$. Observe that the equality $m = n = 1$ is not possible, since otherwise H would contain two vertices x and y with $N_H(x) \subseteq N_H(y)$ that contradicts the fact that H is prime by Proposition 3.2. We may therefore suppose that $n \geq 2$. Furthermore, the graph H cannot contain any dominated vertex, which is again due to Proposition 3.2.

Claim 1. H contains an induced C_6.

Proof of Claim 1. Let $x_1,\ x_2 \in U'$ be given. Since they cannot dominate each other, there exist $a_{12},\ a_{21} \in V \setminus V'$ such that $a_{12} \in N_H(x_1) \setminus N_H(x_2)$ and $a_{21} \in N_H(x_2) \setminus N_H(x_1)$. Choose a vertex $y_1 \in V'$, and, since a_{12} is not dominated, it has a neighbor, say $b_{12} \in U \setminus U'$, such that $b_{12}y_1 \notin E(H)$. However, since H is P_6-free, the edge $b_{12}a_{21}$ must be present in H; hence, the set $\{x_1, y_1, x_2, a_{21}, b_{12}, a_{12}\}$ induces the desired C_6.

Claim 2. For any vertex $x \in U'$, the graph $T = H - N_H[x]$ is $2K_2$-free.

Proof of Claim 2. Assume otherwise that M is an induced matching in T having order at least two. Observe that, if $b \in V \setminus V'$ is an end vertex of an edge in M, since it cannot be dominated by any vertex $y \in V'$, it has a neighbor $c_{(b,y)} \in U \setminus U'$ such that $yc_{(b,y)} \notin E(H)$.

Case 2.1. Suppose that $V(M) \cap U' = \emptyset$ so that M contains edges a_1b_1 and a_2b_2 with $a_1, a_2 \in U \setminus U'$ and $b_1, b_2 \in V \setminus V'$. Since K is dominating, the vertices a_1 and a_2 (respectively, b_1 and b_2) have at least one neighbor in V' (respectively, in U').

Subcase 2.1 (i). Assume that $a_1,\ a_2 \in N_H(y_1)$ and $b_1,\ b_2 \in N_H(x_1)$ for some $y_1 \in V'$ and $x_1 \in U' \setminus \{x\}$. In such a case, we must have $c_{(b_1,y_1)}b_2 \notin E(H)$, since otherwise, the set $\{x, y_1, a_1, b_1, c_{(b_1,y_1)}, b_2\}$ induces a P_6. However, then the set $\{c_{(b_1,y_1)}, b_1, a_1, y_1, a_2, b_2\}$ induces a P_6 in H, a contradiction.

Subcase 2.1 (ii). Assume that $a_1,\ a_2 \in N_H(y_1)$, while there exist distinct vertices $x_1,\ x_2 \in U'$ such that $b_1 \in N_H(x_1) \setminus N_H(x_2)$

and $b_2 \in N_H(x_2) \setminus N_H(x_1)$. If $c_{(b_1,y_1)}b_2 \notin E(H)$, then the set $\{c_{(b_1,y_1)}, b_1, a_1, y_1, x_2, b_2\}$ induces a P_6, and if $c_{(b_1,y_1)}b_2 \in E(H)$, then the set $\{x, y_1, x_2, b_1, c_{(b_1,y_1)}, b_2\}$ induces a P_6 in H, both of which is impossible.

Subcase 2.1 (iii). Assume that $b_1,\ b_2 \in N_H(x_1)$, while there exist distinct vertices $y_1,\ y_2 \in V'$ such that $a_1 \in N_H(y_1) \setminus N_H(y_2)$ and $a_2 \in N_H(y_2) \setminus N_H(y_1)$. If $c_{(b_1,y_1)}y_2 \notin E(H)$, the set $\{c_{(b_1,y_1)}, b_1, a_1, y_1, x, y_2\}$ induces a P_6; hence, $c_{(b_1,y_1)}y_2 \in E(H)$. On the other hand, if $c_{(b_1,y_1)}b_2 \notin E(H)$, then the set $\{b_2, a_2, y_2, c_{(b_1,y_1)}, b_1, a_1\}$ induces a P_6, and if $c_{(b_1,y_1)}b_2 \in E(H)$, then the set $\{x, y_1, a_1, b_1, c_{(b_1,y_1)}, b_2\}$ induces a P_6 in H.

Subcase 2.1 (iv). Assume that there exist distinct vertices $x_1,\ x_2 \in U'$ and $y_1,\ y_2 \in V'$ such that $b_1 \in N_H(x_1) \setminus N_H(x_2)$, $b_2 \in N_H(x_2) \setminus N_H(x_1)$ and $a_1 \in N_H(y_1) \setminus N_H(y_2)$ and $a_2 \in N_H(y_2) \setminus N_H(y_1)$. If $c_{(b_1,y_1)}b_2 \notin E(H)$, then the set $\{c_{(b_1,y_1)}, b_1, a_1, y_1, x_2, b_2\}$ induces a P_6, and if $c_{(b_1,y_1)}b_2 \in E(H)$, then the set $\{x, y_1, x_1, b_1, c_{(b_1,y_1)}, b_2\}$ induces a P_6 in H, both of which are impossible.

Case 2.2. Suppose that $|V(M) \cap U'| = 1$. We may, therefore, assume that M contains edges of the form x_1b_1 and a_2b_2, where $x_1 \in U' \setminus \{x\}$, $a_2 \in U \setminus U'$ and $b_1, b_2 \in V \setminus V'$. Choose a vertex $y_1 \in V'$. If $c_{(b_1,y_1)}b_2 \in E(H)$, then the set $\{x, y_1, x_1, b_1, c_{(b_1,y_1)}, b_2\}$ induces a P_6 in H so that we must have $c_{(b_1,y_1)}b_2 \notin E(H)$. However, it then follows that the edges $b_1c_{(b_1,y_1)}$ and a_2b_2 form an induced matching that shares no vertex with U', which is not possible by Case 2.1.

Case 2.3. Suppose that $|V(M) \cap U'| = 2$, and let M contain the edges x_1b_1 and x_2b_2 such that $x_1, x_2 \in U' \setminus \{x\}$ and $b_1,\ b_2 \in V \setminus V'$. Once again, choose a vertex $y_1 \in V'$. If $c_{(b_1,y_1)}b_2 \notin E(H)$, then the set $\{c_{(b_1,y_1)}, b_1, x_1, y_1, x_2, b_2\}$ induces a P_6, while, if $c_{(b_1,y_1)}b_2 \in E(H)$, then the set $\{x, y_1, x_2, b_2, c_{(b_1,y_1)}, b_1\}$ induces a P_6 in H, both of which are not possible.

