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On some uniqueness questions in primary representations of ideals

By

Douglas H. UNDERWOOD*

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0. Introduction

In a noetherian ring, every ideal can be represented as an irredundant intersection of finitely many primary ideals. There are several uniqueness properties associated with such a representation, for example see [11; Chap. IV, §5, Theorems 6, 7, 8]. The major part of this paper is devoted to constructing examples to show that these uniqueness properties do not hold when an ideal is an infinite irredundant intersection of primary ideals.

We begin with a discussion of the notion of associated prime divisor of an ideal. We consider four definitions of associated prime divisor which appear in the literature, and show that that of Nagata [8; p. 19] is the most general. However the Zariski-Samuel characterization, that P is an associated prime divisor of the ideal A if $A:(x)$ is P-primary for some x, is the one which is relevant when we study irredundant primary representations.

In §2 we study irredundancy in the representation of the radical $\sqrt{A}$ of the ideal A as the intersection of its minimal prime divisors. We find that, if A has infinitely many minimal prime divisors, these

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may be irredundant and redundant primes in almost any combination. If P is a minimal prime divisor of A, P is irredundant in the intersection of all minimal prime divisors of A iff $P=\sqrt{A}:x$ for some $x\notin\sqrt{A}$. We show in §4 that $\sqrt{A}$ has an irredundant primary representation iff $\sqrt{A}$ is the intersection of the irredundant minimal prime divisors of A, and this is the only irredundant primary representation of $\sqrt{A}$.

In §3 we examine briefly the representation of any ideal A as the intersection $\bigcap A(M)$, where M is a maximal associated prime divisor of A and $A(M)=A\cdot R_M\cap R$. We show that $A(M)$ is irredundant in this intersection if M is an irredundant minimal prime divisor of A.

By definition, the representation $A=\bigcap Q_\alpha$ of A as an intersection of primary ideals is irredundant if no Q_α may be omitted and if $\alpha_1\neq\alpha_2$ implies $P_{\alpha_1}\neq P_{\alpha_2}$, where $P_\alpha=\sqrt{Q_\alpha}$. In case $\{\alpha\}$ is finite, the following hold: (1) $\{P_\alpha\}$ is uniquely determined by A. (2) For any α_0, $\bigcap_{Q_\alpha\subset P_{\alpha_0}} Q_\alpha=A(P_{\alpha_0})$. (3) If S is a multiplicative system in the ring R which does not meet A, then $A(S)$ is an intersection of some of the Q_α, where $A(S)=A\cdot R_S\cap R$. (4) If P is a prime ideal, then P contains A iff P contains some P_α. (5) $\sqrt{A}=\bigcap P_\alpha$. In §4 we show that none of these properties holds in general for infinite representations, thus answering some questions of Krull in [6]. We answer another question of Krull by exhibiting an integral domain in which every ideal has an irredundant primary representation, but in which the Q-condition (see §4) does not hold.

We also obtain some conditions under which some of the uniqueness conditions of the previous paragraph hold.

Notation and preliminary observations. By a **ring** R we always mean a commutative ring with identity. An **integral domain** (or simply **domain**) is a ring without zero-divisors. We use $\subset$ for proper or improper containment, and $<$ for proper containment. We write $x\in S\backslash T$ for $x\in S$ and $x\notin T$, and $A:x$ instead of $A:(x)$. If an index set I is specified or understood, $\bigcap B_\alpha$ represents the intersection of all B_α for $\alpha \in I$.

In several of our examples we use the relation between lattice-ordered groups and certain integral domains as given by Jaffard in [4; p. 78]. Let G be an (additive) Abelian group together with a partial order $\leqslant$ compatible with the operation in G. G is **lattice-ordered** if $a, b \in G$ implies $\inf(a, b) \in G$. A **segment** of the lattice-ordered group G is a non-void subset A of $G^+ = \{x \in G \mid x \geqslant 0\}$ which is closed under $\geqslant$ and inf, i.e., $a \in A$ and $b \geqslant a$ imply $b \in A$, and $a, b \in A$ implies $\inf(a, b) \in A$. A is a **prime** segment if $a, b \in G^+ \setminus A$ implies $a + b \notin A$.

Jaffard shows that to each lattice-ordered group G there corresponds an integral domain D. D is obtained from G as follows: Let F be an arbitrary field, and let R be the group ring of G with respect to F. R may be regarded as the set of formal sums $\sum_{i=1}^{n} a_i X^{g_i}$, $a_i \in F$, $g_i \in G$. Define a map ϕ from $R \setminus \{0\}$ onto G by $\phi(\sum a_i X^{g_i}) = \inf\{g_i\}$. R is a domain, and ϕ may be extended to the non-zero elements of the quotient field K of R by $\phi(p/q) = \phi(p) - \phi(q)$. Then $D = \{0\} \cup \{y \in K \mid \phi(y) \geqslant 0\}$. It is easily seen that there is a 1–1 inclusion-preserving correspondence between proper segments in G and proper ideals in D, and that prime segments correspond to prime ideals, and conversely. In fact, if A is a segment of G, it is immediate that $B = \{0\} \cup \phi^{-1}(A)$ is an ideal of D. To see that $\phi(B \setminus \{0\}) = A$ is a segment of G when B is an ideal of D, we must see that A is closed under inf. In [3], Heinzer shows that given $a, b \in B$, there is an element $c \in D$ such that $(c) = (a, b)$, and $\phi(c) = \inf(\phi(a), \phi(b))$, thus showing that A is closed under inf. We construct several examples by finding a lattice-ordered group G with the desired prime segment structure, and then pulling back to D. On occasion, we will apply the language of rings to ordered groups. For example we may speak of the radical of a segment, or of a minimal prime divisor of a segment.

1. Associated prime divisors of an ideal

We set down four definitions of associated prime divisor of an ideal which are found, either as definitions or characterizations, in [1; p. 131], [11; p. 210], [1; p. 165, Ex. 17] and [8; p. 19], respectively.

(1.1) Definitions.

(B) P is an associated prime divisor of A (in the Bourbaki sense) if $P=A:x$ for some $x\in R$.

(Z-S) P is an associated prime divisor of A (in the sense of Zariski-Samuel) if $A:x$ is P-primary for some $x\in R$.

(Bw) P is an associated prime divisor of A (in the weak Bourbaki sense) if P is a minimal prime divisor of $A:x$ for some $x\in R$.

(N) P is an associated prime divisor of A (in the Nagata sense) if $P\cdot R_S$ is a maximal associated prime divisor of $A\cdot R_S$ for some multiplicative system S which does not meet A. (P is a maximal associated prime divisor of A if P contains A and is maximal with respect to the property of being contained in the set $\{x\in R \mid xy\in A$ for some $y\notin A\}$ of zero-divisors modulo A).

The above definitions are equivalent in case R is noetherian. In general we have

(1.2) Theorem. *Let A be an ideal of R, and let P be a prime containing A. Then P satisfies* (B) *implies P satisfies* (Z-S) *implies P satisfies* (Bw) *implies P satisfies* (N).

Proof. If $P=A:x$, then $A:x$ is P-primary, and if $A:x$ is P-primary, then P is a minimal prime divisor of $A:x$, hence the first two implications. For the last, since P is a minimal prime divisor of $A:x$, $A\cdot R_P:x$ has radical $P\cdot R_P$, and it follows that $P\cdot R_P$ is contained in the set of zero-divisors modulo $A\cdot R_P$. Also $P\cdot R_P$ is a maximal ideal of R_P, so we have the third implication.

We note that it follows from [8; p. 20] and [11; p. 210] that definitions (Z-S), (Bw) and (N) are equivalent when the ideal A has a finite primary representation. Example (1.5) shows that (B) is not equivalent to the other definitions in this case.

To show that none of the implications in (1.2) can be reversed, we list the following examples.

(1.3) Example. (N) does not imply (Bw). Let $R=K[x_1, x_2, \cdots]$, a polynomial ring in infinitely many indeterminates x_i over the field K. Let $A=(\{x_i x_j\}_{i>j})$. If P is a prime divisor of A, P must contain x_i or x_j for each pair (i, j) of positive integers. Moreover each prime $P_k=(x_1, x_2, \cdots, x_{k-1}, x_{k+1}, \cdots)$ contains A, so the, $P_k(k=1, 2, \cdots)$ are the minimal prime divisors of A. The maximal ideal $M=(x_1, x_2, \cdots)$ is an associated prime divisor of A in the Nagata sense, for if $f\in M$, we can write $f=\sum_{i=1}^{n} c_i x_i$, $c_i\in R$, and $x_{n+1}\notin A$ but $fx_{n+1}\in A$, so f is a zero-divisor modulo A.

