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STAR OPERATIONS ON PRÜFER v-MULTIPLICATION DOMAINS

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ABSTRACT. Let D be an integrally closed domain, $S(D)$ the set of star operations on D, w the w-operation, and $S_w(D) = \{* \in S(D) \mid w \leq *\}$. Let X be an indeterminate over D and $N_v = \{f \in D[X] \mid c(f)_v = D\}$. In this paper, we show that, if D is a Prüfer v-multiplication domain (PvMD), then $|S_w(D)| = |S_w(D[X])| = |S(D[X]_{N_v})|$. We prove that D is a PvMD if and only if $|\{* \in S_w(D) \mid *$ is of finite type$\}| < \infty$. We then use these results to give a complete characterization of integrally closed domains D with $|S_w(D)| < \infty$.

0. Introduction. Let D be an integral domain that is not a field, and let K be the quotient field of D. Let $S(D)$ be the set of star operations on D and $S_w(D) = \{* \in S(D) \mid w \leq *\}$; so $S_w(D) \subseteq S(D)$. (Definitions related to star operations will be reviewed in Section 1.) It is easy to see that $S_w(D) = S(D)$ if and only if each maximal ideal of D is a t-ideal (Proposition 1.6). So, if $|S(D)| < \infty$, then $S_w(D) = S(D)$ [**16,** Proposition 2.1]. But if $D[X]$ is the polynomial ring over a valuation domain D that is not a field, then $|S(D[X])| = \infty$ [**16,** Corollary 2.3] while $|S(D)| = |S_w(D)| = |S_w(D[X])| \leq 2$ (Theorem 2.6). Note that $d \leq * \leq v$ for any star operation $*$ on D; so $|S(D)| = 1 \Leftrightarrow d = v$; $|S(D)| = 2 \Leftrightarrow d \neq v$ and $S(D) = \{d, v\}$; $|S_w(D)| = 1 \Leftrightarrow w = v$; and $|S_w(D)| = 2 \Leftrightarrow w \neq v$ and $S_w(D) = \{w, v\}$.

In [**13**], Heinzer studied the integral domains D with $|S(D)| = 1$; in particular, he showed that if D is integrally closed, then $|S(D)| = 1$ if and only if D is an h-local Prüfer domain whose maximal ideals are

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invertible [**13,** Theorem 5.1]. As the t-operation analog of Heinzer's result, in [**7**], El Baghdadi and Gabelli studied the integral domains D with $|S_w(D)| = 1$. Among other things, they showed that, if D is integrally closed, then $|S_w(D)| = 1$ if and only if D is an independent ring of Krull type whose maximal t-ideals are t-invertible [**7,** Theorem 3.3]. In [**16**], Houston, Mimouni and Park characterized the integrally closed domains having two star operations. For example, they proved that, if D is integrally closed, then $|S(D)| = 2$ if and only if D is an h-local Prüfer domain with exactly one nondivisorial maximal ideal [**16,** Theorem 3.3]. In [**17,** Theorem 5.3], they also gave a complete characterization of an integrally closed domain D with $|S(D)| < \infty$.

The purpose of this paper is to prove the t-operation analogs of Houston, Mimouni and Park's results ([**16,** Theorems 3.3] and [**17,** Theorem 5.3]). That is, we give some characterizations for integrally closed domains D with $|S_w(D)| < \infty$. In Section 1, we review definitions and notations related to star operations. We also recall some basic results on the (t-)Nagata ring $D[X]_{N_v}$, Prüfer v-multiplication domains (PvMD) and *e.a.b.* star operations, which are essential in the arguments of Sections 2 and 3. In Section 2, we show that, if D is a PvMD, then there are bijections from $S_w(D)$ onto $S(D[X]_{N_v})$ and $S_w(D[X])$, respectively, and hence $|S_w(D)| = |S_w(D[X])| = |S(D[X]_{N_v})|$. We then use this result with [**16,** Theorem 3.1] to show that if D is an independent ring of Krull type, then $|S_w(D)| = 2^{|\mathfrak{U}|}$, where $\mathfrak{U}$ is the set of maximal t-ideals of D that are not v-ideals. In Section 3, we study the integrally closed domains D with $|S_w(D)| < \infty$. We show that, if D is integrally closed, then $|S_w(D)| = 2$ if and only if D is an independent ring of Krull type with exactly one nondivisorial maximal t-ideal, if and only if $|S_w(D[X])| = 2$, if and only if $|S(D[X]_{N_v})| = 2$. We prove that if D is integrally closed, then $|\{* \in S_w(D) \mid * \text{ is of finite type}\}| < \infty$ if and only if D is a PvMD. We then finally give a complete characterization of integrally closed domains D with $|S_w(D)| < \infty$ by using the results of Section 2 and Houston, Mimouni and Park's result [**17,** Theorem 5.3].

1. Star operations and the ring $D[X]_{N_v}$. Let D be an integral domain with quotient field K. Let $\mathbf{F}(D)$ (respectively, $\mathbf{f}(D)$) be the set of nonzero fractional (respectively, finitely generated fractional) ideals of D; so $\mathbf{f}(D) \subseteq \mathbf{F}(D)$. A mapping $I \mapsto I^*$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ is called a *star operation* ($\star$-operation) on D if, for all $0 \neq a \in K$ and $I, J \in \mathbf{F}(D)$, the following conditions are satisfied:

(1) $(aD)^* = aD$ and $(aI)^* = aI^*$,
(2) $I \subseteq I^*$; $I \subseteq J$ implies $I^* \subseteq J^*$, and
(3) $(I^*)^* = I^*$.

Given any star operation $*$ on D, two new star operations $*_f$ and $*_w$ can be constructed by setting $I^{*_f} = \bigcup\{J^* \mid J \subseteq I \text{ and } J \in \mathbf{f}(D)\}$ and $I^{*_w} = \{x \in K \mid xJ \subseteq I \text{ for some } J \in \mathbf{f}(D) \text{ with } J^* = D\}$ for all $I \in \mathbf{F}(D)$. A star operation $*$ on D is said to be of *finite type* if $*_f = *$. Obviously, $(*_f)_f = *_f$ and $(*_w)_f = *_w = (*_f)_w$, and hence $*_f$ and $*_w$ are of finite type. An $I \in \mathbf{F}(D)$ is called a $*$*-ideal* if $I^* = I$, while a $*$-ideal is a *maximal* $*$*-ideal* if it is maximal among proper integral $*$-ideals. Let $*$-Max(D) denote the set of maximal $*$-ideals of D. Although it is possible that $*$-Max$(D) = \emptyset$ even when D is not a field (e.g., v-Max$(V) = \emptyset$ if V is a rank-one nondiscrete valuation domain), it is well known that a maximal $*_f$-ideal is a prime ideal; each prime ideal minimal over a $*_f$-ideal is a $*_f$-ideal; $*_f$-Max$(D) \neq \emptyset$ if D is not a field; $*_f$-Max$(D) = *_w$-Max(D) [**1,** Theorem 2.16]; and $I^{*_w} = \bigcap_{P \in *_f\text{-Max}(D)} ID_P$ for all $I \in \mathbf{F}(D)$ [**1,** Corollary 2.10]. For two star operations $*_1$ and $*_2$ on D, we mean by $*_1 \leq *_2$ that $I^{*_1} \subseteq I^{*_2}$ (equivalently, $(I^{*_1})^{*_2} = (I^{*_2})^{*_1} = I^{*_2}$) for all $I \in \mathbf{F}(D)$. Clearly, if $*_1 \leq *_2$, then $(*_1)_f \leq (*_2)_f$ and $(*_1)_w \leq (*_2)_w$. Also, $*_w \leq *_f \leq *$ for any star operation $*$ on D. An $I \in \mathbf{F}(D)$ is said to be $*$*-invertible* if $(II^{-1})^* = D$, where $I^{-1} = \{x \in K \mid xI \subseteq D\}$. Clearly, I is $*_f$-invertible if and only if $II^{-1} \nsubseteq P$ for all $P \in *_f$-Max(D). Note that $I \in \mathbf{F}(D)$ is $*_f$-invertible if and only if I is $*_w$-invertible, because $*_f$-Max$(D) = *_w$-Max(D). Also, if $*_1 \leq *_2$ are star operations on D, then $*_1$-invertible ideals are $*_2$-invertible.