This completes the proof of Claim 2. Since the graph T is $2K_2$-free and bipartite, it follows that T is a cochordal graph so that $\operatorname{reg}(T) = 1$, which in turn implies that $\operatorname{reg}(H) = 2$, since H is prime.

Now, since H contains an induced C_6 by Claim 1, then either H contains a vertex x such that the graph $H - x$ contains an induced C_6, or else $H \cong C_6$. However, in either case, H cannot be a prime graph since $\operatorname{reg}(H) = \operatorname{reg}(C_6) = 2$, and C_6 is itself not a prime graph.

Assume now that H has a dominating induced 6-cycle C, say on the vertices $x_1, \ldots, x_6$ such that $x_i x_{i+1} \in E(H)$ in the cyclic fashion. Since a 6-cycle itself is not a prime graph, the set $V(H) \setminus V(C)$ is not empty.

Claim 3. Any vertex $x \in V(H) \setminus V(C)$ has at least two neighbors in C, that is, $|N_C(x)| \geq 2$.

Proof of Claim 3. Since $V(C)$ is dominating in H, any such vertex has at least one neighbor in $V(C)$, and, if it has a unique neighbor, then H contains an induced P_6 which is not possible.

Claim 4. For any vertex $x_i \in V(C)$, the graph $H - N_H[x_i]$ is $2K_2$-free.

Proof of Claim 4. Consider the vertex x_5, and suppose that the graph $L = H - N_H[x_5]$ has an induced matching M of cardinality 2.

Subclaim 4.1. $M \cap \{x_1x_2, x_2x_3\} = \emptyset$.

Proof of Subclaim 4.1. Assume, without loss of generality, that $M = \{x_1x_2, xy\}$ for some $x, y \in V(H) \setminus V(C)$. Note that $x, y \notin N_H[x_5]$. Now, if $xx_3 \in E(H)$, then the vertex x_6 must be adjacent to x by Claim 3, together with the fact that H is bipartite. However, we then necessarily have $|N_C(y) \cap V(C)| \leq 1$, which contradicts Claim 3. Furthermore, if neither of the vertices x nor y is not adjacent to x_3, then one of these vertices has no neighbors in $V(C)$, which is not possible, again by Claim 3.

Subclaim 4.2. $V(M) \cap \{x_1, x_2, x_3\} = \emptyset$.

Proof of Subclaim 4.2. Assume, without loss of generality, that $M = \{x_j u, xy\}$ for some $u, x, y \in V(H) \setminus V(C)$ and $j \in [3]$. By symmetry, it suffices to consider the cases only when $j \in \{1, 3\}$ or $j = 2$.

Case 4.2 (i). $j = 1$. In this case, we note that one of x or y is adjacent to either x_2 or x_3. Thus, we let $xx_2 \in E(H)$. It follows that $x_4,\ x_6 \in N_C(y)$. However, this forces $|N_C(x) \cap V(C)| \leq 1$, a contradiction.

Case 4.2 (ii). $j = 2$. It is sufficient to consider the case where $x_1,\ x_3 \in N_C(y)$ and $x_4, x_6 \in N_C(x)$. On the other hand, the vertex u must be adjacent to at least one of the vertices x_4 or x_6. If $ux_4 \in E(H)$, then the set $\{y, x_1, x_2, u, x_4, x_5\}$ induces a P_6 in H, while, if $ux_6 \in E(H)$, then the set $\{x_5, x_6, u, x_2, x_3, y\}$ induces a P_6 in H, both of which are impossible.

We may, therefore, assume that $M = \{xy, ab\}$ for some $x, y, a, b \in V(H) \setminus V(C)$. Again, by Claim 3, we note that at least one of the end vertices of the edges xy and ab has exactly two neighbors in $\{x_1, x_2, x_3\}$. Hence, assume that $x_1,\ x_3 \in N_C(x) \cap N_C(a)$. It then follows that each of the vertices y and b has at least two neighbors in $\{x_2, x_4, x_6\}$. If $x_6 \in N_C(y) \cap N_C(b)$, while $x_4 \notin N_C(y) \cup N_C(b)$, then the set $\{b, x_6, y, x, x_3, x_4\}$ induces a P_6 in H. Thus, we must have that at least one of yx_4 or bx_4 has an edge in H. However, if $yx_4 \in E(H)$, then the set $\{a, x_1, x, y, x_4, x_5\}$, and if $bx_4 \in E(H)$, then the set $\{x_5, x_4, b, a, x_1, x\}$ induces a P_6 in H, any of which is impossible. By symmetry, the case $x_4 \in N_C(y) \cap N_C(b)$, while $x_6 \notin N_C(y) \cup N_C(b)$ can be similarly treated. This completes the proof of Claim 4.

Now, as in the first case, since the graph L is $2K_2$-free and bipartite, it follows that L is a cochordal graph so that $\mathrm{reg}(L) = 1$, which, in turn, implies that $\mathrm{reg}(H) = 2$ since H is prime. However, such a graph cannot be prime, since it contains an induced C_6. □

4. Regularity of Cohen-Macaulay graphs of girth at least five. Our aim in this section is to prove the equality $\mathrm{reg}(G) = \mathrm{im}(G; K_2, C_5)$ when G is a well-covered block-cactus graph **[14]** that, in turn, includes any Cohen-Macaulay graph with girth at least five (see also **[2, 8]**). We recall that a vertex v is a *cut vertex* of a connected graph G if $G - v$ is disconnected. Furthermore, a *block* of a graph G is a maximal connected subgraph of G without a cut-vertex, and a graph G is called a *block-cactus graph*, if each of its blocks is a clique or a cycle. For that purpose, we first introduce a graph class containing all such graphs.

Definition 4.1. Let G be a graph, and let C be an induced cycle of length $4 \leq n \leq 7$ in G. Then, C is called a *basic cycle* of G if one of the following holds:

(i) $n = 4$ and contains two adjacent vertices of degree two in G;
(ii) $n = 5$ and contains no two adjacent vertices of degree three or more in G;
(iii) $n = 6$ or 7 and, if $x, y \in V(C)$ are two vertices such that $\deg_G(x),\ \deg_G(y) \geq 3$, then $d_G(x, y) \geq 3$.

We say that G is in the class $\mathscr{BW}$, if its vertex set can be partitioned into $V(G) = B \cup W$ such that B consists of vertices of basic cycles of G and basic cycles form a partition of B, and W induces a weakly chordal graph in G.

We next recall two operations on graphs from **[3]** under which the regularity remains stable.

Definition 4.2. Let $x,\ y \in V(G)$ be two non-adjacent vertices of a graph G. Then, $\{x, y\}$ is called a $\mathfrak{t}$-*pair* of G, if the following hold:

(i) there exist no vertices $u,\ v \in V \setminus \{x, y\}$ such that $G[\{x, y, u, v\}] \cong 2K_2$;
(ii) there exists a vertex $w \in N_G(x) \cap N_G(y)$ satisfying $N_G[w] \subseteq N_G[x] \cup N_G[y]$.