Now we show that $A=\sqrt{A}$. Since $\sqrt{A}=\cap P_k$, it is sufficient to show that $\cap P_k\subset A$, the opposite inclusion being obvious. Suppose we have $f\in R\backslash A$. We write $f=a_0+\sum_{i=1}^{n(1)} a_{1i} x_1^i+\cdots+\sum_{i=1}^{n(m)} a_{mi} x_m^i+X$, where $a_0, a_{ji}\in K$ and each monomial term in X has a factor of the form $x_i x_j$ with $i>j$, so $X\in A$. Since $f\notin A$, $a_0\neq 0$ or $a_{ji}\neq 0$ for some j, i. if $a_0\neq 0$, f belongs to no P_k. If $a_0=0$, then $a_{ji}\neq 0$ for some j, i, and we see then that $f\notin P_j$. Hence $f\in R\backslash A$ implies $f\notin \cap P_k$, or $\cap P_k=\sqrt{A}\subset A$, and we have $A=\sqrt{A}$.

If $x\in A$, then $A:x=R$, and M is not a minimal prime divisor of $A:x$. If $x\notin A=\sqrt{A}$, then $x\notin P_k$ for some k. But then if $y\in A:x$, $yx\in A\subset P_k$ so $y\in P_k$. Therefore $A:x\subset P_k\subset M$, and we have that M cannot be a minimal prime divisor of $A:x$, so M is not an associated prime divisor of A in the weak Bourbaki sense.

(1.4) Example. (Bw) does not imply (Z-S). Note that any minimal prime divisor P of an ideal $A=A:1$ satisfies (Bw). Let P be a redundant minimal prime divisor of an ideal $A=\sqrt{A}$ (see §2). If $A:x$ is P-primary, then $x\notin A=\sqrt{A}$, and $P=\sqrt{A:x}$. But then

$P=\sqrt{A}:x$ by 2. 3, so P is an irredundant minimal prime divisor of A by 2. 1, a contradiction. Therefore P does not satisfy (Z-S).

(1. 5) Example. (Z-S) does not imply (B). Let $R=R_v$ be a rank one valuation ring with the additive group of real numbers as value group. Let $A=\{z\in R_v \mid v(z)\geqslant 1\}$, and let $P=\{z\in R_v \mid v(z)>0\}$ be the (only) proper prime ideal of R_v. For any $x\in R_v\backslash A$, $A:x$ is P-primary, so P satisfies (Z-S). If $v(x)\geqslant 1$, then $x\in A$ and $A:x=R_v\neq P$. If $0<v(x)<1$, there is an element $y\in P$ such that $v(x)+v(y)<1$. Thus $y\notin A:x$, so $A:x\neq P$. If $v(x)=0$, $A:x=A\neq P$. Therefore $P=A:x$ holds for no $x\in R_v$, so P does not satisfy (B).

2. Semi-prime ideals and irredundant minimal prime divisors

Let us call a minimal prime divisor of the ideal A an **MPD** of A. For any ideal A, $\sqrt{A}$ is the intersection of all MPD's of A. We say that the MPD P of A is an **irredundant** MPD of A if it is irredundant in the intersection of all MPD's of A.

(2. 1) Theorem. *Let A be an ideal of the ring R and let P be a prime ideal of R. Then P is an irredundant MPD of A iff $P=\sqrt{A}:x$ for some $x\notin\sqrt{A}$.*

Proof. If P is an irredundant MPD of A, let $x\in(\bigcap P_\alpha)\backslash P$, where $\{P_\alpha\}$ is the set of all MPD's of A except P. If $y\in\sqrt{A}:x$, then $yx\in\sqrt{A}\subset P$. But $x\notin P$, so $y\in P$, and $\sqrt{A}:x\subset P$. If $z\in P$, then $zx\in P\cap(\bigcap P_\alpha)=\sqrt{A}$, or $z\in\sqrt{A}:x$. Hence $P=\sqrt{A}:x$.

Conversely suppose that $P=\sqrt{A}:x$. P is a prime divisor of A, and since $x\notin\sqrt{A}$ we have $x\notin P_\alpha$ for some MPD P_α of A. Then $y\in P$ implies $yx\in\sqrt{A}\subset P_\alpha$ implies $y\in P_\alpha$, so $P\subset P_\alpha$, and we must have $P=P_\alpha$. Thus P is an MPD of A. If $P_\beta\neq P$ is any MPD of A, there is an element $z\in P\backslash P_\beta$. Then $z\in\sqrt{A}:x$, or $zx\in\sqrt{A}\subset P_\beta$, so $x\in P_\beta$. Therefore x belongs to all MPD's of A except P, so P is irredundant for A.

Theorem 2.1 shows that the irredundant MPD's of $\sqrt{A}$ are just the associated prime divisors of $\sqrt{A}$ in the sense of Bourbaki.

(2.2) **Theorem.** *If P is an MPD of A, P is redundant iff $\sqrt{A}:P=\sqrt{A}$.*

Proof. It is sufficient to show that $\sqrt{A}:P=\bigcap P_\alpha$, where $\{P_\alpha\}$ is the set of all MPD's of A except P. If $x\in\bigcap P_\alpha$, then $xP\subset\sqrt{A}$, so $\bigcap P_\alpha\subset\sqrt{A}:P$. Conversely for each α there is an element $y\in P\backslash P_\alpha$. Hence if $x\in\sqrt{A}:P$, $xy\in\sqrt{A}\subset P_\alpha$, so $x\in P_\alpha$. Hence $\sqrt{A}:P\subset\bigcap P_\alpha$, and we have equality.

If we examine the discussion of associated prime divisors in [11; p. 211], we find that the associated prime divisors of an ideal A which has a finite primary representation are just those primes P of the form $P=\sqrt{A:x}$ for some x. This condition is close to the condition in 2.1. In fact:

(2.3) **Proposition.** *Suppose $x\notin\sqrt{A}$. Then $\sqrt{A:x}\subset\sqrt{A}:x$. If $\sqrt{A:x}$ is prime, $\sqrt{A:x}=\sqrt{A}:x$.*

Proof. Let $y\in\sqrt{A:x}$. Then $y^kx\in A$ for some integer k. Therefore $y^kx^k\in A$, or $y\in\sqrt{A}:x$. Hence $\sqrt{A:x}\subset\sqrt{A}:x$.

Now suppose $\sqrt{A:x}$ is prime. If $y\in\sqrt{A}:x$, we have $(yx)^k\in A$ for some integer k, so $(yx)^k\in A:x$, or $yx\in\sqrt{A:x}$. But $x\in\sqrt{A:x}$ would imply $x\in\sqrt{A}$, contrary to hypothesis. Therefore $y\in\sqrt{A:x}$ since $\sqrt{A:x}$ is prime, hence $\sqrt{A}:x\subseteq\sqrt{A:x}$, and we have equality.

Any prime ideal P of R is an irredundant MPD of some ideal A of R, namely $A=P$. There are some primes P such that if P is an MPD of an arbitrary ideal A, then P is irredundant for A.

(2.4) **Theorem.** *Suppose the prime ideal P is the radical of a finitely generated ideal. If P is an MPD of an ideal A, then P is irredundant for A.*

Proof. Let $P=\sqrt{(x_1,\cdots,x_n)}$ be an MPD of A. Then

$A(P)=A\cdot R_P\cap R=\{r\in R \mid rs\in A$ for some $s\notin P\}$ is P-primary, so $P=\sqrt{A(P)}$. For each x_i there is an integer $m(i)$ such that $x_i^{m(i)}\in A(P)$, and there is an element $r_i\notin P$ such that $r_i x_i^{m(i)}\in A$. Let $r=\prod_{i=1}^{n} r_i$, $m=\sum_{i=1}^{n} m(i)$.

Let $z\in P$. For some integer k we have $z^k\in(x_1,\cdots,x_n)$, say $z^k=t_1x_1+\cdots+t_nx_n$, $t_i\in R$. For each i, $r_ix_i^m\in A$, so $rz^{kj}\in A$ for sufficiently large j, and $z^{kj}\in A:r$. Then we have $z\in\sqrt{A:r}\subset\sqrt{A}:r$ by 2.3. Hence $P\subset\sqrt{A}:r$. But $r\notin P$, so $rs\in\sqrt{A}\subset P$ implies $s\in P$, or $\sqrt{A}:r\subset P$. Therefore $P=\sqrt{A}:r$, so P is irredundant for A by 2.1.

The examples which follow show that redundant and irredundant MPD's may be present in almost any combination when an ideal has infinitely many MPD's.

