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THE DEGREE OF THE ALGEBRA OF COVARIANTS OF A BINARY FORM

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ABSTRACT. We calculate the degree of the algebra of covariants $\mathcal{C}_d$ for binary d-forms. We obtain the integral representation and asymptotic behavior of the degree.

1. Introduction. Let $R = R_0 \oplus R_1 \oplus \cdots$, $R_0 = \mathbb{C}$, be a finitely generated graded commutative $\mathbb{C}$-algebra without zero divisors. Denote by

$$\mathcal{P}(R, z) = \sum_{j=0}^{\infty} \dim R_j z^j,$$

its Poincaré series. Letting r be the transcendence degree of the quotient field of R over $\mathbb{C}$, the number

$$\deg(R) := \lim_{z \to 1} (1 - z)^r \mathcal{P}(R, z),$$

is called the degree of the algebra R. The first two terms of the Laurent series expansion of $\mathcal{P}(R, z)$ at the point $z = 1$ have the following form

$$\mathcal{P}(R, z) = \frac{\deg(R)}{(1 - z)^r} + \frac{\psi(R)}{(1 - z)^{r-1}} + \cdots .$$

The numbers $\deg(R)$ and $\psi(R)$ are important characteristics of the algebra R. For instance, if R is an algebra of invariants of a finite group G, then $\deg(R)^{-1}$ is an order of the group G and $2\frac{\psi(R)}{\deg(R)}$ is the number of pseudo-reflections in G, see **[2]**.

Let V_d be the standard $(d+1)$-dimensional complex representation of SL_2, and let $\mathcal{I}_d := \mathbb{C}[V_d]^{SL_2}$ be the corresponding algebra of invariants. In the language of classical invariant theory the algebra $\mathcal{I}_d$ is called the

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algebra of invariants for binary forms of degree d. The following explicit formula for the degree $\deg(\mathcal{I}_d)$ was derived by Hilbert in [**4**]:

$$\deg(\mathcal{I}_d) = \begin{cases} -\dfrac{1}{4d!} \displaystyle\sum_{0\le e<d/2} (-1)^e \binom{d}{e} \left(\frac{d}{2}-e\right)^{d-3}, & \text{if } d \text{ is odd,} \\ -\dfrac{1}{2d!} \displaystyle\sum_{0\le e<d/2} (-1)^e \binom{d}{e} \left(\frac{d}{2}-e\right)^{d-3}, & \text{if } d \text{ is even.} \end{cases}$$

In [**6, 7**], Springer obtained two different proofs of this result. Also, he found an integral representation and the asymptotic behavior for Hilbert's constants. For this purpose, Springer [**7**] derived an explicit formula for the Poincaré series $\mathcal{P}(\mathcal{I}_d, z)$.

Let $\mathcal{C}_d$ be the algebra of the covariants of binary d-forms, i.e., $\mathcal{C}_d \cong \mathbb{C}[V_1 \oplus V_d]^{SL_2}$. In the present paper, acting in the spirit of Springer's papers, we calculate $\deg(\mathcal{C}_d)$ and $\psi(\mathcal{C}_d)$. The following formulas hold:

$$\deg(\mathcal{C}_d) = \frac{1}{d!} \sum_{0\le j<d/2} (-1)^j \binom{d}{j} \left(\frac{d}{2}-j\right)^{d-1}$$

and

$$\psi(\mathcal{C}_d) = \frac{1}{2} \deg(\mathcal{C}_d).$$

Also, we calculate both an integral representation and the asymptotic behavior of the constants. For this purpose we use the explicit formula for the Poincaré series $\mathcal{P}(\mathcal{C}_d, z)$ derived by the first author in [**1**].

2. Computation of $\deg(\mathcal{C}_d)$**.** The algebra of covariants $\mathcal{C}_d$ is a finitely generated graded algebra

$$\mathcal{C}_d = (\mathcal{C}_d)_0 \oplus (\mathcal{C}_d)_1 \oplus \cdots \oplus (\mathcal{C}_d)_i \oplus \cdots ,$$

where the subspaces $(\mathcal{C}_d)_i$ of covariants of degree i are each finite-dimensional, and $(\mathcal{C}_d)_0 \cong \mathbb{C}$. The formal power series

$$\mathcal{P}(\mathcal{C}_d, z) = \sum_{i=0}^{\infty} \dim((\mathcal{C}_d)_i) z^i,$$

is called the Poincaré series of the algebra of covariants $\mathcal{C}_d$. The finite generation of $\mathcal{C}_d$ implies that its Poincaré series is the power series expansion of a rational function.