The most well-known examples of star operations are the d-, v-, t- and w-operations. The d-operation is just the identity function on $\mathbf{F}(D)$; so $d = d_f = d_w$. The v-operation is defined by $I_v = (I^{-1})^{-1}$ and the t-operation (respectively, w-operation) is given by $t = v_f$ (respectively, $w = v_w$). In particular, a v-ideal is called a *divisorial ideal*. It is known that, if $*$ is a star operation on D, then $d \leq * \leq v$, and hence $d \leq *_f \leq t$ and $d \leq *_w \leq w \leq t \leq v$.

Let $*$ be a star operation on D. As in [**3,** page 224], we say that an overring R of D is $*$*-linked* over D if $I^* = D$ implies $(IR)_v = R$ for all $I \in \mathbf{f}(D)$. Recall that $*$ is *endlich arithmetisch brauchbar* (*e.a.b.*) if $(AB)^* \subseteq (AC)^*$ for all $A, B, C \in \mathbf{f}(D)$ implies $B^* \subseteq C^*$. It is known that if D admits an *e.a.b.* star operation, then D is integrally closed [**11,** Corollary 32.8]. Conversely, if D is integrally closed, then $D = \bigcap V$, where V ranges over all valuation overrings of D, and thus the mapping $I \mapsto I^b = \bigcap IV$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ is an *e.a.b.* star operation of finite type [**11,** pages 396–398]. More generally, we have

Lemma 1.1. ([**4,** Lemma 3.1]). *Let D be an integrally closed domain and $\{V_\alpha\}$ the set of $*$-linked valuation overrings of D. Then the map $*_c : \mathbf{F}(D) \to \mathbf{F}(D)$, given by $I \mapsto I^{*_c} = \bigcap_\alpha IV_\alpha$, is an e.a.b. star operation of finite type on D such that $*_w = (*_c)_w \leq *_c$ and $*_f$-*$\mathrm{Max}\,(D) = *_c$-$\mathrm{Max}\,(D)$. *In particular, $d_c = b$.*

Let X be an indeterminate over D and $D[X]$ the polynomial ring over D. For any $f \in D[X]$, we denote by $c_D(f)$ (simply, $c(f)$) the ideal of D generated by the coefficients of f. The next lemma will be used in Section 2 without references.

Lemma 1.2. ([**12,** Lemma 4.1 and Proposition 4.3]). *Let $* = v, t$ or w and $I \in \mathbf{F}(D)$. Then $(ID[X])^{-1} = I^{-1}D[X]$ and $(ID[X])^* = I^*D[X]$.*

Let $S = \{f \in D[X] \mid c(f) = D\}$. Then $D[X]_S$, denoted by $D(X)$, is called the *Nagata ring* of D [**11,** Section 33]. For the t-operation analog, let $N_v = \{f \in D[X] \mid f \neq 0$ and $c(f)_v = D\}$. Then $D[X]_{N_v}$, called the (t-)*Nagata ring* of D, is an overring of $D(X)$, and $D[X]_{N_v} = D(X)$ if and only if each maximal ideal of D is a t-ideal. The (t-)Nagata ring $D[X]_{N_v}$ has many interesting ring-theoretic properties and, in particular, it is very useful when we study the w-operation on D. For more on Nagata rings, the reader can refer to [**3, 9, 20**].

We next review some basic properties of $D[X]_{N_v}$ that are very useful in the arguments of this paper.

Lemma 1.3. *Let I be a nonzero fractional ideal of D.*

(i) *$ID[X]_{N_v} \cap K = I_w$ and $I_wD[X]_{N_v} = ID[X]_{N_v}$.*
(ii) *$(ID[X]_{N_v})^{-1} = I^{-1}D[X]_{N_v}$, and so $(ID[X]_{N_v})_v = I_vD[X]_{N_v}$.*
(iii) $\mathrm{Max}\,(D[X]_{N_v}) = \{PD[X]_{N_v} \mid P \in t\text{-}\mathrm{Max}\,(D)\}$.

(iv) *I is t-invertible if and only if $ID[X]_{N_v}$ is invertible.*

Proof. (i) appears in [**2,** Lemma 2.1] and [**9,** Proposition 3.4]. For (ii), (iii) and (iv), see [**20,** Corollary 2.3 (3), Proposition 2.1, Corollary 2.5]. $\square$

We say that D is a *Prüfer $*$-multiplication domain* (P$*$MD) if each nonzero finitely generated ideal of D is $*_f$-invertible. Hence, PdMDs are just the Prüfer domains. Obviously, P$*$MD $\Leftrightarrow$ P$*_f$MD $\Leftrightarrow$ P$*_w$MD, because $(*_f)_f = *_f$, $(*_w)_f = *_w$, and $I \in \mathbf{F}(D)$ is $*_f$-invertible if and only if I is $*_w$-invertible. Also, if $*_1 \leq *_2$ are star operations on D, then P$*_1$MDs are P$*_2$MDs; thus, Prüfer domain $\Rightarrow$ P$*$MD $\Rightarrow$ PvMD for any star operations $*$ on D.

Theorem 1.4. *If D is integrally closed, the following statements are equivalent.*

(i) *D is a PvMD.*
(ii) *$v_c = w$.*
(iii) *w is an e.a.b. star operation.*
(iv) *$w = t$.*
(v) *$D[X]$ is a PvMD.*
(vi) *$D[X]_{N_v}$ is a Prüfer domain.*
(vii) *Each ideal of $D[X]_{N_v}$ is extended from D.*
(viii) *$fD[X]_{N_v} = c(f)D[X]_{N_v}$ for all $0 \neq f \in D[X]$.*
(ix) *D_P is a valuation domain for every maximal t-ideal P of D.*

Proof. (i) $\Leftrightarrow$ (ii) appears in [**4,** Corollary 3.8].

(i) $\Leftrightarrow$ (iii) was proved in [**8,** Theorem 3.1] (in a more general setting of semistar operations).

For (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (ix), see [**20,** Theorems 2.12, 3.1, 3.2, 3.5, 3.7]. $\square$

Remark 1.5. Let M be a maximal t-ideal of a PvMD. It is well known and easy to show that $M_v = M$ if and only if M is t-invertible. Also, if M is a maximal ideal of a Prüfer domain, then $M_v = M$ if and only if M is invertible. In this paper, we use this fact without any further comments.

The next simple result shows that $S_w(D) = S(D)$ if each maximal ideal of D is a t-ideal. Thus, if $|S(D)| < \infty$, then $S_w(D) = S(D)$ [**16,** Proposition 2.1]. This result also provides a characterization of DW-domains (i.e., integral domains in which the d-operation coincides with the w-operation) which were introduced and studied by Mimouni [**22**] and Picozza-Tartarone [**23**].