(2.5) **Example.** An ideal with infinitely many MPD's, all irredundant. Example 1.3 is such an example. We saw that the primes P_k $(k=1,2,\cdots)$ are the MPD's of A, and since $x_k\in(\bigcap_{i\neq k}P_i)\setminus P_k$ for each k, each P_k is irredundant.

(2.6) **Example.** An ideal A with infinitely many redundant and infinitely many irredundant MPD's, with the property that $\sqrt{A}$ is equal to the intersection of the irredundant MPD's of A. We obtain our example by constructing a suitable lattice-ordered group G and segment A of G, then using the domain and ideal corresponding to G and A as outlined in §0.

Take G to be the group of all real-valued left-continuous step functions (finitely many points of discontinuity) on $(0,\infty)$ whose points of discontinuity are of the form n or $n-1/k$ $(n=1,2,3,\cdots;\ k=2,3,4,\cdots)$. We call these points **admissible** points, and denote them by α. G is a group under pointwise addition, and is lattice-ordered by the relation $\leqslant$, where $f\leqslant g$ iff $f(x)\leqslant g(x)$ for all $x\in(0,\infty)$.

For each $x\in(0,\infty)$, $P_x=\{f\in G^+ \mid f(x)>0\}$ is a prime segment. In addition, we have the prime segment $P_0=\{f\in G^+ \mid$ there is an M

such that $f(x)>0$ if $x\geqslant M\}$ of functions which are eventually positive. If α is the smallest admissible point which is greater than or equal to x, then $f(x)=f(\alpha)$ for all $f\in G^+$, hence $P_x=P_\alpha$. Thus we have a prime segment for each admissible point, and we have P_0. We will show that there are no other prime segments in G.

First we show that each P_α is maximal. Suppose $M>P_\alpha$ is a proper segment. Then there is an $f'\in M\backslash P_\alpha$, so $f'(\alpha)=0$. Then f' is zero on an interval $(\alpha', \alpha]$ where either α' is an admissible point or $\alpha'=0$. Let $a>0$ be the maximum value attained by f', and let f be defined by $f(x)=0$ on $(\alpha', \alpha]$, $f(x)=a$ elsewhere. Then $f\geqslant f'$, so $f\in M$. Let g be defined by $g(x)=a$ on $(\alpha', \alpha]$, $g(x)=0$ elsewhere. $g\in P_\alpha$, so $g\in M$. Hence $0=\inf(f, g)\in M$, so $M=G^+$ is not a proper segment. Therefore P_α is maximal.

We also see that P_α is minimal, for if not there would be a prime $P<P_\alpha$. Let $f\in P_\alpha\backslash P$. Then $f(\alpha)>0$, so $f(x)=a>0$ for all x in some interval $(\alpha', \alpha]$. Define g by $g(x)=0$ on $(\alpha', \alpha]$, $g(x)=a$ elsewhere. Then $g\notin P_\alpha$, so $g\notin P$, and $h=f+g\notin P$. But $h(x)\geqslant a>0$ everywhere, and $h\notin P$. However for any $f'\in P$ we can find an integer n such that $nh=h+h+\cdots+h\geqslant f'$, which implies $nh\in P$, and hence $h\in P$, a contradiction.

We now show that P_0 is the only proper prime other than the P_α. Suppose P is a proper prime, $P\neq P_\alpha$ for all α. Then $P_\alpha\not\subset P$ for all α, so for each α there is an $f_\alpha\in P_\alpha\backslash P$, i.e., $f_\alpha(\alpha)>0$.

(2.7) **Lemma.** *Given any positive number N, there is an element $g_N\in G^+\backslash P$ with $g_N(x)\geqslant 1$ for all $x\in(0, N]$.*

Proof. We may suppose that N is a positive integer. For any integer n, $1\leqslant n\leqslant N$, we have $f_n\in P_n\backslash P$, so f_n is positive on an interval $(n-1/k_n, n]$. We also have $f_{n-1/k_n}\notin P$ which is positive on $(n-1/(k_n-1), n-1/k_n]$, and so on to $f_{n-1/2}\notin P$ which is positive on $(n-1, n-1/2]$. Then $h_n=f_n+f_{n-1/k_n}+\cdots+f_{n-1/2}$ is not in P, and is positive on $(n-1, n]$. Let $g=h_1+\cdots+h_N$. $g\notin P$, and g is positive on $(0, N]$. Then a suitable multiple $g_N=g+\cdots+g$ can be found which is not in P, and which satisfies $g_N(x)\geqslant 1$ on $(0, N]$. This completes the lemma.

Now each $g\in G$ is eventually constant. Suppose $g\in P$ is eventually zero, say $g(x)=0$ for $x\geqslant N$. There is an integer k such that $kg_N\geqslant g$, so $kg_N\in P$. But $g_N\notin P$, so P is not prime, a contradiction. Therefore $g\in P$ implies g is eventually positive.

If $h\in G^+$ is eventually positive, we may assume that $h(x)\geqslant 1$ for $x\geqslant N$, some N. Then $(g_N+h)(x)\geqslant 1$ for all x. For any $f\in P$, there is an integer k such that $k(g_N+h)\geqslant f$, so $g_N+h\in P$. Since $g_N\notin P$, it follows that $h\in P$, and $P=P_0$.

For our example we take the segment $A=\bigcap P_\alpha$. Then $f\in A$ implies $f(x)>0$ for all x. Therefore $A\subset P_0$. P_0 is an MPD of A, and is certainly redundant for A, so we omit P_0 from further consideration. Since A is an intersection of primes, $A=\sqrt{A}$. We show that P_α is irredundant for A iff α is not an integer.

Let β be an admissible point which is not an integer, and let β' be the largest admissible point which is less than β, or $\beta'=0$ if $\beta=1/2$. Let f_β be defined by $f_\beta(x)=0$ on $(\beta', \beta]$, $f_\beta(x)=1$ elsewhere. Then $f_\beta\notin P_\beta$, but $f_\beta\in P_\alpha$ for all $\alpha\neq\beta$. Therefore P_β is irredundant for A.

Let n be a positive integer. If $f\notin P_n$, then $f(x)=0$ for all x in some interval $(\alpha, n]$, so there is an admissible α', $\alpha<\alpha'<n$, such that $f\notin P_{\alpha'}$. Thus we have $f\in\bigcap_{\alpha\neq n} P_\alpha$ implies $f\in P_n$, or $P_n\supset\bigcap_{\alpha\neq n} P_\alpha$, and P_n is redundant for A.

By the argument of the previous paragraph we see that if $f\in\bigcap_{\alpha\notin Z^+} P_\alpha$, where $Z^+=\{1, 2, 3, \cdots\}$, then $f\in\bigcap P_\alpha=\sqrt{A}$. It follows that $\sqrt{A}=\bigcap_{\alpha\notin Z^+} P_\alpha$, i.e., $\sqrt{A}$ is the intersection of the irredundant MPD's of A.

We now pass to the domain D (see section 0) to obtain the desired example.

(2.8) **Example.** An ideal with infinitely many irredundant and

N (a positive integer) redundant MPD's. It is only necessary to restrict the functions used in 2.6 to the interval $(0, N]$. The prime segments are just the P_α for $\alpha \leqslant N$. P_N takes the place of P_0 as the set of all functions which are eventually positive.

(2.9) Example. An ideal with no irredundant MPD's. Let R be the polynomial domain $K[x_1, x_2, \cdots]$ in infinitely many indeterminates x_i over the field K, and let $A=(x_1x_2, x_3x_4, \cdots, x_{2n-1}x_{2n}, \cdots)$. We see immediately that the primes $P=(x_{1+d_1}, x_{3+d_2}, \cdots, x_{2n-1+d_n}, \cdots)$, where $d_i=0$ or 1 for each i, are the MPD's of A.

Let $f \in R$ be written as a sum of monomial terms, no two of which can be combined. Then no two monomial terms of f', the image of f in R/P, can be combined unless one term is 0, because R/P is a polynomial ring over K in the indeterminates x_i which are not generators of P. We see then that f belongs to a particular prime P (an MPD of A) iff each monomial term of f does.

To see that each P is redundant, it is enough to see that if a monomial f does not belong to one of the MPD's of A, then it does not belong to at least two of the MPD's. Suppose then that f is a monomial not belonging to P_1. Let us illustrate with a specific prime, say $P_1=(x_1, x_3, x_5, \cdots, x_{2n-1}, \cdots)$. Now f can be written $f=ax_2^{k(2)}x_4^{k(4)}\cdots x_{2n}^{k(2n)}$, since $f \notin P_1$. But then also $f \notin P_2=(x_1, x_3, \cdots, x_{2n-1}, x_{2n+2}, x_{2n+3}, \cdots) \neq P_1$. It follows that P_1 is redundant for A, and by symmetry, all MPD's of A are redundant.