The following theorem shows an explicit form for this rational function. Let φ_n, $n \in \mathbb{N}$, be the linear operator that transforms a rational function f in z to a rational function $\varphi_n(f)$ which is defined on the power z^n by

$$(\varphi_n(f))\,(z^n) = \frac{1}{n}\sum_{j=0}^{n-1} f(\zeta_n^j z), \quad \zeta_n = e^{2\pi i/n}.$$

Theorem 2.1 ([**1**]). *The Poincaré series $\mathcal{P}(\mathcal{C}_d, z)$ has the following form*:

$$\mathcal{P}(\mathcal{C}_d, z) = \sum_{0\le j<d/2} \varphi_{d-2j}\left(\frac{(-1)^j z^{j(j+1)}(1+z)}{(z^2,z^2)_j\,(z^2,z^2)_{d-j}}\right),$$

where $(a,q)_n = (1-a)(1-a\,q)\cdots(1-a\,q^{n-1})$ denotes the q-shifted factorial.

It is well known that the transcendence degree of the quotient field for the algebra of covariants $\mathcal{C}_d$ over $\mathbb{C}$ coincides with the order of pole $z=1$ for the rational function $\mathcal{P}(\mathcal{C}_d, z)$ and equals d. Therefore, the first terms of the Laurent series for $\mathcal{P}(\mathcal{C}_d, z)$ at the point $z=1$ are

$$\mathcal{P}(\mathcal{C}_d, z) = \frac{\deg(\mathcal{C}_d)}{(1-z)^d} + \frac{\psi(\mathcal{C}_d)}{(1-z)^{d-1}} + \cdots.$$

In order to calculate the rational coefficients $\deg(\mathcal{C}_d)$ and $\psi(\mathcal{C}_d)$ we shall prove several auxiliary facts.

Lemma 2.2. *The following statements hold*:

(i) *the first terms of the Taylor series for the function $(z^2,z^2)_j$ at $z=1$ are $(z^2,z^2)_j = 2^j j!(1-z)^j - 2^{j-1}j!\,j^2(1-z)^{j+1} + \cdots$;*

(ii) *the first terms of the Laurent series for the function*

$$\frac{(-1)^j z^{j(j+1)}(1+z)}{(z^2,z^2)_j\,(z^2,z^2)_{d-j}} \quad \textit{at } z=1$$

are:

$$\frac{(-1)^j}{2^{d-1} j!(d-j)!} \frac{1}{(1-z)^d} + \frac{(-1)^j (d+1)}{2^d j!(d-j)!} (d-2j-1) \frac{1}{(1-z)^{d-1}} + \cdots .$$

Proof.

(i) We have

$$(z^2, z^2)_j = (1-z^2)(1-z^4) \cdots (1-z^{2j}).$$

Let us expand the polynomial $1-z^n$ in a Taylor series about $z=1$. We have

$$\begin{aligned} 1 - z^n &= -n(z-1) - \frac{n(n-1)}{2!}(z-1)^2 + \cdots \\ &= n(1-z) - \frac{n(n-1)}{2!}(1-z)^2 + O((1-z)^3). \end{aligned}$$

Therefore,

$$\begin{aligned} (z^2, z^2)_j &= (1-z^2)(1-z^4) \cdots (1-z^{2j}) \\ &= (2(1-z) - \frac{2}{2!}(1-z)^2 + \cdots) \\ &\times \left(4(1-z) - \frac{4 \cdot 3}{2!}(1-z)^2 + \cdots\right) \\ &\times \left(2j(1-z) - \frac{2j(2j-1)}{2!}(1-z)^2 + \cdots\right) \\ &= (2 \cdot 4 \cdots 2j(1-z)^j + (1+3+5 \cdots + 2j-1) \\ &\times 2^{j-1} j!(1-z)^{j+1} + \cdots) \\ &= 2^j j!(1-z)^j - 2^{j-1} j!\, j^2 (1-z)^{j+1} + \cdots . \end{aligned}$$