When $\{x, y\}$ is a $\mathfrak{t}$-pair with respect to the vertex w, then w is called the $\mathfrak{t}$-*neighbor* of the pair $\{x, y\}$, and the graph $\mathfrak{t}(G; xy)$ constructed by

$$V(\mathfrak{t}(G; xy)) := (V \setminus \{x, y\}) \cup \{w_{xy}\}$$

and

$$E(\mathfrak{t}(G; xy)) := E(G - \{x, y\}) \cup \{uw_{xy} \colon u \in N_G(x) \cap N_G(y)\}$$

is called the $\mathfrak{t}$-*contraction* of G with respect to the pair $\{x, y\}$.

Definition 4.3. Let z be a non-isolated vertex of a graph G. For any two subsets $A_z, B_z \subseteq V(G - N_G[z])$, we say that $[A_z, B_z]$ is a $\mathfrak{t}$-*pairing* of z in G, if the following hold:

(i) $ab \in E(G)$ for any $a \in A_z$ and $b \in B_z$;
(ii) there exists a vertex $w \in N_G(z)$ satisfying $N_G[w] \subseteq N_G[z] \cup A_z \cup B_z$.

When $[A_z, B_z]$ is a $\mathfrak{t}$-pairing of z, then the graph $\mathfrak{t}(G; z, A_z, B_z)$ (or $\mathfrak{t}(G; z)$, for short) constructed by

$$V(\mathfrak{t}(G; z, A_z, B_z)) := (V \setminus \{z\}) \cup \{x_z, y_z\}$$

and

$$\begin{aligned} E(\mathfrak{t}(G; z, A_z, B_z)) := {} & E(G - z) \cup \{ux_z, uy_z \colon u \in N_G(z)\} \\ & \cup \{ax_z \colon a \in A_z\} \cup \{by_z \colon b \in B_z\} \end{aligned}$$

is called the $\mathfrak{t}$-*expansion* of G with respect to the vertex z and the pairing $[A_z, B_z]$.

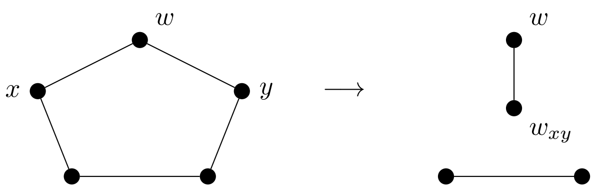

Figure 1. A $\mathfrak{t}$-contraction.

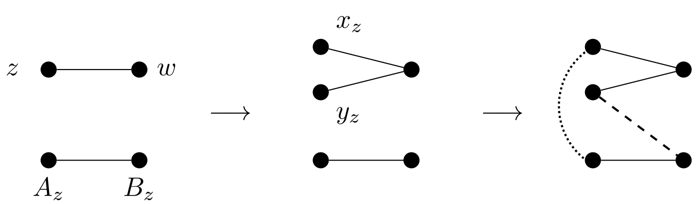

Figure 2. A $\mathfrak{t}$-expansion.

Theorem 4.4. **[3]** *If $\{x, y\}$ is a $\mathfrak{t}$-pair of a graph G, then* $\operatorname{reg}(G) = \operatorname{reg}(\mathfrak{t}(G; xy))$. *If z is a non-isolated vertex of G such that $[A_z, B_z]$ is a $\mathfrak{t}$-pairing in $G - N_G[z]$, then* $\operatorname{reg}(G) = \operatorname{reg}(\mathfrak{t}(G; z))$.

The proof of the main result of this section relies upon the affect of a particular edge contraction operation on graphs to the regularity that we next describe. We first recall that, if $e = xy$ is an edge of a graph G, then the *contraction* of e on G is the graph G/e, defined by

$$V(G/e) = (V(G) \setminus \{x, y\}) \cup \{w\}$$

and

$$E(G/e) = E(G - \{x, y\}) \cup \{wz \colon z \in N_G(x) \cup N_G(y)\}.$$

In particular, when u and v are two non-adjacent vertices of G, we define the *fake-contraction* (or $\mathfrak{f}$-*contraction*) $\mathfrak{f}(G; uv)$ of G with respect to u and v to be the graph $(G \cup uv)/uv$, where the graph $G \cup uv$ is obtained from G by the addition of the edge uv to G.

Proposition 4.5. *Let $\{a, b, c, d\}$ be a set of vertices of a 4-path (not necessarily induced) in G with edges ab, bc and cd such that* $\deg_G(b) = \deg_G(c) = 2$. *If $ad \in E(G)$, then* $\mathrm{reg}(G) = \mathrm{reg}(G - \{a, b, c, d\}) + 1$. *If $ad \notin E(G)$, then* $\mathrm{reg}(G) = \mathrm{reg}(\mathfrak{f}(G - \{b, c\}; da)) + 1$.

Proof. Suppose first that $ad \in E(G)$. We then apply a $\mathfrak{t}$-expansion on the vertex d with respect to the $\mathfrak{t}$-pairing $[\{b\}, \emptyset]$. Observe that, in the resulting graph $\mathfrak{t}(G; d)$, the pair $\{a, c\}$ is a $\mathfrak{t}$-pair with b as a $\mathfrak{t}$-neighbor. The $\mathfrak{t}$-contraction of $\{a, c\}$ provides the graph $\mathfrak{t}(\mathfrak{t}(G; d); ac)$ in which $\{b, y_d\}$ is a $\mathfrak{t}$-pair. When we $\mathfrak{t}$-contract $\{b, y_d\}$, the set $\{x_d, w_{ac}, w_{by_d}\}$ in the graph obtained induces a 3-path such that the vertex w_{by_d} is of degree one, that is, $\{x_d, w_{by_d}\}$ is a $\mathfrak{t}$-pair. Therefore, the $\mathfrak{t}$-contraction of $\{x_d, w_{by_d}\}$ in $\mathfrak{t}(\mathfrak{t}(\mathfrak{t}(G; d); ac); by_d)$ results in a graph isomorphic to $G - \{a, b, c, d\} \cup K_2$ (compare to Figure 3). Now, since

$$\begin{aligned}\mathrm{reg}(G - \{a, b, c, d\} \cup K_2) &= \mathrm{reg}(G - \{a, b, c, d\}) + \mathrm{reg}(K_2)\\ &= \mathrm{reg}(G - \{a, b, c, d\}) + 1,\end{aligned}$$

the claim follows from Theorem 4.4.