(2.10) Example. A suitable modification of 2.9 gives an ideal with N irredundant MPD's and infinitely many redundant MPD's. We merely need to take $R=K[t_1, \cdots, t_N, x_1, x_2, \cdots]$ and $A=(t_1)\cap(t_2)\cap\cdots\cap(t_N)\cap(x_1x_2, x_3x_4, \cdots)$. The prime ideals (t_i) are irredundant for A, and the primes of the form $(x_{1+d_1}, x_{3+d_2}, \cdots)$ as in 2.9 are redundant.

Because of 2.4 we might conjecture that a finitely generated ideal A would have only irredundant MPD's. To see that this is false we need only reduce to R/A in the above examples. Then we have $A'=(0)$, where A' is the image of A in R/A.

3. P-components of an ideal

Let A be an ideal of R. We will call a maximal associated prime divisor of A (in the sense of Nagata, see §1) an **MAPD** of A. If P is an MPD of A, the ideal $A(P)=A\cdot R_P\cap R=\{x\in R\mid xy\in A$ for some $y\notin P\}$ is called the **P-component of A.** An element $x\in R$ which is not a zero-divisor modulo A is said to be **prime to A.** In [5], Krull proves the following:

(3.1) Theorem. *Let A be an ideal of the ring R. Then*:

(a) *Every element or ideal of R which is not prime to A is contained in an MAPD of A.*

(b) If *P is a prime containing A and $x\in R\backslash P$, then x is prime to $A(P)$.*

(c) *Every MPD of A is contained in at least one MAPD of A.*

(d) *$A=\cap A(M)$, where M runs over the set of all MAPD's of A.*

(3.2) Theorem. *If $\{M_\alpha\}$ is the set of all MAPD's of A and M_{α_0} is an irredundant MPD of A, then $A(M_{\alpha_0})$ is irredundant in the representation $A=\cap A(M_\alpha)$.*

Proof. Since M_{α_0} is an irredundant MPD of A, there is an x belonging to all MPD's of A except M_{α_0}, and $M_{\alpha_0}=\sqrt{A}:x$. For each $\alpha\neq\alpha_0$, M_α contains an MPD (not M_{α_0}) of A, hence we see that $x\in\bigcap_{\alpha\neq\alpha_0} M_\alpha$. Note that $x\notin M_{\alpha_0}$ implies $x^k\notin A(M_{\alpha_0})$ for all $k>0$.

Since M_{α_0} is maximal with respect to being contained in the set of zero-divisors modulo A, and since $x\notin M_{\alpha_0}$, the ideal (M_{α_0}, x) contains an element $m+rx(m_\alpha\in M_{\alpha_0}, r\in R)$ which is prime to A. Then $m+rx$ belongs to no MAPD of A, so $m\notin M_\alpha$ for each $\alpha\neq\alpha_0$, because $m\in M_\alpha$ and $x\in M_\alpha$ would imply $m+rx\in M_\alpha$. Now $m\in M_{\alpha_0}=\sqrt{A}:x$, so $m^kx^k\in A$ for some integer k. But $m^k\notin M_\alpha$ for all $\alpha\neq\alpha_0$, so $x^k\in A(M_\alpha)$ for all $\alpha\neq\alpha_0$. Therefore we have $x^k\in(\bigcap_{\alpha\neq\alpha_0} A(M_\alpha))\backslash A(M_{\alpha_0})$, and $A(M_{\alpha_0})$ is irredundant in the representation of 3.1(d).

If M_{α_0} is a redundant MPD of A, $A(M_{\alpha_0})$ may be either redundant or irredundant in the representation above. For examples, see [10; p. 29].

We need one further observation about the ideal $A(P)$.

(3.3) Proposition. *If P is any prime containing A, and if $A=\sqrt{A}$, then $\sqrt{A(P)}=A(P)$.*

Proof. Since for any ideal B and multiplicative system S not meeting B we have $\sqrt{B(S)}=\sqrt{B}(S)$, it follows that $A(P)=\sqrt{A}(P)=\sqrt{A(P)}$.

4. Irredundant primary representation

In this section we consistently use Q for a primary ideal and P for a prime. If we use P_α in context with Q_α, we mean $P_\alpha=\sqrt{Q_\alpha}$. We say that the ideal A has an *irredundant primary representation* $A=\bigcap Q_\alpha$ if (a) for each α_0, $Q_{\alpha_0}\not\supset(\bigcap_{\alpha\neq\alpha_0} Q_\alpha)$, and (b) $P_{\alpha_1}\neq P_{\alpha_2}$ if $\alpha_1\neq\alpha_2$. *Except in Lemma* 4.10, *the notation $A=\bigcap Q_\alpha$ is used to denote an irredundant primary representation of A.* With $A=\bigcap Q_\alpha$, we call $\{P_\alpha\}$ *the set of primes corresponding to this representation.*

In [6] Krull proves that if R is a ring with Q-condition, every ideal of R has an irredundant primary representation. The *Q-condition* is this: for every multiplicative system S of R, and for every ideal A of R, there is an $x\in R$ such that $A(S)=A:x$, where $A(S)=\{r\in R\mid rs\in A$ for some $s\in S\}$. When R is an arbitrary ring Krull then raises questions 4.1 (a–d) below, to which we add 4.1 (e–g).

(4.1) Questions.

(a) If $A=\bigcap Q_\alpha=\bigcap Q_\beta$, is $\{P_\alpha\}=\{P_\beta\}$?

(b) Suppose $A=\bigcap Q_\alpha$. Let M be a non-void subset of $\{P_\alpha\}$

with the property that $P_{\alpha_1} \subset P_{\alpha_2}$ and $P_{\alpha_2} \in M$ implies $P_{\alpha_1} \in M$. For each such M, is $\bigcap_{P_\alpha \in M} Q_\alpha$ independent of the particular Q_α used in the primary representation of A?

(c) If $A = \bigcap Q_\alpha$, is every ideal $A(S)$, where S is a multiplicative system not meeting A, an intersection of some set of the Q_α?

(d) Is there a ring R which does not have the Q-condition, but in which every ideal has an irredundant primary representation?

(e) If $A = \bigcap Q_\alpha$ and P is a prime ideal containing A, does P contain some P_α?

(f) If $A = \bigcap Q_\alpha$, is $\sqrt{A} = \bigcap P_\alpha$?

(g) If $A = \bigcap Q_\alpha = \bigcap Q_\beta$, is $\bigcap P_\alpha = \bigcap P_\beta$?

In the case of finite primary representations, the answers to these questions are yes, except for (d). We will show that in general the answer to (d) is yes, and the answers to the other questions are no. Our examples, moreover, are in integrally closed integral domains.

(4.2) **Proposition.** *Suppose $A = \bigcap Q_\alpha$. For each α there is an element x such that $A:x = Q_\alpha : x$ and $A:x$ is P_α-primary. Each P_α is an associated prime divisor of A in the* (Z-S) *sense, hence also in the sense of Nagata.*

Proof. Fix α_0, and let $x \in (\bigcap_{\alpha \neq \alpha_0} Q_\alpha) \setminus Q_{\alpha_0}$. Then $A:x = (\bigcap Q_\alpha):x = \bigcap (Q_\alpha : x)$. But if $x \in Q_\alpha$, $Q_\alpha : x = R$, so $A:x = Q_{\alpha_0}:x$. Since $x \notin Q_{\alpha_0}$, $Q_{\alpha_0}:x$ is P_{α_0}-primary.

We see then that if P is a prime corresponding to an irredundant primary representation of A, there is an element x such that $P = \sqrt{A:x}$.

(4.3) **Theorem.** *Suppose $A = \bigcap Q_\alpha$. If P is a prime ideal of the form $\sqrt{A:x}$, in particular if $A:x$ is P-primary, then $P \subset P_\alpha$ for some α.*

Proof. From $A = \bigcap Q_\alpha$ we have $A:x = \bigcap (Q_\alpha : x) = \bigcap_{x \notin Q_\alpha} (Q_\alpha : x)$. Since $x \notin Q_\alpha$ implies $\sqrt{Q_\alpha : x} = P_\alpha$, we take radicals and get

(4. 4) $$P=\sqrt{A:x}=\sqrt{\bigcap_{x\notin Q_\alpha}(Q_\alpha:x)}\subset\bigcap_{x\notin Q_\alpha}\sqrt{Q_\alpha:x}=\bigcap_{x\notin Q_\alpha}P_\alpha\,.$$

Since $x\notin Q_\alpha$ for some α (otherwise we would have $x\in A$ and $\sqrt{A:x}=R\neq P$), P is contained in at least one P_α.