Proposition 1.6. *The following statements are equivalent.*

(i) $S_w(D) = S(D)$.
(ii) *Each maximal ideal of* D *is a* t*-ideal.*
(iii) $d = w$ *on* D.

Proof. (i) $\Leftrightarrow$ (iii). This follows directly from the fact that $d \leq *$ for any star operation $*$ on D.

(ii) $\Leftrightarrow$ (iii). Let $\text{Max}(D)$ be the set of maximal ideals of D. Clearly, each maximal ideal of D is a t-ideal if and only if $t\text{-Max}(D) = \text{Max}(D)$. Hence, if each maximal ideal of D is a t-ideal, then $I_d = \bigcap_{M \in \text{Max}(D)} ID_M = \bigcap_{M \in t\text{-Max}(D)} ID_M = I_w$ for all $I \in \mathbf{F}(D)$. Thus, $d = w$. Conversely, if $w = d$, then $t\text{-Max}(D) = w\text{-Max}(D) = \text{Max}(D)$, and thus each maximal ideal of D is a t-ideal. $\square$

2. Prüfer v-multiplication domains. Let D be an integral domain with quotient field K, $S(D)$ the set of star operations on D and $S_w(D) = \{* \in S(D) \mid w \leq *\}$. Let X be an indeterminate over D, $D[X]$ the polynomial ring over D and $N_v = \{f \in D[X] \mid c(f)_v = D\}$.

In this section, we show that if D is a PvMD, then $|S_w(D)| = |S_w(D[X])| = |S(D[X]_{N_v})|$. This will be proved by a series of lemmas (Lemmas 2.1–2.5).

Lemma 2.1. *Let* $* \in S_w(D)$ *and* $J \in \mathbf{F}(D)$. *If* J *is* w*-invertible, then* $(JI)^* = (JI^*)_w$ *for all* $I \in \mathbf{F}(D)$.

Proof. Since $w \leq *$, we have $(JI^*)_w \subseteq (JI^*)^* = (JI)^*$. For the reverse containment, note that $(JJ^{-1})^* = D$ because J is w-invertible and $w \leq *$; so $J^{-1}(JI)^* \subseteq (J^{-1}(JI)^*)^* = (J^{-1}JI)^* = ((JJ^{-1})^*I)^* = I^*$, and hence, $(J^{-1}(JI)^*)_w \subseteq (I^*)_w = I^*$. Thus, $(JI)^* \subseteq (JI^*)_w$ by the assumption that J is w-invertible and $w \leq *$. $\square$

Lemma 2.2. *Let D be a PvMD with quotient field K and $* \in S_w(D)$.*

(i) *If $A \in \mathbf{F}(D[X]_{N_v})$, then $A = ED[X]_{N_v}$ for some $E \in \mathbf{F}(D)$. Moreover, $A \cap K = E_w$.*

(ii) *For each $A \in \mathbf{F}(D[X]_{N_v})$, if we let*

$$A^{*_{N_v}} = E^* D[X]_{N_v},$$

where $E \in \mathbf{F}(D)$ with $A = ED[X]_{N_v}$ as in (i)*, then $*_{N_v}$ is a star operation on $D[X]_{N_v}$ such that $(ID[X]_{N_v})^{*_{N_v}} \cap K = I^*$ for all $I \in \mathbf{F}(D)$.*

Proof.

(i) Since $A \in \mathbf{F}(D[X]_{N_v})$, there is an $0 \neq f \in D[X]$ such that $fA \subseteq D[X]_{N_v}$; so $(c(f)D[X]_{N_v})A = fA = ID[X]_{N_v}$ for some $I \in \mathbf{F}(D)$ by Theorem 1.4. Hence, $A = c(f)^{-1}ID[X]_{N_v}$ because $c(f)D[X]_{N_v}$ is invertible. Thus, if we set $E = c(f)^{-1}I$, then $E \in \mathbf{F}(D)$ and $A = ED[X]_{N_v}$. Moreover, $A \cap K = ED[X]_{N_v} \cap K = E_w$ by Lemma 1.3 (i).

(ii) Clearly, (a) $(D[X]_{N_v})^{*_{N_v}} = D[X]_{N_v}$, (b) if $A, B \in \mathbf{F}(D[X]_{N_v})$, then $A \subseteq A^{*_{N_v}}$, and $A \subseteq B$ implies $A^{*_{N_v}} \subseteq B^{*_{N_v}}$, and (c) $(A^{*_{N_v}})^{*_{N_v}} = A^{*_{N_v}}$ for all $A \in \mathbf{F}(D[X]_{N_v})$. So, to prove that $*_{N_v}$ is a star operation, it suffices to show that, if $0 \neq f, g \in D[X]$ and $A \in \mathbf{F}(D[X]_{N_v})$, then $(f/gA)^{*_{N_v}} = f/gA^{*_{N_v}}$.

Note that $f/gD[X]_{N_v} = (c(f)c(g)^{-1})D[X]_{N_v}$ by Theorem 1.4 and $A = ED[X]_{N_v}$ for some $E \in \mathbf{F}(D)$ by (i); so $f/gA = (c(f)c(g)^{-1}E)D[X]_{N_v}$. Also, by Lemma 2.1, $(c(f)c(g)^{-1}E)^* = ((c(f)c(g)^{-1})E^*)_w$ because $c(f)c(g)^{-1}$ is w-invertible. Thus,

$$\begin{aligned}
\left(\frac{f}{g}A\right)^{*_{N_v}} &= (c(f)c(g)^{-1}E)^* D[X]_{N_v} \\
&= ((c(f)c(g)^{-1})E^*)_w D[X]_{N_v} \\
&= ((c(f)c(g)^{-1})E^*) D[X]_{N_v} \\
&= ((c(f)c(g)^{-1})D[X]_{N_v})(E^* D[X]_{N_v}) \\
&= \left(\frac{f}{g}D[X]_{N_v}\right)(ED[X]_{N_v})^{*_{N_v}} \\
&= \frac{f}{g}A^{*_{N_v}}.
\end{aligned}$$

Moreover, $(ID[X]_{N_v})^{*_{N_v}} \cap K = I^*D[X]_{N_v} \cap K = (I^*)_w = I^*$ because $w \leq *$. $\square$

Lemma 2.3. *Let D be a PvMD with quotient field K and $\star$ a star operation on $D[X]_{N_v}$. For each $I \in \mathbf{F}(D)$, if we set*

$$I^* = (ID[X]_{N_v})^\star \cap K,$$

then $*$ is a star operation on D such that $* \in S_w(D)$ and $*_{N_v} = \star$.

Proof. It is routine to check that $*$ is a star operation on D. Also, note that $E_w = ED[X]_{N_v} \cap K \subseteq (ED[X]_{N_v})^\star \cap K = E^*$ for every $E \in \mathbf{F}(D)$. Thus, $w \leq *$.