Assume next that $ad \notin E(G)$. We apply a $\mathfrak{t}$-expansion on the vertex d with respect to the $\mathfrak{t}$-pairing $[N_G(a) \setminus N_G(d),\ \{a\}]$ having the vertex c as a $\mathfrak{t}$-neighbor. If we denote by $\{x_d, y_d\}$ the resulting $\mathfrak{t}$-pair in $\mathfrak{t}(G; d)$, then $\{a, c\}$ is a $\mathfrak{t}$-pair in $\mathfrak{t}(G; d)$ with $\mathfrak{t}$-neighbor b. Once we

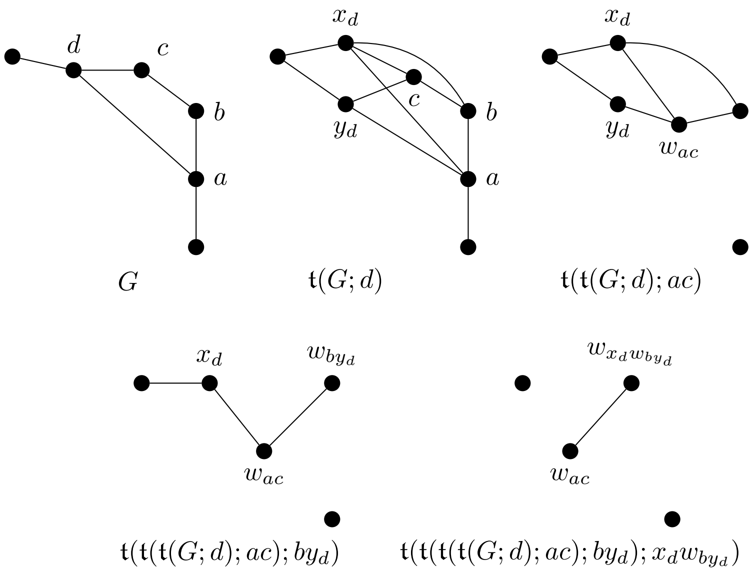

FIGURE 3. The first phase of expansions and contractions in Proposition 4.5.

contract this $\mathfrak{t}$-pair in $\mathfrak{t}(G;d)$ and denote the newly created vertex by w_{ac}, then $\{b, y_d\}$ becomes a $\mathfrak{t}$-pair in $\mathfrak{t}(\mathfrak{t}(G;d);ac)$ with a $\mathfrak{t}$-neighbor the vertex w_{ac}. Finally, the $\mathfrak{t}$-contraction of $\{b, y_d\}$ in $\mathfrak{t}(\mathfrak{t}(G;d);ac)$ yields a graph isomorphic to $\mathfrak{f}((G-\{b,c\});da)\cup K_2$, where the isolated edge is induced by w_{ac} and w_{by_d} (see Figure 4). Now, since

$$\begin{aligned}\operatorname{reg}(\mathfrak{f}((G-\{b,c\});da)\cup K_2) &= \operatorname{reg}(\mathfrak{f}((G-\{b,c\});da)) + \operatorname{reg}(K_2)\\ &= \operatorname{reg}(\mathfrak{f}((G-\{b,c\});da)) + 1,\end{aligned}$$

the claim follows from Theorem 4.4. $\square$

The following is the main result of this section:

Theorem 4.6. *If* $G \in \mathscr{BW}$, *then* $\operatorname{reg}(G) = \operatorname{im}(G; K_2, C_5) \leq 2\operatorname{im}(G)$.

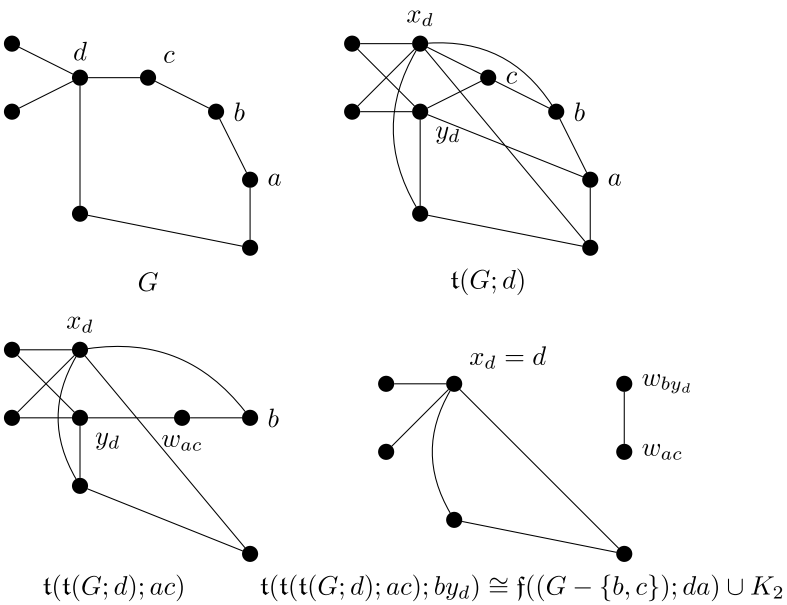

FIGURE 4. The second phase of expansions and contractions in Proposition 4.5.

Proof. We proceed by induction on the order of G. We first note that, if $B = \emptyset$, then $\mathrm{reg}(G) = \mathrm{im}(G)$ so that we may assume $B \neq \emptyset$. Let L be the set of vertices of a basic cycle, say, of length $n \geq 4$. If every vertex in L has degree two in G, we consider any 4-path (not necessarily induced) on $\{a,b,c,d\} \subseteq L$. Otherwise, L contains at least one vertex, say d, of degree at least three. In this case, we consider a 4-path on $\{a,b,c,d\}$ in L such that the adjacent vertices b and c are of degree two in G. If $ad \in E(G)$, that is, $n = 4$, then $\mathrm{reg}(G) = \mathrm{reg}(G - \{a,b,c,d\} \cup K_2)$ by Proposition 4.5. Therefore, if we define $B' := B \setminus L$ and $W' := W \cup (L - \{a,b,c,d\}) \cup V(K_2)$, then the graph $G - \{a,b,c,d\} \cup K_2$ belongs to the class $\mathscr{BW}$ with the partition $V(G - \{a,b,c,d\} \cup K_2) = B' \cup W'$ so that $\mathrm{reg}(G) = \mathrm{reg}(G - \{a,b,c,d\}) + 1 = \mathrm{im}(G - \{a,b,c,d\}; K_2, C_5) + 1 = \mathrm{im}(G; K_2, C_5)$.

We may, therefore, assume that $ad \notin E(G)$, that is, $n \geq 5$. Note that

$$\operatorname{reg}(G) = \operatorname{reg}(\mathfrak{f}(G - \{b, c\}; da) \cup K_2)$$

by Proposition 4.5. We next examine two cases separately:

Case 1. $n = 5$ or 6. If we define $B' := B \setminus L$ and $W' := W \cup (L - \{a, b, c\}) \cup V(K_2)$, then the decomposition $V(\mathfrak{f}((G - \{b, c\}); da) \cup K_2) = B' \cup W'$ satisfies the required conditions for which $\mathfrak{f}((G - \{b, c\}); da) \cup K_2 \in \mathscr{BW}$.