It follows that

$$\begin{aligned} (z^2, z^2)_j (z^2, z^2)_{d-j} &= (2^j j!(1-z)^j - 2^{j-1} j!\, j^2 (1-z)^{j+1} + \cdots) \\ &\times (2^{d-j}(d-j)!(1-z)^{d-j} - 2^{d-j-1}(d-j)! \\ &\times (d-j)^2 (1-z)^{d-j+1} + \cdots) \\ &= 2^d j!(d-j)!(1-z)^d - 2^{d-1} j!(d-j)! \\ &\times ((d-j)^2 + j^2)(1-z)^{d+1} + \cdots . \end{aligned}$$

(ii) To find the first terms of the Laurent series for the function

$$\frac{(-1)^j z^{j(j+1)}(1+z)}{(z^2,z^2)_j\,(z^2,z^2)_{d-j}},$$

we expand the numerator in the Taylor series expressed in terms of powers of $(1-z)$. We have:

$$\begin{aligned} 1+z &= 2-(1-z), \\ z^{j(j+1)} &= 1-j(j+1)(1-z)+\cdots, \\ (1+z)z^{j(j+1)} &= 2-(2j(j+1)+1)(1-z)+\cdots. \end{aligned}$$

It is easy to check that the following decomposition holds:

$$\frac{a_0+a_1x+\cdots}{b_0+b_1x+\cdots}=\frac{a_0}{b_0}+\frac{a_1b_0-a_0b_1}{b_0^2}x+\cdots, b_0\neq 0.$$

Then

$$\begin{aligned} &\frac{(-1)^j z^{j(j+1)}(1+z)}{(z^2,z^2)_j\,(z^2,z^2)_{d-j}} \\ &= \frac{2-(2j(j+1)+1)(1-z)+\cdots}{2^d j!(d-j)!(1-z)^d-2^{d-1}j!(d-j)!((d-j)^2+j^2)(1-z)^{d+1}+\cdots} \\ &= \frac{1}{(1-z)^d}\,\frac{2-(2j(j+1)+1)(1-z)+\cdots}{2^d j!(d-j)!-2^{d-1}j!(d-j)!((d-j)^2+j^2)(1-z)+\cdots} \\ &= \frac{1}{(1-z)^d}\left(\frac{1}{2^{d-1}j!(d-j)!}+\frac{(-1)^j(d+1)}{2^d j!(d-j)!}\,(d-2j-1)\,(1-z)+\cdots\right). \square \end{aligned}$$

The following lemma shows how the function φ_n acts on the negative powers of $1-z$.

Lemma 2.3. *For $h\in\mathbb{N}$,*

$$\varphi_n\left(\frac{1}{(1-z)^h}\right)=\sum_{i=0}^{h}\frac{\alpha_{n\,i}}{(1-z)^i},$$

where $\alpha_{nh}=n^{h-1}$ and $\alpha_{n,h-1}=-n^{h-2}(n-1)h/2$.

Proof. Using article [**1,** Lemma 4], we get

$$\varphi_n\left(\frac{1}{(1-z)^h}\right) = \frac{\varphi_n\left((1+z+z^2+\cdots+z^{n-1})^h\right)}{(1-z)^h}.$$

Obviously, α_{nh} is the remainder after the division of $\varphi_n((1+z+z^2+\cdots+z^{n-1})^h)$ by $(1-z)$.

Using the definition of the function φ_n we get

$$\begin{aligned}\varphi_n & \left((1+z+z^2+\cdots+z^{n-1})^h\right) \\ &= \frac{1}{n}\sum_{j=0}^{n-1}\left(1+\zeta_n^j z+(\zeta_n^j)^2z^2+\cdots+(\zeta_n^j)^{n-1}z^{(n-1)}\right)^h \Big|_{z^n=z}.\end{aligned}$$

The remainder of division of this polynomial by $(1-z)$ is equal to its value at the point $z=1$. Thus,

$$\begin{aligned}\varphi_n & \left((1+z+z^2+\cdots+z^{n-1})^h\right)\Big|_{z=1} \\ &= \frac{1}{n}\sum_{j=0}^{n-1}\left(1+\zeta_n^j+(\zeta_n^j)^2+\cdots+(\zeta_n^j)^{n-1}\right)^h = n^{h-1}.\end{aligned}$$