Next, to prove the equality $*_{N_v} = \star$, let $A \in \mathbf{F}(D[X]_{N_v})$. Then $A = ID[X]_{N_v}$ and $A^\star = JD[X]_{N_v}$ for some $I, J \in \mathbf{F}(D)$ by Lemma 2.2 (i). So $I^* = A^\star \cap K = JD[X]_{N_v} \cap K = J_w$, and thus $A^{*_{N_v}} = (ID[X]_{N_v})^{*_{N_v}} = I^*D[X]_{N_v} = J_wD[X]_{N_v} = JD[X]_{N_v} = A^\star$. Thus, $*_{N_v} = \star$. $\square$

We say that a nonzero prime ideal Q of $D[X]$ is an *upper to zero in* $D[X]$ if $Q \cap D = (0)$. It is known that an upper to zero Q in $D[X]$ is a maximal t-ideal if and only if Q is t-invertible, if and only if $Q \cap N_v \neq \emptyset$ [**19,** Theorem 1.4]. Also, D is a PvMD if and only if D is integrally closed and each upper to zero in $D[X]$ is a maximal t-ideal [**19,** Proposition 3.2]. We know that, if Q is a maximal t-ideal of $D[X]$, then either $Q \cap D = (0)$ or $Q \cap D \neq (0)$, $Q \cap D$ is a maximal t-ideal of D, and $Q = (Q \cap D)[X]$ (cf., [**19,** Proposition 1.1]). So, if D is a PvMD, then t-Max$(D[X]) = \{P[X] \mid P \in t\text{-Max}(D)\} \cup \{Q \mid Q$ is an upper to zero in $D[X]\}$, and hence $A_w = AD[X]_{N_v} \cap AK[X]$, $A_wD[X]_{N_v} = AD[X]_{N_v}$, and $A_wK[X] = AK[X]$ for all $A \in \mathbf{F}(D[X])$.

Lemma 2.4. *Let D be a PvMD with quotient field K and $* \in S_w(D)$.*

(i) *If $A \in \mathbf{F}(D[X])$, then $A_w = (Q_1^{k_1} \cdots Q_n^{k_n}(ED[X]))_w$ for some Q_i an upper to zero in $D[X]$, k_i a nonzero integer, and $E \in \mathbf{F}(D)$. Moreover, this expression is unique up to the w-operation on $D[X]$.*

(ii) *For each $A \in \mathbf{F}(D[X])$, if we let*

$$A^{*_{D[X]}} = (Q_1^{k_1} \cdots Q_n^{k_n}(E^*D[X]))_w,$$

where Q_i, k_i, and E are as in (i), *then* $*_{D[X]} \in S_w(D[X])$ *such that* $(ID[X])^{*_{D[X]}} \cap K = I^*$ *for all* $I \in \mathbf{F}(D)$.

Proof.

(i) Since $A \in \mathbf{F}(D[X])$, there is a $0 \neq f \in D[X]$ such that $fA \subseteq D[X]$. Recall that t-$\mathrm{Max}(D[X]) = \{P[X] \mid P \in t$-$\mathrm{Max}(D)\} \cup \{Q \mid Q$ is an upper to zero in $D[X]\}$ and that fA is contained in only finitely many uppers to zero in $D[X]$ (because $K[X]$ is a principal ideal domain (PID)), say, $q_1, \dots, q_k$. Note also that each upper to zero in $D[X]$ is t-invertible; hence, there are positive integers e_i such that, if we let $B = q_1^{e_1} \cdots q_k^{e_k}$, then B is t-invertible, $fAB^{-1} \subseteq D[X]$ and fAB^{-1} is not contained in any upper to zero in $D[X]$. Hence, $(fAB^{-1})_w = (fAB^{-1})_{N_v} \cap (fAB^{-1})K[X] = (fAB^{-1})_{N_v} \cap K[X] = ID[X]_{N_v} \cap K[X] = ID[X]_{N_v} \cap IK[X] = (ID[X])_w$ for some $I \in \mathbf{F}(D)$ by Lemma 2.2 (i), and thus $(fA)_w = (B(ID[X]))_w$. The same argument also shows that $fD[X] = (p_1^{n_1} \cdots p_s^{n_s}(JD[X]))_w$ for some p_i an upper to zero in $D[X]$, n_i a positive integer and $J \in \mathbf{F}(D)$. Clearly, J and $p_j^{n_j}$ are t-invertible, and thus $A_w = (q_1^{e_1} \cdots q_k^{e_k} \cdot p_1^{-n_1} \cdots p_s^{-n_s}(J^{-1}I)D[X])_w$.

For uniqueness up to the w-operation on $D[X]$, assume that

$$(Q_1^{k_1} \cdots Q_n^{k_n}(ID[X]))_w = (p_1^{e_1} \cdots p_m^{e_m}(JD[X]))_w,$$

where Q_i and p_j are uppers to zero in $D[X]$, k_i and e_j are nonzero integers, and $I, J \in \mathbf{F}(D)$. Then

$$\begin{aligned}(Q_1^{k_1} \cdots Q_n^{k_n})K[X] &= (Q_1^{k_1} \cdots Q_n^{k_n}(ID[X]))K[X] \\ &= (Q_1^{k_1} \cdots Q_n^{k_n}(ID[X]))_w K[X] \\ &= (p_1^{e_1} \cdots p_m^{e_m}(JD[X]))_w K[X] \\ &= (p_1^{e_1} \cdots p_m^{e_m})K[X].\end{aligned}$$

Note that $K[X]$ is a PID and both $Q_iK[X]$ and $p_jK[X]$ are prime ideals of $K[X]$; so the expression of $(Q_1^{k_1} \cdots Q_n^{k_n})K[X]$ is unique, and thus $n = m$, $Q_i = p_i$ and $e_i = k_i$ by rearranging the order of $p_i, \dots, p_m$ (if necessary). Also,

$$\begin{aligned}ID[X]_{N_v} &= (Q_1^{k_1} \cdots Q_n^{k_n}(ID[X]))D[X]_{N_v} \\ &= (Q_1^{k_1} \cdots Q_n^{k_n}(ID[X]))_w D[X]_{N_v}\end{aligned}$$

$$= (p_1^{e_1} \cdots p_m^{e_m}(JD[X]))_w D[X]_{N_v}$$
$$= JD[X]_{N_v}.$$

Thus, $I_w = ID[X]_{N_v} \cap K = JD[X]_{N_v} \cap K = J_w$.

(ii) For $I, J \in \mathbf{F}(D)$, if $I_w = J_w$, then $I^* = (I_w)^* = (J_w)^* = J^*$ because $w \leq *$. Hence, $*_{D[X]}$ is well defined by (i). Also, it is clear that $A_w \subseteq A^{*_{D[X]}}$ for all $A \in \mathbf{F}(D[X])$ and $(ID[X])^{*_{D[X]}} \cap K = I^*D[X] \cap K = I^*$ for all $I \in \mathbf{F}(D)$. So it suffices to show that $*_{D[X]}$ is a star operation on $D[X]$.