Case 2. $n = 7$. In this case, we note that L can contain at most one other vertex of degree three or more in G. If there exists such a vertex, we may assume, without loss of generality, that this vertex is a. Therefore, if we set $B' := B - \{a, b, c\}$ and $W' := W \cup V(K_2)$, the resulting decomposition $V(\mathfrak{f}((G - \{b, c\}); da) \cup K_2) = B' \cup W'$ satisfies the required conditions for which $\mathfrak{f}((G - \{b, c\}); da) \cup K_2 \in \mathscr{BW}$.

In either case, we conclude that $\operatorname{reg}(G) = \operatorname{reg}(\mathfrak{f}((G - \{b, c\}); da) \cup K_2)$ so that the claim follows from induction. $\square$

Remark 4.7. The inequality $\operatorname{reg}(G) \leq 2 \operatorname{im}(G)$ for any graph in the class $\mathscr{BW}$ can also be seen from its structural characterization without the aid of Theorem 4.6. Indeed, when $G \in \mathscr{BW}$, by removing a vertex of a basic cycle whose degree is three or more together with all of its neighbors leaves a graph such that the gap between the regularity of G and the resulting graph is at most two. Therefore, once we destroy every basic cycle of G in such a way, the resulting graph is the disjoint union of a weakly chordal graph and paths, and the induced matching number of this final graph exactly equals to that of G.

The following are a direct consequence of Theorem 4.6.

Corollary 4.8. *If G is a well-covered block-cactus graph, then reg$(G) = \operatorname{im}(G; K_2, C_5)$.*

Corollary 4.9. *If G is a Cohen-Macaulay graph of girth at least five, then* $\operatorname{reg}(G) = \operatorname{im}(G; K_2, C_5)$.

5. Graphs with $\mathrm{im}(G) < \mathrm{reg}(G) = \mathrm{m}(G)$**.** In this section, we verify that the five cycle graph C_5 is the only connected graph satisfying the inequality $\mathrm{im}(G) < \mathrm{reg}(G) = \mathrm{m}(G)$, as promised.

Lemma 5.1. *Let H be a graph with* $\mathrm{reg}(H) = \mathrm{m}(H)$*. If $\{H_1, \dots, H_k\}$ is a prime factorization of H, then* $\mathrm{reg}(H_i) = \mathrm{m}(H_i)$ *for each $i \in [k]$.*

Proof. If there exists a $j \in [k]$ such that $\mathrm{reg}(H_j) < \mathrm{m}(H_j)$, then

$$\begin{aligned} \mathrm{m}(H) = \mathrm{reg}(H) &= \mathrm{reg}(H_1) + \cdots + \mathrm{reg}(H_k) \\ &< \mathrm{m}(H_1) + \cdots + \mathrm{m}(H_k) \leq m(H), \end{aligned}$$

a contradiction. □

Theorem 5.2. *If G is a connected graph satisfying* $\mathrm{im}(G) < \mathrm{reg}(G) = \mathrm{m}(G)$*, then $G \cong C_5$.*

Proof. We first prove the claim for prime graphs. Therefore, let G be a prime graph satisfying $\mathrm{im}(G) < \mathrm{reg}(G) = \mathrm{m}(G)$. We proceed by induction on the order of G. If $v \in V(G)$, we then have $\mathrm{reg}(G) = 1 + \mathrm{reg}(G - N_G[v])$. Observe that $\mathrm{reg}(G - N_G[v]) = \mathrm{m}(G - N_G[v])$ since, otherwise, $\mathrm{reg}(G) = 1 + \mathrm{reg}(G - N_G[v]) < 1 + \mathrm{m}(G - N_G[v]) \leq \mathrm{m}(G)$, which is not possible by the assumption.

Now, let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a prime factorization of $H := G - N_G[v]$. It then follows from Lemma 5.1 that $\mathrm{reg}(H_i) = \mathrm{m}(H_i)$ for each $i \in [k]$. On the other hand, if $\mathrm{im}(H_i) < \mathrm{reg}(H_i)$ for some $i \in [k]$, we have $H_i \cong C_5$ by the induction hypothesis. Furthermore, if $\mathrm{im}(H_j) = \mathrm{reg}(H_j)$ for some $j \in [k]$, we then have $H_j \cong K_2$ since, in such a case, the equality $\mathrm{im}(H_j) = \mathrm{m}(H_j)$ implies that the graph H_j must contain a (closed) dominated vertex by **[2,** Theorem 30], which is only possible when $H_j \cong K_2$ by Proposition 3.2. Therefore, the prime factorization $\mathcal{H}$ can be divided into two pieces $\mathcal{H}_1 := \{H_1, \dots, H_l\}$ and $\mathcal{H}_2 := \{H_{l+1}, \dots, H_k\}$, where $H_i \cong K_2$ for any $1 \leq i \leq l$ and $H_j \cong C_5$ for any $l < j \leq k$.

Case 1. $l < k$. If we define $T := \cup_{j=l+1}^{k} V(H_j)$, then the set $U := V(H) \setminus T$ is an independent set, due to the fact that $\mathrm{reg}(H) = \mathrm{m}(H)$. Moreover, by the same reason, no vertex $u \in U$ can have a neighbor in T. However, this implies that each $H_j \cong C_5$ in $\mathcal{H}_2$ is a connected component of H. On the other hand, the vertex v has at least two neighbors, say w and z in G, by Proposition 3.2 and, since G is connected, at least one of these two vertices is adjacent to a vertex in T. Now, if $wp \in E(G)$ for some $p \in T$, then $\mathrm{reg}(G) < \mathrm{m}(G)$ since the addition of the edges wp and vz increases the matching number of G by two, a contradiction.

Case 2. $l = k$. In this case, we necessarily have $\mathrm{im}(H) = \mathrm{reg}(H) = \mathrm{m}(H)$. If we define $L := \cup_{i=1}^{k} V(H_i)$, the set $W := V(H) \setminus L$ is an independent set as in the previous case. Once again, by [**2,** Theorem 30], the graph H contains two vertices, say p and q, such that $N_H[p] \subseteq N_H[q]$. However, since G is prime, such a vertex p cannot be dominated in G; hence, there exists an $x \in N_G(v)$ such that $xp \in E(G)$, while $xq \notin E(G)$. Since W is an independent set, one end of the edge pq must be contained in L. We also note that the vertex v has at least one neighbor y other than x, again by Proposition 3.2.

Subcase 2.1. If $q \in L$, while $p \notin L$, then the set $\mathcal{H} \cup \{vy, xp\}$ is a matching in G of size one more than $\mathrm{reg}(G) = \mathrm{m}(G)$, which is not possible.