Note that the containment in 4. 4 becomes equality if the intersection is finite. Since $P=\bigcap_{i=1}^{n}P_i$ implies $P=P_i$ for some i, 4. 2 and 4. 3 yield the well-known result that if an ideal A is a finite irredundant intersection of primaries, the corresponding primes are uniquely determined, and are just those primes of the form $\sqrt{A:x}$.

Several corollaries to 4. 3 show that some primes corresponding to a particular irredundant primary representation of A must correspond to every such representation. In 4. 5—4. 8, we suppose that we have two representations $A=\bigcap Q_\alpha=\bigcap Q_\beta$ of A. We omit the proofs of 4. 5—4. 7.

(**4. 5**) **Corollary.** *Any P_α which is maximal in $\{P_\alpha\}$ is in $\{P_\beta\}$, and is maximal in $\{P_\beta\}$.*

(**4. 6**) **Corollary.** *If the P_α and the P_β are all MPD's of A, then $\{P_\alpha\}=\{P_\beta\}$.*

(**4. 7**) **Corollary.** *If the P_α are all MPD's of A, then so are the P_β, and $\{P_\alpha\}=\{P_\beta\}$.*

(**4. 8**) **Corollary.** *If only finitely many P_α are not MPD's of A, then every P_β is a P_α.*

Proof. Fix a P_β. $P_\beta=\sqrt{A:x}$ for some x, and by 4. 4 we have $P_\beta\subset\bigcap_{x\notin Q_\alpha}P_\alpha$. If $x\notin Q_\alpha$ for only finitely many α, the inclusion becomes equality, and P_β must equal some P_α. If $x\notin Q_\alpha$ for infinitely many α, then there is an α_0 such that $x\notin Q_{\alpha_0}$ and P_{α_0} is an MPD of A. Moreover $P_\beta\subset P_{\alpha_0}$, so $P_\beta=P_{\alpha_0}$ by minimality of P_{α_0}.

(**4. 9**) **Corollary.** *If $A=\bigcap Q_\alpha$ and M is an MAPD of A of the form $M=\sqrt{A:x}$, then M is a P_α.*

Proof. By 4. 3 $M\subset P_\alpha$ for some α. But P_α is an associated prime divisor of A in the sense of Nagata, and M is maximal among the associated prime divisors of A, so $M=P_\alpha$.

We insert the following lemma for later reference.

(4.10) Lemma. *Suppose Q is primary and $Q=\bigcap Q_\alpha$ (not necessarily an irredundant representation). If $Q_{\alpha_0}\not\supset \bigcap_{\alpha\neq\alpha_0} Q_\alpha$, then $\sqrt{Q}=\sqrt{Q_{\alpha_0}}$.*

Proof. We may assume that there is more than one Q_α. Since $Q_{\alpha_0}\not\supset\bigcap_{\alpha\neq\alpha_0} Q_\alpha$, there is an $x\in(\bigcap_{\alpha\neq\alpha_0} Q_\alpha)\backslash Q_{\alpha_0}$. As in the proof of 4.2 we have $Q:x=\bigcap_{x\notin Q_\alpha}(Q_\alpha:x)=Q_{\alpha_0}:x$. But $\sqrt{Q:x}=\sqrt{Q}$, and $\sqrt{Q_{\alpha_0}:x}=\sqrt{Q_{\alpha_0}}$, so $\sqrt{Q}=\sqrt{Q:x}=\sqrt{Q_{\alpha_0}:x}=\sqrt{Q_{\alpha_0}}$.

We return for a moment to the problem of semi-prime ideals which we considered in section 2.

(4.11) Proposition. *Suppose $A=\sqrt{A}=\bigcap P_\alpha$, an irredundant prime representation. Then $\{P_\alpha\}=\{$irredundant MPD's of $A\}$.*

Proof. Fix P_{α_0}. By irredundancy there is an $x\in(\bigcap_{\alpha\neq\alpha_0} P_\alpha)\backslash P_{\alpha_0}$, and $A{:}x$ is P_{α_0}-primary. Since $x\notin P_{\alpha_0}$, $x\notin\sqrt{A}=A$. But then $P_{\alpha_0}=\sqrt{A:x}=\sqrt{A}:x$ by 2.3, and by 2.1 P_{α_0} is an irredundant MPD of A.

No intersection of MPD's of A can equal $\sqrt{A}=A$ if an irredundant MPD of A is missing, so we see that $\{P_\alpha\}$ is the set of all irredundant MPD's of A.

(4.12) Theorem. *If $A=\sqrt{A}=\bigcap Q_\alpha$, then $\{Q_\alpha\}=\{$irredundant MPD's of $A\}$.*

Proof. By 4.11, it is sufficient to show that $Q_\alpha=P_\alpha$ for each α. Fix Q_α. With x an element of all the primaries except Q_α, we have $\sqrt{A:x}=P_\alpha$ by 4.2. But $x\notin A=\sqrt{A}$, so $P_\alpha=\sqrt{A}:x$ by 2.3, and P_α is an irredundant MPD of A by 2.1.

Since P_α is an MPD of A, $A(P_\alpha)$ is P_α-primary. Moreover $y\in A(P_\alpha)$ implies $yz\in A\subset Q_\alpha$ for some $z\notin P_\alpha$. But then $y\in Q_\alpha$, so

$A(P_\alpha)\subset Q_\alpha \subset P_\alpha$. By 3.3 $A(P_\alpha)=\sqrt{A(P_\alpha)}=P_\alpha$, so we have $P_\alpha \subset Q_\alpha \subset P_\alpha$, or $Q_\alpha = P_\alpha$, and the theorem is proved.

Our next theorem and its corollary have a relation to question 4.1 (b). In a finite irredundant primary representation $A=\bigcap_{i=1}^{n} Q_i$, we have $\bigcap_{Q_i\subset P_1} Q_i = A(P_1)$. Although the "component" $\bigcap_{Q_i\subset P_1} Q_i$ is not generally independent of the particular representation in the infinite case, we see that $\bigcap_{Q_i\subset P_1} Q_i$ may be replaced by $A(P_1)$ to yield an irredundant (not primary) representation of A in the sense of the next theorem.

(4.13) Theorem. *Let $A=\bigcap Q_\alpha$, and fix α_0. Then*

$$(*) \qquad A=A(P_{\alpha_0})\cap\Big(\bigcap_{Q_\alpha\not\subset P_{\alpha_0}} Q_\alpha\Big)$$

is an irredundant representation of A in the sense that if $A(P_{\alpha_0})$ or any $Q_\alpha \not\subset P_{\alpha_0}$ is omitted, the intersection of the remaining ideals is not A.

Proof. We first note that $A(P_{\alpha_0})\subset \bigcap_{Q_\alpha\subset P_{\alpha_0}} Q_\alpha$. This follows immediately from the fact that $A\subset Q_\alpha\subset P_{\alpha_0}$ implies $A(P_{\alpha_0})\subset Q_\alpha$. Since $A\subset A(P_{\alpha_0})$, it follows that (*) is valid. Also $\bigcap_{Q_\alpha\subset P_{\alpha_0}} Q_\alpha \not\supset \bigcap_{Q_\alpha\not\subset P_{\alpha_0}} Q_\alpha$, so $A(P_{\alpha_0})\not\supset \bigcap_{Q_\alpha\not\subset P_{\alpha_0}} Q_\alpha$, and $A(P_{\alpha_0})$ is irredundant in (*).

If (*) were not irredundant, there would be some $Q_{\alpha_1}\not\subset P_{\alpha_0}$ such that $Q_{\alpha_1}\supset \Big[A(P_{\alpha_0})\cap\Big(\bigcap_{\substack{Q_\alpha\not\subset P_{\alpha_0}\\ \alpha\neq\alpha_1}} Q_\alpha\Big)\Big]$. Let $z\in\Big(\bigcap_{\alpha\neq\alpha_1} Q_\alpha\Big)\backslash Q_{\alpha_1}$. If z were in $A(P_{\alpha_0})$, we would have $z\in \Big[A(P_{\alpha_0})\cap\Big(\bigcap_{\substack{Q_\alpha\not\subset P_{\alpha_0}\\ \alpha\neq\alpha_1}} Q_\alpha\Big)\Big]\subset Q_{\alpha_1}$, a contradiction, so $z\notin A(P_{\alpha_0})$. Let $s\in Q_{\alpha_1}$. Then $sz\in A$, and since $z\notin A(P_{\alpha_0})$ we have $s\in P_{\alpha_0}$. But then $Q_{\alpha_1}\subset P_{\alpha_0}$, a contradiction. Therefore (*) is irredundant.