Let $A, B \in \mathbf{F}(D[X])$ and $0 \neq f, g \in D[X]$. By (i), $A_w = (A_1(E_1D[X]))_w$ and $B_w = (B_1(E_2D[X]))_w$, where both A_1 and B_1 are products of uppers to zero in $D[X]$ and $E_i \in \mathbf{F}(D)$. Clearly, $(D[X])^{*_{D[X]}} = D[X]$, $A \subseteq A^{*_{D[X]}}$ and $(A^{*_{D[X]}})^{*_{D[X]}} = A^{*_{D[X]}}$. By (i), $fD[X] = (C_1(I_1D[X]))_w$ and $gD[X] = (C_2(I_2D[X]))_w$, where C_i are products of uppers to zero in $D[X]$ and $I_i \in \mathbf{F}(D)$. Clearly, I_1 and I_2 are t-invertible. Also, $(f/gA)_w = (C_1C_2^{-1}A_1((I_1I_2^{-1}E_1)D[X]))_w$, where $C_1C_2^{-1}A_1$ is a product of uppers to zero in $D[X]$ and $I_1I_2^{-1}E_1 \in \mathbf{F}(D)$. Thus,

$$\begin{aligned}\left(\frac{f}{g}A\right)^{*_{D[X]}} &= (C_1C_2^{-1}A_1((I_1I_2^{-1}E_1)^*D[X]))_w \\ &= (C_1C_2^{-1}A_1((I_1I_2^{-1}(E_1)^*)_w D[X]))_w \\ &= (C_1C_2^{-1}A_1((I_1I_2^{-1}(E_1)^*)D[X]))_w \\ &= ((C_1(I_1D[X]))(C_2(I_2D[X]))^{-1}(A_1((E_1)^*D[X])))_w \\ &= \left(\frac{f}{g}(A_1((E_1)^*D[X]))\right)_w \\ &= \frac{f}{g}A^{*_{D[X]}},\end{aligned}$$

where the second equality follows from Lemma 2.1. Finally, assume that $A \subseteq B$. Then $A_w \subseteq B_w$, and so $(E_1D[X])_{N_v} = (A_w)_{N_v} \subseteq (B_w)_{N_v} = (E_2D[X])_{N_v}$. Thus, $(A_1((E_1)^*D[X]))_{N_v} = ((E_1)^*D[X])_{N_v} \subseteq ((E_2)^*D[X])_{N_v} = (B_1((E_2)^*D[X]))_{N_v}$ by Lemma 2.2 (ii). Also,

$$\begin{aligned}(A_1((E_1)^*D[X]))K[X] &= (A_1)K[X] \\ &= (A_1(E_1D[X]))K[X] \\ &= (A_1(E_1D[X]))_w K[X]\end{aligned}$$

$$\begin{aligned} &\subseteq (B_1(E_2D[X]))_w K[X] \\ &= (B_1((E_2)^* D[X]))K[X]. \end{aligned}$$

Thus,

$$\begin{aligned} A^{*D[X]} &= (A_1((E_1)^* D[X]))_w \\ &= (A_1((E_1)^*)D[X]))_{N_v} \cap (A_1((E_1)^* D[X]))K[X] \\ &\subseteq (B_1((E_2)^* D[X]))_{N_v} \cap (B_1((E_2)^* D[X]))K[X] \\ &= (B_1((E_2)^* D[X]))_w \\ &= B^{*_{D[X]}}. \end{aligned}$$ □

Lemma 2.5. *Let D be a PvMD with quotient field K and $\star \in S_w(D[X])$. For each $E \in \mathbf{F}(D)$, if we set*

$$E^* = (ED[X])^\star \cap K,$$

then $* \in S_w(D)$ and $*_{D[X]} = \star$.

Proof. It is routine to check that $*$ is a star operation on D (or see [**15,** Proposition 2.1]). Moreover, since $w \le \star$ on $D[X]$, $E_w = E_w D[X] \cap K = (ED[X])_w \cap K \subseteq (ED[X])^\star \cap K = E^*$ for all $E \in \mathbf{F}(D)$. Thus, $* \in S_w(D)$.

Next, if $A \in \mathbf{F}(D[X])$, then $A_w = (B(ED[X]))_w$ for some B a product of uppers to zero in $D[X]$ and $E \in \mathbf{F}(D)$ by Lemma 2.4 (i), and so $(B^{-1}A)_w = (ED[X])_w = E_w D[X]$. By Lemma 2.1, $(B^{-1}A^\star)_w = (B^{-1}A)^\star = (ED[X])^\star$ and $(B^{-1}A)^\star \cap K = E^*$. Note that $(B^{-1}A)^\star \in \mathbf{F}(D[X])$ and $E_w D[X] \subseteq (B^{-1}A)^\star \subseteq E_v D[X]$, and hence $(B^{-1}A)^\star = (JD[X])_w = J_w D[X]$ for some $J \in \mathbf{F}(D)$ by Lemma 2.4 (i). So $E^* = (B^{-1}A)^\star \cap K = J_w D[X] \cap K = J_w$, and thus $(B^{-1}A)^\star = E^* D[X]$. Therefore, $A^\star = (BB^{-1}A^\star)_w = (B(B^{-1}A^\star)_w)_w = (B(B^{-1}A)^\star)_w = (B(E^*D[X]))_w = (B(ED[X]))^{*_{D[X]}} = A^{*_{D[X]}}$. Thus, $\star = *_{D[X]}$. □

The next result is the main result of this paper, which is crucial for studying integrally closed domains D with $|S_w(D)| < \infty$. Its proof is now a straightforward consequence of the previous preliminary lemmas.

Theorem 2.6. *Let D be a PvMD.*

(i) *The map* $* \mapsto *_{N_v}$ *of* $S_w(D)$ *into* $S(D[X]_{N_v})$ *is bijective and* $S(D[X]_{N_v}) = \{*_{N_v} \mid * \in S_w(D)\}$.
(ii) *The map* $* \mapsto *_{D[X]}$ *of* $S_w(D)$ *into* $S_w(D[X])$ *is bijective and* $S_w(D[X]) = \{*_{D[X]} \mid * \in S_w(D)\}$.

Hence, $|S_w(D)| = |S_w(D[X])| = |S(D[X]_{N_v})|$.

Proof.

(i) This follows directly from Lemmas 2.2 and 2.3.
(ii) This is an immediate consequence of Lemmas 2.4 and 2.5. $\square$

Corollary 2.7. *If D is a Prüfer domain, then* $|S(D)| = |S(D(X))|$.

Proof. This follows from Theorem 2.6 and Proposition 1.6 because Prüfer domains are PvMDs in which each maximal ideal is a t-ideal. $\square$

Remark 2.8.

(1) Let $*$ be a star operation on D and, for $A \in \mathbf{F}(D[X])$, let

$$A^{\blacktriangle^*} = \bigcap \left\{ z^{-1} \left(\sum_{f \in zA} c(f) \right)^* [X] \mid 0 \neq z \in (K[X] : A) \right\}.$$

Then $\blacktriangle^*$ is a star operation on $D[X]$ such that $(E[X])^{\blacktriangle^*} = E^*[X]$ for all $E \in \mathbf{F}(D)$, and, moreover, if $* \in S_w(D)$, then $\blacktriangle^* \in S_w(D[X])$ [**5,** Theorem 2.1 and Corollary 2.4] (or see the proof of [**6,** Proposition 3.4]). Clearly, if D is an integral domain such that $t = v$ on D but $t \neq v$ on $D[X]$, then $S_w(D[X]) \neq \{\blacktriangle^* \mid * \in S_w(D)\}$ (see [**5,** Remark 2.7 (b)]). Hence, in general, Theorem 2.6 does not hold.

(2) A *strong Mori domain* (SM domain) is an integral domain that satisfies the ascending chain condition on integral w-ideals. It was proved in [**6,** Theorem 3.2 and Corollary 3.21] that, if D is an SM domain with $|S_w(D[X])| < \infty$, then $|S_w(D)| = |S_w(D[X])| = |S(D[X]_{N_v})|$, while there is a one-dimensional local Noetherian domain D (hence an SM domain on which $d = w$) such that $|S(D)| \lneq |S(D(X))| = \infty$ [**6,** Example 3.7].