Obviously, $\alpha_{n,h-1}$ is the coefficient of $(1-z)$ in the Taylor series expansion for

$$\varphi_n\left((1+z+z^2+\cdots+z^{n-1})^h\right)$$

at the point $z=1$. Therefore,

$$\alpha_{n,h-1} = -\lim_{z\to 1}(\varphi_n\left((1+z+z^2+\cdots+z^{n-1})^h\right))'.$$

We have

$$\begin{aligned}&\left(\frac{1}{n}\sum_{j=0}^{n-1}\left(1+\zeta_n^j z+(\zeta_n^j)^2z^2+\cdots+(\zeta_n^j)^{n-1}z^{(n-1)}\right)^h\right)' \\ &\quad = \frac{h}{n}\sum_{j=0}^{n-1}(1+\zeta_n^j z+(\zeta_n^j)^2z^2+\cdots+(\zeta_n^j)^{n-1}z^{(n-1)})^{h-1}\times \\ &\qquad\qquad \times(\zeta_n^j+2(\zeta_n^j)^2z+\cdots+(n-1)(\zeta_n^j)^{n-1}z^{(n-2)}).\end{aligned}$$

It now follows that

$$\lim_{z\to 1}\left(\frac{1}{n}\sum_{j=0}^{n-1}\Bigl(1+\zeta_n^j z+(\zeta_n^j)^2z^2+\cdots+(\zeta_n^j)^{n-1}z^{(n-1)}\Bigr)^h\right)'$$
$$=\frac{h}{n}\sum_{j=0}^{n-1}\left(\sum_{k=0}^{n-1}(\zeta_n^j)^k\right)^{h-1}(\zeta_n^j+2(\zeta_n^j)^2+\cdots+(n-1)(\zeta_n^j)^{n-1})$$
$$=\frac{h}{n}n^{h-1}(1+2+\ldots+(n-1))=\frac{1}{2}h(n-1)n^{h-1}.$$

By using the relation

$$\lim_{z\to 1}\left(f(z^n)|_{z^n=z}\right)'=\frac{1}{n}\lim_{z\to 1}f'(z^n),$$

we get

$$\begin{aligned}\alpha_{n,h-1}&=-\lim_{z\to 1}(\varphi_n\left((1+z+z^2+\cdots+z^{n-1})^h\right))'\\&=-\frac{1}{n}\lim_{z\to 1}\left(\frac{1}{n}\sum_{j=0}^{n-1}\Bigl(1+\zeta_n^j z+(\zeta_n^j)^2z^2+\cdots+(\zeta_n^j)^{n-1}z^{(n-1)}\Bigr)^h\right)'\\&=-\frac{1}{2}h(n-1)n^{h-2}.\end{aligned}$$

□

Now we can compute $\deg(\mathcal{C}_d)$ and $\psi(\mathcal{C}_d)$.

Theorem 2.4.

$$\begin{aligned}\deg(\mathcal{C}_d)&=\lim_{z\to 1}(1-z)^d\mathcal{P}(\mathcal{C}_d,z)\\&=\frac{1}{d!}\sum_{0\le j<d/2}(-1)^j\binom{d}{j}\left(\frac{d}{2}-j\right)^{d-1}\end{aligned}$$

and

$$\psi(\mathcal{C}_d)=\lim_{z\to 1}\left(-(1-z)^d\mathcal{P}(\mathcal{C}_d,z)\right)'_z=\frac{1}{2}\deg(\mathcal{C}_d).$$