(4.14) Corollary. *Let $A=\bigcap Q_\alpha$. If P_{α_0} is an MPD of A, then A has the irredundant primary representation $A=A(P_{\alpha_0})\cap\Big(\bigcap_{\alpha\neq\alpha_0} Q_\alpha\Big)$.*

Corollary 4.14 *shows that if A has an irredundant primary representation $A=\bigcap Q_\alpha$, any Q_α whose radical P_α is an MPD of A may be replaced by $A(P_\alpha)$, and the result is still an irredundant* *primary representation of A. In the finite case, Q_α must be $A(P_\alpha)$, but this is not true for infinite representations.*

There is a condition which implies an affirmative answer to 4.1 (b) *in case M is of the form $M=\{P_\alpha \mid P_\alpha \subset P_{\alpha_0}$ for some $\alpha_0\}$.*

(4.15) **Proposition.** *Let $A=\cap Q_\alpha$ and fix α_0. If $P_{\alpha_0} \not\supset \bigcap_{Q_\alpha \not\subset P_{\alpha_0}} Q_\alpha$, then $\bigcap_{Q_\alpha \subset P_{\alpha_0}} Q_\alpha = A(P_{\alpha_0})$.*

Proof. We have seen that $A(P_{\alpha_0}) \subset \bigcap_{Q_\alpha \subset P_{\alpha_0}} Q_\alpha$. Let $y \in (\bigcap_{Q_\alpha \not\subset P_{\alpha_0}} Q_\alpha) \setminus P_{\alpha_0}$. For any $z \in \bigcap_{Q_\alpha \subset P_{\alpha_0}} Q_\alpha$, $yz \in A$. But $y \notin P_{\alpha_0}$, so $z \in A(P_{\alpha_0})$. Therefore $\bigcap_{Q_\alpha \subset P_{\alpha_0}} Q_\alpha \subset A(P_{\alpha_0})$, and we have equality.

The Q-condition implies that certain primaries in a representation $A=\cap Q_\alpha$ are uniquely determined by A. To prove this we need the following lemma. This lemma is a generalization of a result of Nakano in [9]. Nakano's proof is easily generalized to our case with the help of 3.1 (b).

(4.16) **Lemma.** *If P is a prime ideal containing A, and if there is an x such that $A(P)=A:x$, then $x \notin \sqrt{A(P)}$.*

(4.17) **Theorem.** *Suppose R has Q-condition, $A=\cap Q_\alpha$ is an ideal of R, and suppose that P_{α_0} is an MPD of A which is contained in no other P_α. Then $Q_{\alpha_0}=A(P_{\alpha_0})$.*

Proof. We know that $A(P_{\alpha_0}) \subset Q_{\alpha_0}$.

By the Q-condition we have $A(P_{\alpha_0})=A:z$ for some $z \notin \sqrt{A(P_{\alpha_0})} = P_{\alpha_0}$. If $\alpha \neq \alpha_0$, $A(P_{\alpha_0}) \not\subset P_\alpha$, as $A(P_{\alpha_0}) \subset P_\alpha$ implies $P_{\alpha_0} \subset P_\alpha$. Thus for any $\alpha \neq \alpha_0$ there is an element $y \in A(P_{\alpha_0}) \setminus P_\alpha$. Since $y \in A(P_{\alpha_0}) = A:z$, $yz \in A \subset Q_\alpha$. But $y \notin P_\alpha$, so $z \in Q_\alpha$. Hence $z \in (\bigcap_{\alpha \neq \alpha_0} Q_\alpha) \setminus P_{\alpha_0}$, or $P_{\alpha_0} \not\supset \bigcap_{\alpha \neq \alpha_0} Q_\alpha$. Then by 4.15, $Q_{\alpha_0}=A(P_{\alpha_0})$.

We are now ready to give the examples which answer the questions 4.1. We use $K[t, x_1, x_2, \cdots]$ to denote the polynomial ring in the indeterminates $t, x_1, x_2, \cdots$ over the field K of characteristic zero.

(4. 18) **Example.** This example shows that the answers to 4. 1 (a, c, e, f, g) are no.

Let $R=K[t, x_1, x_2, \cdots]$, $Q_0=(x_2^2, x_3, x_4, x_5, \cdots)$, and $Q_k=(t^k, x_1-k, x_2, x_3, \cdots, x_{k+1}, x_{k+2}^2, x_{k+3}, \cdots)$ for all $k\geqslant 1$.

The Q_i are primary with radicals $P_0=(x_2, x_3, x_4, \cdots)$ and $P_k=(t, x_1-k, x_2, x_3, \cdots)$ for $k\geqslant 1$. Let $A=\bigcap_{i=0}^{\infty} Q_i$. The P_i are all different, and for each $k\geqslant 0$ we have $x_{k+2}\in(\bigcap_{i\neq k} Q_i)\backslash Q_k$, so $A=\bigcap_{i=0}^{\infty} Q_i$ is an irredundant primary representation.

For $k\geqslant 1$, let, $\overline{Q}_k=(t^k, x_1-k, x_2^2, x_3, x_4, \cdots, x_{k+2}, \cdots)$. $\overline{Q}_k$ is P_k-primary, hence so is $Q'_k=Q_k\cap\overline{Q}_k$ for $k\geqslant 1$. Moreover $x_{k+2}\in(\bigcap_{i\neq k} Q'_i)\backslash Q'_k$ for each $k\geqslant 1$. It can be seen that $Q_0=\bigcap_{i=1}^{\infty}\overline{Q}_i$, and it follows that $A=\bigcap_{i=1}^{\infty} Q'_i$ is another irredundant primary representation of A.

Then $A=\bigcap_{i=0}^{\infty} Q_i=\bigcap_{i=1}^{\infty} Q'_i$. P_0 corresponds to the first, but not to the second, of these representations, giving a negative answer to 4. 1 (a).

Consider $A=\bigcap_{i=1}^{\infty} Q'_i$. We have $P_0\supset A$, but $P_0\not\supset P_i$ for $i\geqslant 1$, hence a negative answer to 4. 1 (e).

We see that $t\in\bigcap_{i=1}^{\infty}\sqrt{Q'_i}$, but $t\notin\bigcap_{i=0}^{\infty}\sqrt{Q_i}$, so we have a counterexample to 4. 1 (g), so also certainly to 4. 1 (f). In fact it is easy to see that $\sqrt{A}=P_0$, because if $f\in P_0$, $f^2\in A$.

Turning to 4. 1 (c), we consider the representation $A=\bigcap_{i=1}^{\infty} Q'_i$, and let $S=R-P_0$. Then $A(S)=A(P_0)$. P_0 is an MPD of A, so $A(P_0)$ is primary for P_0; but $A(P_0)$ is not the intersection of any set of the Q'_i, as such an intersection would be irredundant, and by 4. 10 we would have $P_0=P_i$ for some i. Hence the negative answer to 4. 1 (c).

Professor Ohm has suggested how the above example might be modified to give a counterexample to 4. 1 (a) in a **two-dimensional** domain. In a one-dimensional domain, every proper prime ideal is maximal, and from 4. 7 we see that 4. 1 (a) cannot be contradicted in dimension one.

First we omit the indeterminate t from 4.18. With this change we still have a counterexample to 4.1 (a). Choose K to be the field $k_0(t_3, t_4, t_5, \cdots)$, where the t_i are indeterminates over the field k_0. Let $P=(x_2^2-t_3x_3^2, x_2^2-t_4x_4^2, \cdots, x_2^2-t_nx_n^2, \cdots)$. $P\subset Q_i$ for $i\geqslant 0$, so $P\subset A$. We will show that P is prime, and that there are not three prime ideals O_1, O_2, O_3 in R such that $P<O_1<O_2<O_3<R$. Then it follows that $D=R/P$ is a two-dimensional domain, and reduced modulo P 4.18 gives the counterexample to 4.1 (a) in a two-dimensional domain.

To see that P is prime, let $T=\{t_3, t_4, \cdots\}$, $X=\{x_1, x_2, x_3, \cdots\}$, and consider the ideal $P'=(x_2^2-t_3x_3^2, x_2^2-t_4x_4^2, \cdots)$ in $R'=k_0[T, X]$. Let $k_0[X]^*$ be the multiplicative system of non-zero elements of $k_0[X]$, and let $P''=P'\cdot R''$, where R'' is the ring of quotients $R'_{k_0[X]^*}$. If we regard R'' as $k_0(X)[T]$, we see that the generators $x_2^2-t_ix_i^2$ of P'' in R'' are first-degree polynomials in T, so P'' is prime in R''. It follows that P' is prime in R'. P is the extension of P' to $R=R'_{k_0[T]^*}$, so P is prime in R.