We say that D is of *finite character* (respectively, *finite t-character*) if each nonzero element of D is contained in only finitely many maximal ideals (respectively, maximal t-ideals) of D. We say that D is *h-local* if D is of finite character and each nonzero prime ideal is contained in a unique maximal ideal. The D is called an *independent ring of Krull type* if D is a PvMD of finite t-character and each nonzero prime t-ideal is contained in a unique maximal t-ideal. Clearly, an h-local Prüfer domain is an independent ring of Krull type, and conversely, an independent ring of Krull type whose maximal ideals are t-ideals is an h-local Prüfer domain.

Lemma 2.9. *The following statements are equivalent.*

(i) *D is an independent ring of Krull type.*
(ii) *$D[X]$ is an independent ring of Krull type.*
(iii) *$D[X]_{N_v}$ is an h-local Prüfer domain.*

Proof. This follows directly from Theorem 1.4 and [**10,** Corollary 2.3]. □

Let D be an h-local Prüfer domain. It is known that, if $\mathcal{U}$ is the set of maximal ideals of D that are not v-ideals, then $|S(D)| = 2^{|\mathcal{U}|}$ [**16,** Theorem 3.1]. We next give the independent ring of Krull type analog of this result.

Corollary 2.10. *Let D be an independent ring of Krull type. If $\mathfrak{U}$ is the set of maximal t-ideals of D that are not v-ideals, then $|S_w(D)| = |S_w(D[X])| = |S(D[X]_{N_v})| = 2^{|\mathfrak{U}|}$.*

Proof. By Lemma 2.9, $D[X]_{N_v}$ is an h-local Prüfer domain. Note that each maximal ideal of $D[X]_{N_v}$ is of the form $P[X]_{N_v}$ for some maximal t-ideal P of D and $(P[X]_{N_v})_v = P_v[X]_{N_v}$ for all nonzero prime ideals P of D by Lemma 1.3; so $P[X]_{N_v}$ is a v-ideal if and only if P is a v-ideal. Thus, $|S_w(D)| = |S_w(D[X])| = |S(D[X]_{N_v})| = 2^{|\mathfrak{U}|}$ by [**16,** Theorem 3.1] and Theorem 2.6. □

Corollary 2.10 shows that, if D is an independent ring of Krull type, then $2^{|\mathfrak{U}|} = |S_w(D)|$. We next show that $2^{|\mathfrak{U}|} \leq |S_w(D)|$ for any integral domain D.

Proposition 2.11. *If $\mathfrak{U}$ is the set of maximal t-ideals of an integral domain D that are not v-ideals, then $2^{|\mathfrak{U}|} \leq |S_w(D)|$. Hence, if $|S_w(D)| < \infty$, then $|\mathfrak{U}| < \infty$.*

Proof. For each $P \in \mathfrak{U}$, if we set

$$E^{*_P} = (P : (P : E))$$

for all $E \in \mathbf{F}(D)$, then $*_P$ is a star operation [**14,** Proposition 3.2]. Note that $x \in (P : E) \Leftrightarrow xE \subseteq P \Rightarrow xE_w \subseteq P_w = P \Leftrightarrow x \in (P : E_w)$; so $(P : E) \subseteq (P : E_w)$. Hence, $(P : E) = (P : E_w)$, and thus $E^{*_P} = (E_w)^{*_P}$. Thus, $w \leq *_P$. Note also that, if $Q \in \mathfrak{U}$ with $P \neq Q$, then $(P : Q) = P$ because $P_v = Q_v = D$, and thus $Q^{*_P} = (P : P) = D \neq Q = Q^{*_Q}$.

Next, for $\emptyset \neq \Delta \subseteq \mathfrak{U}$, let $E^{*_\Delta} = \bigcap_{P \in \Delta} E^{*_P}$ for all $E \in \mathbf{F}(D)$. Clearly, $*_\Delta$ is a star operation on D with $w \leq *_\Delta$ by the previous paragraph. Let Δ_1 and Δ_2 be two distinct nonempty subsets of $\mathfrak{U}$, say, $\Delta_1 \nsubseteq \Delta_2$, and choose $P \in \Delta_1 \setminus \Delta_2$. Then $P^{*_{\Delta_1}} = P \neq D = P^{*_{\Delta_2}}$ by the previous paragraph, and hence $*_{\Delta_1} \neq *_{\Delta_2}$. This also shows that $*_\Delta \neq v$ for every $\emptyset \neq \Delta \subseteq \mathfrak{U}$. Thus, $2^{|\mathfrak{U}|} = |\{v\} \cup \{*_\Delta \mid \emptyset \neq \Delta \subseteq \mathfrak{U}\}| \leq |S_w(D)|$. $\square$

3. Integrally closed domains D with $|S_w(D)| < \infty$. Throughout, D denotes an integral domain with quotient field K, $S(D)$ (respectively, $SF(D)$) be the set of star operations (respectively, star operations of finite type) on D, $S_w(D) = \{* \in S(D) \mid w \leq *\}$, and $SF_w(D) = S_w(D) \cap SF(D)$.

In this section, we study an integrally closed domain D with $|S_w(D)| < \infty$. First, in Corollaries 3.1 and 3.2, we give some characterizations of the integrally closed domains D with $|S_w(D)| \leq 2$.

Corollary 3.1. *If D is integrally closed, the following statements are equivalent.*

(i) $|S_w(D)| = 1$.
(ii) $v_c = v$.
(iii) *D is a PvMD on which $t = v$.*
(iv) *D is an independent ring of Krull type whose maximal t-ideals are t-invertible.*

(v) $|S_w(D[X])| = 1$.
(vi) $D[X]$ *is a PvMD on which* $t = v$.
(vii) $|S(D[X]_{N_v})| = 1$.
(viii) $D[X]_{N_v}$ *is an h-local Prüfer domain whose maximal ideals are invertible.*

Proof. (i) $\Rightarrow$ (ii) is clear because $w \le v_c$ by Lemma 1.1.

(ii) $\Rightarrow$ (iii). If $v_c = v$, then $t = v$ and v is an *e.a.b.* star operation, and hence each nonzero finitely generated ideal of D is t-invertible [**11,** Theorem 34.6]. Thus, D is a PvMD.

(iii) $\Rightarrow$ (i). If D is a PvMD, then $w = t$ by Theorem 1.4, and thus $w = v$.

(iii) $\Leftrightarrow$ (iv). [**18,** Theorem 3.1].

(iii) $\Leftrightarrow$ (vi). Note that, if D is integrally closed, then $t = v$ on D if and only if $t = v$ on $D[X]$ [**18,** Proposition 4.6]. Thus, the result follows from Theorem 1.4.

(iv) $\Leftrightarrow$ (viii). This is an immediate consequence of Lemmas 1.3 and 2.9.

(v) $\Leftrightarrow$ (vi). This follows from the equivalence of (i) and (iii) because $D[X]$ is integrally closed.

(vii) $\Leftrightarrow$ (viii). Clearly, $D[X]_{N_v}$ is integrally closed. Thus, the result follows directly from Heinzer's result [**13,** Theorem 5.1]. $\square$

The following corollary is the t-operation version of [**16,** Theorem 3.3] that $|S(D)| = 2$ if and only if D is an h-local Prüfer domain with exactly one non-invertible maximal ideal.

Corollary 3.2. *If D is integrally closed, the following statements are equivalent.*

(i) $|S_w(D)| = 2$.
(ii) D *is an independent ring of Krull type with exactly one nondivisorial maximal* t*-ideal.*
(iii) $|S_w(D[X])| = 2$.
(iv) $D[X]$ *is an independent ring of Krull type with exactly one nondivisorial maximal* t*-ideal.*
(v) $|S(D[X]_{N_v})| = 2$.