Proof. Using Lemmas 1 and 2 we get

$$\mathcal{P}(\mathcal{C}_d,z)=\sum_{0\le j<d/2}\varphi_{d-2j}\left(\frac{(-1)^j z^{j(j+1)}(1+z)}{(z^2,z^2)_j\,(z^2,z^2)_{d-j}}\right)$$

$$
\begin{aligned}
&= \sum_{0\le j<d/2} \varphi_{d-2j}\left(\frac{(-1)^j}{2^{d-1} j!(d-j)!}\frac{1}{(1-z)^d}+\cdots\right)\\
&= \sum_{0\le j<d/2} \frac{(-1)^j}{2^{d-1} j!(d-j)!}\varphi_{d-2j}\left(\frac{1}{(1-z)^d}\right)\\
&\quad + \sum_{0\le j<d/2} \frac{(-1)^j(d+1)}{2^{d-1} j!(d-j)!}\left(\frac{1}{2}d-j-\frac{1}{2}\right)\\
&\times \varphi_{d-2j}\left(\frac{1}{(1-z)^{d-1}}\right)+\cdots\\
&= \frac{1}{(1-z)^d}\sum_{0\le j<d/2} \frac{(-1)^j(d-2j)^{d-1}}{2^{d-1} j!(d-j)!}\\
&\quad - \frac{1}{(1-z)^{d-1}}\frac{1}{2}\sum_{0\le j<d/2} \frac{(-1)^j}{2^{d-1} j!(d-j)!}\\
&\quad \times (d-2j)^{d-2}(d-2j-1)(d-1) + \frac{1}{(1-z)^{d-1}}\frac{1}{2}\sum_{0\le j<d/2}\\
&\quad \times \frac{(-1)^j}{2^{d-1} j!(d-j)!}(d+1)(d-2j-1)(d-2j)^{d-2}+\cdots .
\end{aligned}
$$

Thus, the coefficient of $\frac{1}{(1-z)^d}$ is

$$
\begin{aligned}
\deg(\mathcal{C}_d) &= \sum_{0\le j<d/2} \frac{(-1)^j(d-2j)^{d-1}}{2^{d-1} j!(d-j)!}\\
&= \frac{1}{d!}\sum_{0\le j<d/2}(-1)^j\binom{d}{j}\left(\frac{d}{2}-j\right)^{d-1},
\end{aligned}
$$

and the coefficient of $\frac{1}{(1-z)^{d-1}}$ is

$$
\psi(\mathcal{C}_d) = \frac{1}{2d!}\sum_{0\le j<d/2}(-1)^j\binom{d}{j}\left(\frac{d}{2}-j\right)^{d-1}. \qquad \square
$$

3. Asymptotic behavior of $\deg(\mathcal{C}_d)$**.** Let us establish an integral representation for the degree $\deg(\mathcal{C}_d)$. We denote by

$$c_d := \deg(\mathcal{C}_d) \cdot d! = \sum_{0 \le j < d/2} (-1)^j \binom{d}{j} \left(\frac{d}{2} - j\right)^{d-1}.$$

The following statement holds:

Lemma 3.1.

(i) $c_d = 2\pi^{-1}(d-1)! \int\limits_0^\infty \frac{\sin^d x}{x^d} dx$,

(ii) $\deg(\mathcal{C}_d) > 0$.

Proof.

(i) We have:

$$\begin{aligned} 2c_d = & \sum_{0 \le j < d/2} (-1)^j \binom{d}{j} \left(\frac{d}{2} - j\right)^{d-1} \\ & + \sum_{0 \le j < d/2} (-1)^j \binom{d}{j} \left(\frac{d}{2} - j\right)^{d-1} \\ = & \sum_{0 \le j < d/2} (-1)^j \binom{d}{j} \left(\frac{d}{2} - j\right)^{d-1} \\ & + \sum_{d/2 \le j < d} (-1)^j \binom{d}{j} \operatorname{sign}\left(\frac{d}{2} - j\right) \left(\frac{d}{2} - j\right)^{d-1} \\ = & \sum_{j=0}^{d} (-1)^j \binom{d}{j} \operatorname{sign}\left(\frac{d}{2} - j\right) \left(\frac{d}{2} - j\right)^{d-1}. \end{aligned}$$

We use that

$$\frac{\pi}{2} \operatorname{sign}(a) = \int\limits_0^\infty \frac{\sin ax}{x} dx.$$