Subcase 2.2. Let $p \in L$ and $q \notin L$. Then, there exists a $t \in L$ such that the edge pt is in the prime factorization $\mathcal{H}$. It then follows from [**11,** Lemma 1] that $tq \in E(H)$ and $\deg_H(p) = \deg_H(t) = 2$. Now, the set $\mathcal{H}' \cup \{xp, vy\}$, where $\mathcal{H}' := (\mathcal{H} \setminus \{pq\}) \cup \{tq\}$, is then a matching in G of size one more than $\mathrm{reg}(G) = \mathrm{m}(G)$, which is not possible.

Subcase 2.3. Assume that $p, q \in L$. Observe that the case where the vertices p and q have a common neighbor in H is not possible by subcase 2.2. However, it then follows from [**11,** Lemma 1] that we must have $\deg_H(p) = 1$. Now, if there exists an $h \in W$ such that $qh \in E(H)$, then the set $\mathcal{H}'' := (\mathcal{H} \setminus \{pq\}) \cup \{xp, qh, vy\}$ is a matching in G of size one more than $\mathrm{reg}(G) = \mathrm{m}(G)$, a contradiction. Thus, we may further assume that $\deg_H(q) = 1$. However, this forces that the set W is empty, that is, $H \cong kK_2$. On the other hand, since G is prime, the vertex q must have a neighbor, say y, in $N_G(v)$. If the vertex v has a neighbor s other than x and y, then the set

$$\mathcal{H}''' := (\mathcal{H} \setminus \{pq\}) \cup \{xp, qy, vs\}$$

is a matching in G of size one more than $\text{reg}(G) = \text{m}(G)$; hence, $N_G(v) = \{x, y\}$. Therefore, we must have either $k = 1$ so that $G \cong C_5$ or else the graph $G - N_G[y]$ is a star, that is, $G - N_G[y] \cong K_{1,l}$ for some $l \geq 1$. However, the latter case contradicts the fact that G is a prime graph.

Finally, assume that G is not a prime graph, and let $\mathcal{G} = \{G_1, \ldots, G_r\}$ be a prime factorization of G. It then follows from Lemma 5.1 that $\text{reg}(G_i) = \text{m}(G_i)$ for any $i \in [r]$. Now, if $\text{im}(G_i) = \text{reg}(G_i)$ for each $i \in [r]$, then $G_i \cong K_2$ for by Proposition 3.2 and **[2,** Theorem 30] so that $r = \text{im}(G) = \text{reg}(G) = \text{m}(G)$, a contradiction. Therefore, there exists a $j \in [r]$ such that $\text{im}(G_j) < \text{reg}(G_j)$. Since G_j is prime, then $G_j \cong C_5$ by the above argument. However, since G is not prime, the set $V(G) \setminus \cup_{i=1}^{r} V(G_i)$ cannot be empty. On the other hand, since G is connected, we can use any vertex in this set together with those in $V(G_j)$ to create a matching in G of size larger than $\text{m}(G)$, a contradiction. $\square$

6. Vertex decomposable perfect prime graphs. In this section, we first prove that a special ear addition to a prime graph gives rise to a new prime graph under which the regularity increases exactly by one. In particular, such an operation enables us to prove that there exists an infinite family of vertex decomposable prime graphs of arbitrarily high regularity. Note that, since the regularity of vertex decomposable graphs is independent of the characteristic of the coefficient field, the graphs we construct are necessarily perfect primes.

We recall that a vertex x of G is called a *shedding vertex* if, for every independent set S in $G - N_G[x]$, there is some vertex $v \in N_G(x)$ so that $S \cup \{v\}$ is independent. A graph G is called *vertex-decomposable* if either it is an edgeless graph or it has a shedding vertex x such that $G - x$ and $G - N_G[x]$ are both vertex-decomposable.

Definition 6.1. Let $G = (V, E)$ be a graph and $e = xy$ an edge of G, and let P_3 be a (disjoint) 3-path on $\{x', v, y'\}$. We define a new graph $P(G; e)$, the 3-*ear addition* to G with respect to the edge e, by the addition of P_3 to G on the end vertices of e, that is,

$$V(P(G; e)) := V \cup \{x', v, y'\}$$

and

$$E(P(G;e)) := E \cup \{xx', x'v, vy', y'y\}.$$

Before we describe the affect of a 3-ear addition $P(G;e)$ on the regularity of G, we need the following technical results.

Lemma 6.2. *If $N_G(x) = \{y\}$ in G, then either* $\operatorname{reg}(G) = \operatorname{reg}(G-x)$ *or else* $\operatorname{reg}(G) = \operatorname{reg}(G - N_G[y]) + 1$.

Proof. We apply Lemma 3.1 at the vertex y. If $\operatorname{reg}(G) = \operatorname{reg}(G - N_G[y])+1$, there is nothing to prove. Assume that $\operatorname{reg}(G) = \operatorname{reg}(G-y)$. Since x is an isolated vertex of $G-y$, we have $\operatorname{reg}(G) = \operatorname{reg}(G-y) = \operatorname{reg}(G - \{x,y\}) \leq \operatorname{reg}(G-x)$, that is, $\operatorname{reg}(G) = \operatorname{reg}(G-x)$ as claimed. □

Lemma 6.3. *Let x, y, z be three vertices of a graph G with $xy, yz \in E$. If* $\deg_G(x) = 1$ *and* $\deg_G(y) = 2$, *then* $\operatorname{reg}(G) = \operatorname{reg}(G-z)$.

Proof. Suppose, to the contrary, that $\operatorname{reg}(G) > \operatorname{reg}(G-z)$. This implies that $\operatorname{reg}(G) = \operatorname{reg}(G - N_G[z]) + 1$ by Lemma 3.1. If we define $T := G - (N_G[z] \cup \{x\})$, we note that $\operatorname{reg}(G - N_G[z]) = \operatorname{reg}(T)$ since x is an isolated vertex of $G - N_G[z]$. On the other hand, since $\{T, xy\}$ is an induced decomposition of $G-z$, we have $\operatorname{reg}(G-z) \geq \operatorname{reg}(T)+1$ by Theorem 3.3. It follows that $\operatorname{reg}(G-z) \geq \operatorname{reg}(G)$, a contradiction. □

Proposition 6.4. *If G is a prime graph, then* $\operatorname{reg}(P(G;e)) = \operatorname{reg}(G) + 1$ *for any edge e of G.*

Proof. Suppose that $G = (V,E)$ is a prime graph and $e = xy$ is an edge of G. In order to ease the notation, we write P_e instead of $P(G;e)$.