Suppose there are prime ideals O_i such that $P<O_1<O_2<O_3<R$. Let $f_1\in O_1\backslash P$, $f_2\in O_2\backslash O_1$, $f_3\in O_3\backslash O_2$, $f_4\in R\backslash O_3$, and choose n so that $f_i\in K[x_1, \cdots, x_n]=\overline{R}$, $i=1, 2, 3, 4$. Denoting $B\cap\overline{R}$ by $\overline{B}$ for any ideal B of R, we have $\overline{P}<\overline{O}_1<\overline{O}_2<\overline{O}_3<\overline{R}$. We see that $\overline{P}=(x_2^2-t_3x_3^2, \cdots, x_2^2-t_nx_n^2)$. Let $\overline{P}_{(i)}$ be the prime ideal of $\overline{R}$ generated by the first i generators of $\overline{P}$ for $i=1, \cdots, n-3$. Then $(0)<\overline{P}_{(1)}<\overline{P}_{(2)}<\cdots<\overline{P}_{(n-3)}<\overline{P}<\overline{O}_1<\overline{O}_2<\overline{O}_3<\overline{R}$, a chain of $n+2$ primes (including (0)) in $\overline{R}$. But this implies that $\overline{R}$ has dimension greater than or equal to $n+1$, while we know that $\overline{R}$ has dimension n. We conclude that $D=R/P$ has dimension two, and we have the modification of 4.18.

We turn now to 4.1 (b). We see immediately that 4.1 (b) must be altered because of the negative answer to 4.1 (a). If we consider the two representations $A=\bigcap_{i=0}^{\infty}Q_i=\bigcap_{i=1}^{\infty}Q'_i$ from 4.18, the set $\{P_0\}$ is a valid choice for M in the first representation, but not in the second.

In fact the only situation in which 4.1 (b) is an unambiguous question is when we have $A=\bigcap Q_\alpha=\bigcap Q_\beta$ with $\{P_\alpha\}=\{P_\beta\}$, and in this case the question becomes: is $\bigcap_{P_\alpha\in M} Q_\alpha=\bigcap_{P_\beta\in M} Q_\beta$? Let us, then, strengthen 4.1 (b) to the following: Suppose $A=\bigcap Q_\alpha=\bigcap Q_\beta$ implies $\{P_\alpha\}=\{P_\beta\}$. Is $\bigcap_{P_\alpha\in M} Q_\alpha=\bigcap_{P_\beta\in M} Q_\beta$? The answer is still no. In 4.20 we will have an ideal A with the irredundant primary representation $A=\bigcap Q_\alpha$, with each P_α an MPD of A. It follows from 4.7 that if $A=\bigcap Q_\beta$, $\{P_\alpha\}=\{P_\beta\}$. We will show that one of the primaries Q_{α_0} is not equal to $A(P_{\alpha_0})$. However, we know from 4.14 that $A=A(P_{\alpha_0})\cap(\bigcap_{\alpha\neq\alpha_0} Q_\alpha)$ is another irredundant primary representation of A, and we may take $M=\{P_{\alpha_0}\}$ to find our counterexample.

We need the following proposition.

(4.19) Proposition. *Suppose that* $A=\bigcap Q_\alpha$, P_{α_0} *is an MPD of* A, *and* (a) $P_{\alpha_0}\supset(\bigcap_{\alpha\neq\alpha_0} P_\alpha)$, *and* (b) $Q_{\alpha_0}\not\subset(\bigcup_{\alpha\neq\alpha_0} Q_\alpha)$. *Then* $Q_{\alpha_0}\neq A(P_{\alpha_0})$.

Proof. Let $x\in Q_{\alpha_0}\setminus(\bigcup_{\alpha\neq\alpha_0} Q_\alpha)$. If $xy\in A$, then $xy\in Q_\alpha$ for each α. But if $\alpha\neq\alpha_0$, $x\notin Q_\alpha$, so $y\in P_\alpha$. Hence $xy\in A$ implies $y\in(\bigcap_{\alpha\neq\alpha_0} P_\alpha)\subset P_{\alpha_0}$, so $x\notin A(P_{\alpha_0})$, and $Q_{\alpha_0}\neq A(P_{\alpha_0})$.

(4.20) Example. We find a counterexample to the strong form of question 4.1 (b) by finding an ideal $A=\bigcap Q_\alpha$ satisfying the hypotheses of 4.19, and with all P_α MPD's of A. By the remarks preceding 4.19, this gives the required counterexample.

This example is based on an example given by Gilmer in [2]. Let $R=K[\{X_a\}]$, where a runs over all rationals in the interval $(0,1)$, and the X_a are indeterminates over the field K. For the remainder of this example, a subscript a, b, or r will denote a rational in $(0,1)$.

For the primes we take $P=(\{X_a\}_{\text{all } a})$ and, for each r, $P_r=(\{X_a\}_{a<r}, \{X_a-1\}_{r<a})$.

Our primary ideals are $Q=(\{X_a^{m(a)}\}_{\text{all } a})$, where $m(a)=2$ if $a\leqslant 1/2$, and $m(a)=1$ if $a>1/2$, and, for each r, $Q_r=(\{X_a^{n(a)}\}_{a<r}, \{X_a-1\}_{r<a})$,

where $n(a)=1$ if $a\leqslant 1/2$, and $n(a)=2$ if $a>1/2$. Q is P-primary, and Q_r is P_r-primary for each r.

Let $A=Q\cap(\cap Q_r)$. To simplify the discussion, we list several facts about these primaries.

(1) For each r, $X_{3/4}\notin Q_r$, but $X_{3/4}\in Q$.

(2) $X_{1/4}\notin Q$, and $X_{1/2}-1\notin P$, hence $X_{1/4}(X_{1/2}-1)\notin Q$.

(3) $X_{1/4}\in Q_r$ if $1/4<r$. $X_{1/2}-1\in Q_r$ if $r<1/2$. Therefore $X_{1/4}(X_{1/2}-1)\in Q_r$ for all r.

(4) For fixed r_1, $X_{r_1}^2\in Q$, so $X_{r_1}^2(X_{r_1}-1)\in Q$.

(5) For fixed r_1, $X_{r_1}^2\in Q_r$, if $r_1<r$. $X_{r_1}-1\in Q_r$ if $r_1>r$. Hence $X_{r_1}^2(X_{r_1}-1)\in Q_r$ for all $r\neq r_1$.

(6) $X_{r_1}^2\notin Q_{r_1}$, and $X_{r_1}-1\notin P_{r_1}$, so $X_{r_1}^2(X_{r_1}-1)\notin Q_{r_1}$.

(7) If $0<a<b<1$, $X_a^2\in Q$, so $X_a^2(X_b-1)\in Q$. Moreover $X_a^2\in Q_r$ if $a<r$, and $X_b-1\in Q_r$ if $r<b$. It follows that $X_a^2(X_b-1)\in Q_r$ for all r, so $X_a^2(X_b-1)\in A$.

We now show that the hypotheses of 4.19 are satisfied with $Q_{\alpha_0}=Q$, $P_{\alpha_0}=P$, and that all P_r are MPD's of A.

Irredundancy. The primes P, P_r are distinct. By (2), (3), $Q\not\supset\cap Q_r$, so Q is irredundant. By (4), (5), (6), we have $Q_{r_1}\not\supset Q\cap(\bigcap_{r\neq r_1} Q_r)$, so each Q_{r_1} is irredundant.

***P* is an MPD of *A*.** Suppose $P'\subset P$ is a prime divisor of A. Given $a\in(0,1)$ there is a $b\in(0,1)$ such that $a<b$. By (7), $X_a^2(X_b-1)\in A\subset P'$. But $X_b-1\notin P'$, so $X_a\in P'$. Therefore $P=(\{X_a\}_{\text{all } a})\subset P'\subset P$, and $P'=P$. Therefore P is an MPD of A.

Condition (a). Proved in [2; p. 196].

Condition (b). By (1), $X_{3/4}\in Q\setminus(\cup Q_r)$, so $Q\not\subset(\cup Q_r)$.

To see that each P_r is an MPD of A, suppose $P_r'\subset P_r$ is a prime divisor of A. For any $a<r$, $X_a^2(X_r-1)\in A\subset P_r'$, but $X_r-1\notin P_r'$, so $X_a\in P_r'$. Similarly for $b>r$, $X_r^2(X_b-1)\in P_r'$, but $X_r^2\notin P_r'$, so $X_b-1\in P_r'$. Then $P_r=(\{X_a\}_{a<r}, \{X_b-1\}_{r<b})\subset P_r'\subset P_r$, so $P_r'=P_r$ and P_r is an MPD of A.

This completes example 4.20.

Just as we found a counterexample to 4.1 (b) in case the primes corresponding to an irredundant primary representation of A are uniquely determined by A, we may find counterexamples to 4.1 (c, e, f) in a similar situation.