Then

$$\begin{aligned}\pi c_d &= \frac{\pi}{2}\sum_{j=0}^{d}(-1)^j\binom{d}{j}\ \mathrm{sign}\left(\frac{d}{2}-j\right)\left(\frac{d}{2}-j\right)^{d-1}\\ &= \sum_{j=0}^{d}(-1)^j\binom{d}{j}\left(\frac{d}{2}-j\right)^{d-1}\int\limits_0^{\infty}\frac{\sin(d/2-j)x}{x}\,dx\\ &= \int\limits_0^{\infty}\ \mathrm{Im}\left(\sum_{j=0}^{d}(-1)^j\binom{d}{j}\left(\frac{d}{2}-j\right)^{d-1}e^{i(d/2-j)x}\right)\frac{dx}{x},\quad i^2=-1.\end{aligned}$$

This follows by the same method as in [**7,** Lemma 3.4.7]. We have:

$$\begin{aligned}\sin^d\frac{x}{2} &= \left(\frac{e^{ix/2}-e^{-ix/2}}{2i}\right)^d\\ &= \frac{1}{2^d i^d}\sum_{j=0}^{d}\binom{d}{j}\left(e^{ix/2}\right)^{d-j}\left(e^{-ix/2}\right)^j\\ &= \frac{1}{2^d i^d}\sum_{j=0}^{d}(-1)^j\binom{d}{j}e^{ix(d/2-j)}.\end{aligned}$$

Differentiating $d-1$ times with respect to x, we obtain

$$\left(\sin^d\frac{x}{2}\right)^{(d-1)} = \frac{i^{d-1}}{2^d i^d}\sum_{0\le j\le d}(-1)^j\binom{d}{j}\left(\frac{d}{2}-j\right)^{d-1}e^{ix(d/2-j)}.$$

Hence,

$$\mathrm{Im}\left(\sum_{j=0}^{d}(-1)^j\binom{d}{j}\left(\frac{d}{2}-j\right)^{d-1}e^{i(d/2-j)x}\right) = 2^d\left(\sin^d\frac{x}{2}\right)^{d-1}.$$

Thus,

$$c_d = \frac{1}{\pi}\int\limits_0^{\infty}2^d\left(\sin^d\frac{x}{2}\right)^{d-1}\frac{dx}{x} = \frac{2}{\pi}\int\limits_0^{\infty}\left(\sin^d x\right)^{d-1}\frac{dx}{x}.$$

Integrating by parts $d-1$ times, we obtain

$$c_d = \frac{2(d-1)!}{\pi} \int\limits_0^\infty \frac{\sin^d x}{x^d}\, dx.$$

(ii) It is enough to prove that

$$\int_0^\infty \frac{\sin^d x}{x^d}\, dx > 0.$$

First of all, we prove that the integral is absolutely convergent. Let us split the integral into two parts:

$$\int\limits_0^\infty \frac{\sin^d x}{x^d}\, dx = \int\limits_0^1 \frac{\sin^d x}{x^d}\, dx + \int\limits_1^\infty \frac{\sin^d x}{x^d}\, dx$$

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, the function $(\frac{\sin x}{x})^p$ is continuous on $[0,1]$. Thus, the first integral is convergent. Since

$$\left|\frac{\sin^d x}{x^d}\right| \le \left|\frac{1}{x^d}\right|,$$

then the second integral is absolutely convergent for $d > 1$.

Now the integral can be represented in the form

$$\begin{aligned}
&\int\limits_0^\infty \frac{\sin^d x}{x^d}\, dx \\
&= \sum_{j=0}^\infty \left(\int_{2j\pi}^{(2j+1)\pi} \frac{\sin^d x}{x^d}\, dx + \int_{(2j+1)\pi}^{4j\pi} \frac{\sin^d x}{x^d}\, dx \right) \\
&= \sum_{j=0}^\infty \left(\int_{2j\pi}^{(2j+1)\pi} \frac{\sin^d x}{x^d}\, dx + \int_{2j\pi}^{(2j+1)\pi} \frac{\sin^d (x+\pi)}{(x+\pi)^d}\, dx \right) \\
&\ge \sum_{j=0}^\infty \int_{2j\pi}^{(2j+1)\pi} \left(\frac{\sin^d x}{x^d} - \frac{\sin^d x}{(x+\pi)^d} \right) dx \\
&= \sum_{j=0}^\infty \int_{2j\pi}^{(2j+1)\pi} \frac{\sin^d x}{x^d (x+\pi)^d} \left((x+\pi)^d - x^d \right) dx > 0, \quad d > 1.
\end{aligned}$$

For the case $d=1$, we have

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} > 0. \qquad \square$$

Condition $\deg(\mathcal{C}_d) > 0$ is equivalent to the statement that the transcendence degree of the field of fractions of the algebra $\mathcal{C}_d$ is equal to d.