Assume that $\operatorname{reg}(G) = m$, and let $S \subseteq V$ be a minimal subset satisfying $\widetilde{H}_{m-1}(G[S]) \neq 0$. If $x \notin S$, we may define $S^* := S \cup \{x', v\}$ so that $P_e[S^*] \simeq \Sigma(G[S])$; hence, $\widetilde{H}_m(P_e[S^*]) \neq 0$, that is, $\operatorname{reg}(P_e) \geq m+1$. The case for which $y \notin S$ can be similarly treated. We may, therefore, assume that $x, y \in S$. In this case, we define $S^* :=$

$S \cup \{x', v, y'\}$ and consider the associated Mayer-Vietoris sequence of the pair $(P_e[S^*], v)$:

$$\cdots \longrightarrow \widetilde{H}_m(P_e[S^*]) \longrightarrow \widetilde{H}_{m-1}(P_e[S^*] - N_{P_e[S^*]}[v]) \\ \longrightarrow \widetilde{H}_{m-1}(P_e[S^*] - v) \longrightarrow \widetilde{H}_{m-1}(P_e[S^*]) \longrightarrow \cdots .$$

The graph $P_e[S^*] - N_{P_e[S^*]}[v]$ is isomorphic to $G[S]$; hence, we have $\widetilde{H}_{m-1}(P_e[S^*] - N_{P_e[S^*]}[v]) \neq 0$. On the other hand, the graph $P_e[S^*] - v$ is contractible by **[1,** Proposition 5.1] that, in turn, implies that $\widetilde{H}_{m-1}(P_e[S^*] - v) = 0$ so that $\widetilde{H}_m(P_e[S^*]) \neq 0$ by the exactness, that is, $\mathrm{reg}(P_e) \geq m + 1$.

Assume now that $\mathrm{reg}(P_e) = k$. We analyze these three cases separately.

Case 1. There exists a minimal subset $R \subseteq V$ such that $x,\ y \in R$ and $\widetilde{H}_{k-1}(P_e[R]) \neq 0$. Observe first that the intersection $R \cap \{x', v, y'\}$ cannot be empty. Suppose, otherwise, that $R \cap \{x', v, y'\} = \emptyset$, and define $R^* := R \cup \{x', v, y'\}$. It then follows that the graph $P_e[R^*] - v$ is contractible, again by **[1,** Proposition 5.1], so $P_e[R^*] \simeq \Sigma(P_e[R^*] - N_{P_e[R^*]}[v])$ (see **[12]** for details). However, the graph $P_e[R^*] - N_{P_e[R^*]}[v]$ is isomorphic to $P_e[R]$, that is,

$$\widetilde{H}_k(P_e[R^*]) \cong \widetilde{H}_{k-1}(P_e[R]) \neq 0,$$

which forces $\mathrm{reg}(P_e) > k$, a contradiction. We therefore have $R \cap \{x', v, y'\} \neq \emptyset$.

Now, if $\{x', v, y'\} \subseteq R$, then $P_e[R] \simeq \Sigma(P_e[R] - N_{P_e[R]}[v])$ as above, while $P_e[R] - N_{P_e[R]}[v] \cong G[R \cap V]$; hence, we conclude $\mathrm{reg}(G) \geq k - 1$. On the other hand, if $|R \cap \{x', v, y'\}| = 1$, then the vertex in $\{x', v, y'\}$ that R contains must be either x' or y'. Assume, without loss of generality, that $x' \in R$. It then follows from Lemma 6.2, together with the minimality of R, that $\mathrm{reg}(P_e) = \mathrm{reg}(P_e[R]) = \mathrm{reg}(P_e[R] - N_{P_e[R]}[x']) + 1$. However, this implies that $\mathrm{reg}(G) \geq k - 1$ since $V(P_e[R] - N_{P_e[R]}[x']) \subseteq V$.

Suppose now that $|R \cap \{x', v, y'\}| = 2$. We note that $v \in R$, since otherwise the graph $P_e[R]$ is contractible by **[1,** Proposition 5.1], which is impossible. It then follows that the set R contains either $\{x', v\}$ or $\{v, y'\}$. Assume that $x',\ v \in R$. However, this implies that

$P_e[L] \simeq P_e[L] - x$, while $P_e[L] - x \cong (G - x)[R \cap V] \cup K_2$, that is, $P_e[L] \simeq \Sigma(G[(R \setminus \{x\} \cap V])$. We, therefore, have $\operatorname{reg}(G) \geq k - 1$, as expected.

Case 2. There exists a minimal subset $L \subseteq V$ such that $|L \cap \{x, y\}| = 1$ and $\widetilde{H}_{k-1}(P_e[L]) \neq 0$. Once again, we may assume, without loss of generality, that $x \in L$ while $y \notin L$. As in Case 1, we must have $R \cap \{x', v, y'\} \neq \emptyset$ since, otherwise, we can define $L^* := L \cup \{v, y'\}$ so that $P_e[L^*] \cong P_e[L] \cup K_2$; hence, we would have $P_e[L^*] \simeq \Sigma(P_e[L])$, a contradiction. If $\{x', v, y'\} \subseteq L$, it follows from Lemma 6.3, together with the minimality of L, that $\operatorname{reg}(P_e) = \operatorname{reg}(P_e[L]) = \operatorname{reg}(P_e[L] - x')$. However, we have $P_e[L] - x' \cong G[L \cap V] \cup K_2$, where the component K_2 is induced by $\{v, y'\}$. But, then $P_e[L] - x' \simeq \Sigma(G[L \cap V])$; hence, we must have $\operatorname{reg}(G) \geq k - 1$. We, therefore, only need to check for the case in which $L \cap \{x', v, y'\} = \{x'\}$ by Lemma 6.3 together with the minimality of L. In such a case, we must have $\operatorname{reg}(P_e) = \operatorname{reg}(P_e[L]) = \operatorname{reg}(P_e[L] - N_{P_e[L]}[x']) + 1$ by Lemma 6.2, together with the minimality of L, that is, $\operatorname{reg}(G) \geq k - 1$, since $V(P_e[L] - N_{P_e[L]}[x']) \subseteq V$.

Case 3. There exists a minimal subset $K \subseteq V$ such that $K \cap \{x, y\} = \emptyset$ and $\widetilde{H}_{k-1}(P_e[K]) \neq 0$. We note that the minimality of such a set K forces $|K \cap \{x', v, y\}| \geq 2$, and in any possible case, we have $P_e[K] \simeq \Sigma(G[K \cap V])$ so that $\operatorname{reg}(G) \geq k - 1$. $\square$

We next verify that the primeness is preserved under a 3-ear addition on the end vertices of any edge of a prime graph.

Theorem 6.5. *If G is a prime graph, then so is $P(G; e)$ for any edge e of G.*

Proof. Suppose that G is prime, and let $a \in V(P_e)$ be any vertex. Once again, we divide the proof into several cases.