(vi) $D[X]_{N_v}$ *is an h-local Prüfer domain with exactly one nondivisorial maximal ideal.*

Proof. (i) $\Rightarrow$ (v). Note that $w, v_c, t, v \in S_w(D)$ and $w \leq v_c \leq t \leq v$; so, if $|S_w(D)| = 2$, then either $w = v_c$ or $v_c = t = v$. But, if $v_c = v$, then $|S_w(D)| = 1$ by Corollary 3.1. So $w = v_c$, and hence D is a PvMD by Theorem 1.4. Thus, $|S(D[X]_{N_v})| = 2$ by Theorem 2.6.

(v) $\Leftrightarrow$ (vi). Clearly, $D[X]_{N_v}$ is integrally closed. Thus, $|S(D[X]_{N_v})| = 2$ if and only if $D[X]_{N_v}$ is an h-local Prüfer domain with exactly one nondivisorial maximal ideal [**16,** Theorem 3.3].

(vi) $\Leftrightarrow$ (ii). This is an immediate consequence of Lemmas 2.9 and 1.3 (ii).

(ii) $\Rightarrow$ (i). This follows from Corollary 2.10.

(ii) $\Leftrightarrow$ (iv). Let Q be a maximal t-ideal of $D[X]$. If $Q \cap D = (0)$, then Q is t-invertible [**19,** Theorem 1.4], and hence Q is a v-ideal. If $Q \cap D \neq (0)$, then $Q \cap D$ is a maximal t-ideal of D and $Q = (Q \cap D)[X]$ [**19,** Proposition 1.1]. Also, recall that $((Q \cap D)[X])_v = (Q \cap D)_v[X]$; so Q is a v-ideal if and only if $Q \cap D$ is a v-ideal. Thus, the result follows directly from Lemma 2.9.

(iii) $\Leftrightarrow$ (iv). This follows from the equivalence of (i) and (ii) because $D[X]$ is integrally closed. $\square$

Remark 3.3. If D is an independent ring of Krull type with $|S_w(D)| < \infty$, then $|S_w(D)| = 2^n$ for some integer $n \geq 0$ by Corollary 2.10. But, if we let $D = \mathbb{Z}_{2\mathbb{Z} \cup 3\mathbb{Z}} + X\mathbb{Q}[\![X]\!]$, where $\mathbb{Q}[\![X]\!]$ is the ring of a formal power series over $\mathbb{Q}$, then D is a Prüfer domain with $|S(D)| = 4$, but D is not h-local [**16,** Example 3.7]. Hence, if $|S_w(D)| \geq 3$, then D need not be an independent ring of Krull type even though $|S_w(D)| = 2^n$ for an integer $n \geq 0$.

The next result is the PvMD analog of [**17,** Theorem 3.1] that, if D is integrally closed, then $|SF(D)| = 1$ if and only if $|SF(D)| < \infty$, if and only if D is a Prüfer domain. The proof is a simple modification of that of [**17,** Theorem 3.1].

Proposition 3.4. *The following statements are equivalent for an integrally closed domain D.*

(i) $|SF_w(D)| = 1$.
(ii) $|SF_w(D)| < \infty$.
(iii) *D is a PvMD.*

Proof. (i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (iii). Let P be a maximal t-ideal of D and $0 \neq \alpha \in K$. Let $m \geq n \geq 1$ be integers. If $x \in (1, \alpha^m)^{-1}$, then $x, x\alpha^m \in D$, and so $(x\alpha^n)^m = x^{m-n}(x\alpha^m)^n \in D$. Since D is integrally closed, $x\alpha^n \in D$, and hence $x \in (1, \alpha^n)^{-1}$. Hence, $(1, \alpha^m)^{-1} \subseteq (1, \alpha^n)^{-1}$, and thus, $\alpha^n \in (1, \alpha^n)_v \subseteq (1, \alpha^m)_v = (1, \alpha^m)_t$.

Next, for each integer $n \geq 1$, if we set

$$E^{*_n} = ED_P[\alpha^n] \cap E_t$$

for all $E \in \mathbf{F}(D)$, then $*_n$ is a star operation of finite type [**16,** Proposition 2.7]. Also, since $E_w \subseteq E_w D_P \cap E_t = ED_P \cap E_t \subseteq ED_P[\alpha^n] \cap E_t = E^{*_n}$, we have $w \leq *_n$. Hence, by (ii), there are integers $m \gneq n \geq 1$ such that $*_m = *_n$. Note that $\alpha^n \in (1, \alpha^m)D_P[\alpha^n] \cap (1, \alpha^m)_t = (1, \alpha^m)^{*_n}$. So $\alpha^n \in (1, \alpha^m)^{*_m} \subseteq (1, \alpha^m)D_P[\alpha^m]$, and thus, $\alpha^n = f(\alpha^m) + \alpha^m g(\alpha^m)$ for some polynomials $f, g \in D_P[X]$. So if we let $h(X) = f(X^m) + X^m g(X^m) - X^n$, then $h \in D_P[X] \setminus PD_P[X]$ and $h(\alpha) = 0$, and thus α or α^{-1} is in D_P [**11,** Lemma 19.14]. Hence, D_P is a valuation domain. Therefore, D is a PvMD by Theorem 1.4.

(iii) $\Rightarrow$ (i). If D is a PvMD, then $t = w$ by Theorem 1.4, and since t is the largest star operation of finite type, we have $|SF_w(D)| = 1$. □

Corollary 3.5. *Let D be integrally closed and $\mathfrak{U}$ the set of maximal t-ideals of D that are not v-ideals. If $|S_w(D)| < \infty$, then D is a PvMD and $2^{|\mathfrak{U}|} \leq |S_w(D)| < \infty$.*

Proof. This follows directly from Propositions 2.11 and 3.4. □

An integral domain is called a *Mori domain* if it satisfies the ascending chain condition on integral v-ideals. Hence, Noetherian domains, SM domains and Krull domains are Mori domains. Also, a Mori domain is a Krull domain if and only if it is a PvMD [**21,** Theorem 3.2].

Corollary 3.6. *If D is an integrally closed Mori domain with $|S_w(D)| < \infty$, then D is a Krull domain, and hence $|S_w(D)| = 1$.*

Proof. By Proposition 3.4, D is a PvMD, and, since D is a Mori domain, D is a Krull domain and $t = v$. Note that $w = t$ on PvMDs. Thus, $w = v$. □

We next give a complete characterization of integrally closed domains D with $|S_w(D)| < \infty$. This result can be proved by using Theorem 2.6 and Houston, Mimouni and Park's result [**17,** Theorem 5.3]; so we first recall their result.

Definition 3.7. ([**17,** Definition and Notation 3.4]). Let D be a Prüfer domain that is not a field. Two maximal ideals M, N of D are said to be *dependent* if $M \cap N$ contains a nonzero prime ideal. This defines an equivalent relation on $\mathrm{Max}(D)$, the set of maximal ideals of D. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be the corresponding partition of $\mathrm{Max}(D)$; and, for each $\lambda \in \Lambda$, let P_λ be the largest prime ideal of D contained in $\bigcap_{M \in A_\lambda} M$, and set $S_\lambda = \bigcap_{M \in A_\lambda} D_M$.