Interestingly, in the general case, the Wolstenholme formula holds:

$$\int_0^\infty \frac{\sin^p x}{x^s} = \frac{(-1)^{p-s/2}}{(s-1)!}\frac{\pi}{2^p}\sum_{p-2j>0}(-1)^j\binom{p}{j}(p-2j)^{s-1},$$

if $p-s$ is even, see [**3,** Problem 1033].

Finally, we deal with the asymptotic behavior of $\deg(\mathcal{C}_d)$ as d tends to infinity. By the previous lemma, it is enough to determine the asymptotic behavior of

$$\int\limits_0^\infty \frac{\sin^d x}{x^d}\,dx.$$

Theorem 3.2.

$$\lim_{d\to\infty} d^{1/2}\int_0^\infty \frac{\sin^d x}{x^d}\,dx = \frac{(6\pi)^{1/2}}{2}.$$

Proof. Write

$$I = \lim_{d\to\infty} d^{1/2}\int_0^\infty \frac{\sin^d x}{x^d}\,dx,$$

and split the limit into two parts:

$$\begin{aligned} I &= \lim_{d\to\infty} d^{1/2}\int_0^\infty \frac{\sin^d x}{x^d}\,dx \\ &= \lim_{d\to\infty} d^{1/2}\int_0^{\pi/2} \frac{\sin^d x}{x^d}\,dx \\ &\quad + \lim_{d\to\infty} d^{1/2}\int_{\pi/2}^\infty \frac{\sin^d x}{x^d}\,dx. \end{aligned}$$

Since

$$\left|\int_{\pi/2}^{\infty} \frac{\sin^d x}{x^d} dx\right| \leq \int_{\pi/2}^{\infty} x^{-d} dx = \lim_{b\to\infty} \frac{x^{-d+1}}{1-d}\bigg|_{\pi/2}^{b}$$
$$= \frac{1}{(d-1)x^{d-1}} \longrightarrow 0,$$

it follows that

$$I = \lim_{d\to\infty} d^{1/2} \int_0^{\pi/2} \frac{\sin^d x}{x^d}\, dx.$$

Fix $\varepsilon > 0$ sufficiently small. Since $\sin x/x$ is monotonically decreasing as $0 \leq x \leq \pi/2$, it follows that

$$\frac{\sin x}{x} \leq \frac{\sin \varepsilon}{\varepsilon} = 1 - \frac{\varepsilon^2}{3!} + \frac{\varepsilon^4}{5!} - \cdots,$$

as $\varepsilon \leq x \leq \pi/2$. It readily follows that there exists a strictly positive constant a such that

$$\int_{\varepsilon}^{\pi/2} \frac{\sin^d x}{x^d}\, dx = O\left(e^{-a\cdot d\varepsilon^2}\right).$$

For $0 \leq x \leq \varepsilon$, we have

$$\left(\frac{\sin x}{x}\right)^d = \left(1 - \frac{1}{6}x^2 + O\left(\varepsilon^4\right)\right)^d = e^{-1/6dx^2 + O\left(d\varepsilon^4\right)}.$$

Hence,

$$\int_0^{\varepsilon} e^{-1/6dx^2} dx = \frac{e^{O\left(d\varepsilon^4\right)}}{d^{1/2}} \int_0^{\varepsilon d^{1/2}} e^{-1/6x^2} dx.$$

Now choose $\varepsilon = \ln d/\sqrt{d}$. Then the limit reduces to the Euler-Poisson integral:

$$I = \lim_{d\to\infty} d^{1/2} \int_0^{\pi/2} \frac{\sin^d x}{x^d}\, dx = \sqrt{6} \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{6\pi}}{2}. \qquad \square$$

Thus, the asymptotic behavior of $\deg(\mathcal{C}_d)$ as $d \to \infty$ is as follows:

$$\deg(\mathcal{C}_d) = \frac{c_d}{d!} \sim \sqrt{\frac{6}{\pi}} \frac{1}{d^{3/2}}.$$

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