Case 1. $a \notin \{x, y, x', v, y'\}$. In this case, we have

$$\operatorname{reg}(P_e - a) = \operatorname{reg}(P((G - a); e)) = \operatorname{reg}(G - a) + 1 = \operatorname{reg}(G) < \operatorname{reg}(P_e),$$

where the second and third equalities are due to Proposition 6.4 and the primeness of G, respectively.

Case 2. $a = x$ or $a = y$. Assume that $a = x$. It then follows from Lemma 6.3 that

$$\operatorname{reg}(P_e - x) = \operatorname{reg}((P_e - x) - y') = \operatorname{reg}(G - x) + 1 = \operatorname{reg}(G) < \operatorname{reg}(P_e)$$

since $(P_e - x) - y' \cong (G - x) \cup K_2$, where the component K_2 is induced by $\{x', v\}$. The case $a = y$ can be verified similarly.

Case 3. $a = x'$ or $a = y'$. Due to the symmetry, we only verify the case $a = x'$. Once again, Lemma 6.3 implies that $\operatorname{reg}(P_e - x') = \operatorname{reg}((P_e - x') - y) = \operatorname{reg}(G - y) + 1 = \operatorname{reg}(G) < \operatorname{reg}(P_e)$ since $(P_e - x') - y \cong (G - y) \cup K_2$, where the component K_2 is induced by $\{v, y'\}$.

Case 4. $a = v$. In this case, we repeatedly apply Lemma 6.2. Note that $\operatorname{reg}(P_e - v)$ equals either $\operatorname{reg}((P_e - v) - x')$ or $\operatorname{reg}((P_e - v) - N_{(P_e - v)}[x]) + 1$. For the latter case, we note that the graph $(P_e - v) - N_{(P_e - v)}[x]$ is isomorphic to $(G - N_G[x]) \cup \{y'\}$ in which y' is an isolated vertex, that is,

$$\begin{aligned}\operatorname{reg}(P_e - v) &= \operatorname{reg}((P_e - v) - N_{(P_e - v)}[x]) + 1 \\ &= \operatorname{reg}(G - N_G[x]) + 1 = \operatorname{reg}(G) < \operatorname{reg}(P_e).\end{aligned}$$

If $\operatorname{reg}(P_e - v) = \operatorname{reg}((P_e - v) - x')$, we then have either $\operatorname{reg}((P_e - v) - x') = \operatorname{reg}(((P_e - v) - x') - y')$ or else $\operatorname{reg}((P_e - v) - x') = \operatorname{reg}((P_e - v) - x') - N_{(P_e - v) - x')}[y]) + 1$. For the former, we conclude that $\operatorname{reg}(P_e - v) = \operatorname{reg}(((P_e - v) - x') - y') = \operatorname{reg}(G) < \operatorname{reg}(P_e)$ by Proposition 6.4. On the other hand, since $((P_e - v) - x') - N_{((P_e - v) - x')}[y] \cong G - N_G[y]$, we must have

$$\begin{aligned}\operatorname{reg}(P_e - v) &= \operatorname{reg}((P_e - v) - x') \\ &= \operatorname{reg}((P_e - v) - x') - N_{(P_e - v) - x')}[y]) + 1 \\ &= \operatorname{reg}(G) < \operatorname{reg}(P_e).\end{aligned}$$ □

Corollary 6.6. *For each integer $n \geq 1$, there exists a vertex decomposable prime graph G_n such that* $\operatorname{reg}(G_n) = n$.

Proof. We let $G_1 := K_2$, and define $G_n := P(G_{n-1}; e)$ for some edge $e \in E(G_{n-1})$ for any $n > 1$. Note that $G_2 \cong C_5$, and any induced cycle in G_n for $n \geq 2$ is of fixed length 5. Therefore, the set

$\{G_n : n \geq 1\}$ provides a desired family of vertex decomposable prime graphs by [**16,** Theorem 1], Proposition 6.4 and Theorem 6.5. $\square$

Remark 6.7. The construction within the proof of Corollary 6.6 also works when $n \geq 2$ if we choose $G_2 := \overline{C_k}$, the complement of the k-cycle for some $k \geq 5$, even if the verification of the vertex decomposability of the resulting graphs requires some extra work.

The Corollary 6.6 allows us to construct a family of vertex decomposable prime graphs with an arbitrary gap between their regularities and induced matching numbers.

Corollary 6.8. *For each integer $s \geq 1$, there exists a vertex decomposable prime graph H_s such that* $\operatorname{reg}(H_s) - \operatorname{im}(H_s) = s$.

Proof. For a given graph G, we denote by $\operatorname{Ear}(G)$ the graph obtained from G by applying a 3-ear addition to each edge of G. We first claim that $\operatorname{im}(\operatorname{Ear}(G)) = |E(G)| = m$. We let a_e, b_e and c_e be the vertices added to G corresponding to the 3-ear addition with respect to the edge $e = xy \in E(G)$. Observe that any maximum induced matching of $\operatorname{Ear}(G)$ can share at most one edge with the five cycle induced by $\{x, a_e, b_e, c_e, y\}$. It then follows that $\operatorname{im}(\operatorname{Ear}(G)) \leq |E(\operatorname{Ear}(G))|/5 = 5m/5 = m$. On the other hand, the set $\{a_e b_e : e \in E(G)\}$ forms an induced matching of $\operatorname{Ear}(G)$ of required size.

Now, suppose that G_s is the vertex decomposable prime graph with $\operatorname{reg}(G_s) = s$, the existence of which is guaranteed by Corollary 6.6. Observe that $\operatorname{reg}(\operatorname{Ear}(G_s)) = \operatorname{reg}(G_s) + |E(G_s)|$ as a result of Proposition 6.4 since G_s is a prime graph and any 3-ear addition to a prime graph preserves primeness by Theorem 6.5. We, therefore, conclude that $\operatorname{reg}(\operatorname{Ear}(G_s)) - \operatorname{im}(\operatorname{Ear}(G_s)) = s$, as claimed. $\square$

Remark 6.9. Even if we prove the existence of an infinite family of vertex decomposable prime graphs of arbitrarily high regularity, when we impose some restriction, such graphs become rare. For instance, any (C_4, C_5)-free vertex decomposable prime graph must be isomorphic to a K_2 (compare to [**2,** Theorem 24]). On the other hand, it is not difficult to prove that any graph G of minimum degree at least two with $\mathrm{girth}(G) \geq 6$ cannot contain a shedding vertex, that is, such a graph cannot be vertex decomposable.

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2. Cadde, 12/9, Ankara, 06500 Turkey
Email address: **tbiyikoglu@gmail.com**

Suleyman Demirel University, Department of Mathematics, Isparta, 32260 Turkey
Email address: **yusufcivan3@gmail.com**