Theorem 3.8. ([**17,** Theorem 5.3]). *The following statements are equivalent for an integrally closed domain D that is not a field.*

(i) $|S(D)| < \infty$.
(ii) *D is a Prüfer domain satisfying the following conditions*:
 (a) *D is of finite character;*
 (b) *$|A_\lambda| = 1$ for almost all $\lambda \in \Lambda$;*
 (c) *$|\mathrm{Spec}\,(D/P_\lambda)| < \infty$ for all $\lambda \in \Lambda$;*
 (d) *D has only finitely many nondivisorial maximal ideals.*

Moreover, under the above equivalent conditions,

$$|S(D)| = \prod_{\lambda \in \Lambda} |S(S_\lambda)|.$$

We next need the PvMD analog of Definition 3.7.

Definition 3.9. Let D be a PvMD that is not a field. For two maximal t-ideals M, N of D, we mean by $M \sim N$ that $M \cap N$ contains a nonzero prime ideal. Clearly, $\sim$ is an equivalent relation on t-$\mathrm{Max}(D)$. Let $\{B_\alpha\}_{\alpha \in \Theta}$ be the corresponding partition of t-$\mathrm{Max}(D)$; and, for each $\alpha \in \Theta$, let P_α be the largest prime ideal of D contained in $\bigcap_{M \in B_\alpha} M$ and set $T_\alpha = \bigcap_{M \in B_\alpha} D_M$.

Theorem 3.10. *The following statements are equivalent for an integrally closed domain D that is not a field.*

(i) $|S_w(D)| < \infty$.
(ii) *D is a PvMD of finite t-character such that*
  (a) $|B_\alpha| = 1$ *for almost all* $\alpha \in \Theta$,
  (b) *the number of prime t-ideals of D containing P_α is finite for all* $\alpha \in \Theta$,
  (c) *D has only finitely many nondivisorial maximal t-ideals.*
(iii) $|S_w(D[X])| < \infty$.
(iv) $|S(D[X]_{N_v})| < \infty$.

Moreover, in this case, each T_α is a Prüfer domain with a finite number of maximal ideals and $|S_w(D)| = \prod_{\alpha\in\Theta} |S(T_\alpha)|$.

Proof. (i) $\Leftrightarrow$ (ii). If $|S_w(D)| < \infty$, then D is a PvMD by Proposition 3.4, and so we may assume that D is a PvMD, and hence $D[X]_{N_v}$ is a Prüfer domain with $|S_w(D)| = |S(D[X]_{N_v})|$ by Theorems 1.4 and 2.6. Thus, the result follows from Theorem 3.8 because (1) $\mathrm{Spec}(D[X]_{N_v}) = \{P[X]_{N_v} \mid P = (0)$ or P is a t-ideal of $D\}$, (2) $\mathrm{Max}(D[X]_{N_v}) = \{P[X]_{N_v} \mid P \in t\text{-}\mathrm{Max}(D)\}$ and (3) $(P[X]_{N_v})_v = P_v[X]_{N_v}$.

(i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). By Theorem 2.6, it suffices to show that D is a PvMD. Note that $D[X]$ and $D[X]_{N_v}$ are integrally closed. Thus D is a PvMD by Proposition 3.4 (respectively, Theorems 3.8 and 1.4) if $|S_w(D)| < \infty$ (respectively, $|S_w(D[X])| < \infty$ or $|S(D[X]_{N_v})| < \infty$).

For the "moreover" part, note that $|B_\alpha| < \infty$ for all $\alpha \in \Theta$ because D is of finite t-character. So, if we let $B_\alpha = \{M_1, \ldots, M_k\}$, then T_α is a finite intersection of valuation domains D_{M_i}. Hence, T_α is a Prüfer domain with maximal ideals $M_iD_{M_i} \cap T_\alpha$ **[11,** Theorem 22.8], and so $\bigcap_{M\in B_\alpha} (D[X]_{N_v})_{M[X]_{N_v}} = \bigcap_{i=1}^k D[X]_{M_i[X]} = \bigcap_{i=1}^k D_{M_i}(X) = T_\alpha(X)$. Thus, by Theorems 2.6 and 3.8, $|S_w(D)| = |S(D[X]_{N_v})| = \prod_{\alpha\in\Theta} |S(T_\alpha(X))| = \prod_{\alpha\in\Theta} |S(T_\alpha)|$. $\square$

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REFERENCES

1. D.D. Anderson and S.J. Cook, *Two star-operations and their induced lattices*, Comm. Alg. **28** (2000), 2461–2475.

2. G.W. Chang, *Strong Mori domains and the ring* $D[X]_{N_v}$, J. Pure Appl. Alg. **197** (2005), 293–304.

3. ______, $*$*-Noetherian domains and the ring* $D[X]_{N_*}$, J. Alg. **297** (2006), 216–233.

4. ______, *Prüfer* $*$*-multiplication domains, Nagata rings, and Kronecker function rings*, J. Alg. **319** (2008), 309–319.

5. G.W. Chang, M. Fontana and M.H. Park, *Polynomial extensions of semistar operations*, J. Alg. **390** (2013), 250–263.

6. G.W. Chang, E. Houston and M.H. Park, *Star operations on strong Mori domains*, Houston J. Math.. to appear.

7. S. El Baghdadi and S. Gabelli, w*-Divisorial domains*, J. Alg. **285** (2005), 335–355.

8. M. Fontana, P. Jara and E. Santos, *Prüfer* $*$*-multiplication domains and semistar operations*, J. Alg. Appl. **2** (2003), 21–50.

9. M. Fontana and K.A. Loper, *Nagata rings, Kronecker function rings and related semistar operations*, Comm. Alg. **31** (2003), 4775–4805.

10. S. Gabelli, E. Houston and G. Picozza, w*-Divisoriality in polynomial rings*, Comm. Alg. **37** (2009), 1117–1127.

11. R. Gilmer, *Multiplicative ideal theory*, Queen's Papers Pure Appl. Math. **90**, Queen's University, Kingston, Ontario, 1992.

12. J. Hedstrom and E. Houston, *Some remarks on star-operations*, J. Pure Appl. Alg. **18** (1980), 37–44.

13. W. Heinzer, *Integral domains in which each nonzero ideal is divisorial*, Mathematika **15** (1968), 164–170.

14. W. Heinzer, J.A. Huckaba and I. Papick, m*-canonical ideals in integral domains*, Comm. Alg. **26** (1998), 3021–3043.

15. E. Houston, S. Malik and J. Mott, *Characterizations of* $*$*-multiplication domains*, Canad. Math. Bull. **27** (1984), 48–52.

16. E. Houston, A. Mimouni and M.H. Park, *Integral domains which admit at most two star operations*, Comm. Alg. **39** (2011), 1907–1921.

17. ______, *Integrally closed domains with only finitely many star operations*, Comm. Alg. **42** (2014), 5264–5286.

18. E. Houston and M. Zafrullah, *Integral domains in which each* t*-ideal is divisorial*, Michigan Math. J. **35** (1988), 291–300.

19. ______, *On* t*-invertibility* II, Comm. Alg. **17** (1989), 1955–1969.

20. B.G. Kang, *Prüfer* v*-multiplication domains and the ring* $R[X]_{N_v}$, J. Alg. **123** (1989), 151–170.

21. ______, *On the converse of a well-known fact about Krull domains*, J. Alg. **124** (1989), 284–299.

22. A. Mimouni, *Integral domains in which each ideal is a w-ideal*, Comm. Alg. **33** (2005), 1345–1355.

23. G. Picozza and F. Tartarone, *When the semistar operation $\tilde{\star}$ is the identity*, Comm. Alg. **36** (2008), 1954–1975.

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