(4.21) Example. Another counterexample to 4.1 (c, e, f). Let $R=K[x_1, x_2, x_3, \cdots]$, and $Q_2=(x_1^2, x_3, x_4, \cdots)$, $Q_3=(x_1^3, x_2, x_4, \cdots)$, $\cdots$, $Q_k=(x_1^k, x_2, \cdots, x_{k-1}, x_{k+1}, \cdots)$, $\cdots$. For each $k\geqslant 2$, Q_k is primary for $P_k=(x_1, x_2, \cdots, x_{k-1}, x_{k+1}, \cdots)$.

Let $A=\bigcap_{i=2}^{\infty} Q_i$. Since $x_k \in (\bigcap_{i\neq k} Q_i)\backslash Q_k$ for each $k\geqslant 2$, and the P_k are distinct, we have an irredundant primary representation of A.

Examination of the Q_k shows us that $x_1^k x_k \in A$ for each $k\geqslant 2$, and $x_j x_k \in A$ for $2\leqslant j<k$. Then it is easy to see that each P_k is an MPD of A, so by 4.7 $\{P_k\}_{k=2}^{\infty}$ is the set of primes corresponding to any irredundant primary representation of A.

Since no power of x_1 belongs to all Q_k, $x_1 \notin \sqrt{A}$. But $x_1 \in P_k$ for all $k\geqslant 2$, so there must be an MPD P of A such that $x_1 \notin P$. Thus $P\supset A$, but P contains no P_k, so 4.1 (e) fails, and in this example P cannot correspond to any irredundant primary representation of A. Also $x_1 \in (\bigcap_{k=2}^{\infty} P_k)\backslash \sqrt{A}$, so $\sqrt{A} \neq \bigcap_{k=2}^{\infty} P_k$, giving a counterexample to 4.1 (f).

Finally we see that $A(P)$ is primary, but cannot be an intersection of any set of the Q_k by 4.10, so 4.1 (c) fails.

(4.22) Example. We give an example of an integral domain in which every ideal has an irredundant primary representation, but which does not have the Q-condition, thus showing that the answer to 4.1 (d) is yes. Mott [7] has given such an example in a ring with zero-divisors.

We obtain our domain from a lattice-ordered group by applying Jaffard's theorem. Let G be the additive group of real-valued left-continuous step functions defined on $(0, 1]$ with jumps at the points $\alpha=1-1/k$, $k=2, 3, 4, \cdots$. Our discussion of 2.6 carries over, with the modifications necessary because of the restricted domain of the functions, to show that $P_\alpha=\{f\in G^+|f(\alpha)>0\}$ and $P_1=\{f\in G^+|f(1)>0\}$ are the only proper prime segments of G. Moreover P_1 is redundant and the P_α are irredundant in the intersection $P_1\cap(\cap P_\alpha)$.

Let D be the domain corresponding to G. The non-zero prime ideals of D are $\overline{P}_1$ and the $\overline{P}_\alpha$, which correspond to the prime segments P_1 and P_α of G. Each $\overline{P}_\alpha$ is irredundant in any intersection of primes in which it occurs.

Let A be any non-zero ideal of D, $A\neq D$. Any prime $\overline{P}$ of D which contains A is both an MPD and an MAPD of A. Let $\{\overline{P}_\beta\}$ be the set of prime divisors of A, where each β is either 1 or an α. By 3.1 (d), $A=\cap A(\overline{P}_\beta)$, and each $A(\overline{P}_\beta)$ is primary. For $\beta\neq 1$, $A(\overline{P}_\beta)$ is irredundant in this representation by 3.2. If $\overline{P}_1$ is one of the $\overline{P}_\beta$, $A(\overline{P}_1)$ may be redundant, in which case we omit it, or irredundant, in which case we retain it. In either case, A has an irredundant primary representation.

Now we wish to see that D does not have the Q-condition. Let $B=\overline{P}_1\cap(\cap\overline{P}_\alpha)$. $B\neq(0)$, because the corresponding intersection of prime segments of G is non-void. We also see that $B=\sqrt{B}$, and $\overline{P}_1$ is an MPD of B. Let $S=D-\overline{P}_1$. Then $B(S)=B(\overline{P}_1)$, and also $B(\overline{P}_1)$ is $\overline{P}_1$-primary. Since $B=\sqrt{B}$, 3.3 implies that $B(\overline{P}_1)=\overline{P}_1$. If $x\in B$, $B:x=D\neq B(\overline{P}_1)$. If $x\notin B=\sqrt{B}$, the equality $B:x=\overline{P}_1$ would imply that $\overline{P}_1$ is an irredundant MPD of B by 2.1, which is not the case. Therefore $B:x\neq B(\overline{P}_1)$ for any x, so D does not have Q-condition, and we have our example.

We conclude this paper by raising some questions.

(4.23) Questions. Suppose that the ideal A has an irredundant primary representation $A=\cap Q_\alpha$.

(a) What characterizes those prime ideals P with the property that there is an irredundant primary representation $A=\cap Q_\beta$ with $P=P_\beta$ for some β?

(b) What characterizes those primary ideals Q with the property that there is an irredundant primary representation $A=\bigcap Q_\beta$ with $Q=Q_\beta$ for some β?

We know that if $A=\bigcap Q_\beta$, for each β there is an element $x\notin A$ and a primary ideal $Q(=Q_\beta)$ such that $A:x=Q:x$ is P_β-primary. This leads to the **conjecture**: If $A=\bigcap Q_\alpha$ and if P is a prime such that $A:x=Q:x$ is P-primary for some $x\notin A$ and some P-primary ideal Q, then there is an irredundant primary representation $A=\bigcap Q_\beta$ with $P=P_\beta$ for some β.

Looking at 4.23 (b) we might even conjecture that if $A:x=Q:x$ is primary, then A has representation $A=\bigcap Q_\beta$ with $Q=Q_\beta$ for some β. This is false, even in a noetherian ring. To see this, suppose we have prime ideals P_1, P_2 in a noetherian ring, with $P_1<P_2$. Suppose further that $B=Q_1\cap Q_2$ is an irredundant primary representation, with Q_i primary for P_i. By [11; p. 231] there is a P_2-primary ideal $Q_2'<Q_2$ such that $B=Q_1\cap Q_2'$. Let $x\in Q_1\backslash Q_2$. Then $x\notin Q_2'$, and $B:x=(Q_1\cap Q_2):x=(Q_1:x)\cap(Q_2:x)=Q_2:x$. But also $B:x=(Q_1\cap Q_2'):x=Q_2':x$. Now let $A=Q_2$. Then there is an element $x\notin A$ and a primary ideal Q_2' such that $A:x=Q_2':x$, but Q_2' appears in no irredundant primary representation of A because A has the unique representation $A=Q_2$.

Let us consider examples 4.18, 20, 21 to test our conjecture concerning the characterization of corresponding primes. In 4.20 and 4.21 the corresponding primes are all MPD's of the ideal in question, so by 4.7 every irredundant primary representation must have the same set of corresponding primes. In 4.18 we can see that the only primes of the form $\sqrt{A:z}$ with $A:z$ primary are just $P_0, P_1, P_2, \cdots$. If $P=\sqrt{A:z}$ and $z\notin P_0$, then $y\in P$ implies $y^k z\in A$ for some k. But then $y^k z\in P_0$, so $y\in P_0$. Hence $P\subset P_0$, so $P=P_0$ since P_0 is an MPD of A.

If $P=\sqrt{A:z}$ with $z\in P_0$, we can write $z=r_2x_2+\cdots+r_nx_n$, $r_i\in R$. But then $z\in Q_k$ for $k>n$, and $Q_k:z=R$ for $k>n$. Therefore

$A:z=(Q_0:z)\cap(Q_1:z)\cap\cdots\cap(Q_n:z)$. This finite intersection of primary ideals can be reduced to an irredundant intersection by omitting any redundant ideals. Then we see by 4.10 that $A:z$ is not primary unless $\sqrt{A:z}=P_i$ for some $i=0, 1, \cdots, n$. Thus our conjecture is not contradicted.

One question in [6] remains unsettled. Is there a simple criterion (C) such that a ring R, in which every ideal has an irredundant primary representation, satisfies the Q-condition iff (C) is satisfied? Our discussion of 4.22 shows that if R has an ideal with a redundant MPD, then R does not satisfy the Q-condition. Hence a necessary condition for the Q-condition is that every MPD of each ideal of R be irredundant. Moreover, if this is the case, every semi-prime ideal of R has an irredundant primary representation.

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Louisiana State University, Baton Rouge, Louisiana
and Whitman College, Walla Walla, Washington