J. Math. Kyoto Univ. (JMKYAZ)
21-2 (1981) 331-349

On the group of self-homotopy equivalences of H-spaces of rank 2

By

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(Received January 19, 1980)

Dedicated to Professor T. Kudo on his 60-th birthday

§1. Introduction.

The set $\mathcal{E}(X)$ of homotopy classes of homotopy equivalences of a based space X to itself forms a group under composition of maps. This group is called the group of self-homotopy equivalences of X. The group $\mathcal{E}(X)$ has been studied by several authors (e.g. [2], [5], [10], [12], [14]).

In the present paper, we study the group $\mathcal{E}(X)$ for a simply connected, finite H-complex X of rank 2. The classification of simply connected, finite H-complexes of rank 2 has been given in [8] as follows: X is homotopy equivalent to one of $S^3\times S^3$, $SU(3)$, E_k ($k=0, 1, 3, 4, 5$), $S^7\times S^7$ and $G_{2,b}$ ($-2\leqq b\leqq 5$). Here E_k is the principal S^3-bundle over S^7 with the characteristic class $k\omega\in\pi_7(BS^3)\cong \boldsymbol{Z}_{12}$, ω a generator. For example $E_0=S^3\times S^7$, $E_1=Sp(2)$ and $G_{2,b}$ is the principal S^3-bundle over the Stiefel manifold $V_{7,2}=SO(7)/SO(5)$ induced by a suitable map f_b: $V_{7,2}\to BS^3$ (see §3 for details) such that $G_{2,0}=G_2$ is the compact, exceptional Lie group G_2 of rank 2.

For torsion free, finite H-complexes X of rank 2 which have been classified in [4], [17], the group $\mathcal{E}(X)$ is already known, that is, for $S^i\times S^j$ ($i, j=1, 3, 7$) in [13], [14], for $SU(3)$ and $Sp(2)$ in [10], and for E_k ($k\neq 0, 1$) in [12]. So we will determine $\mathcal{E}(G_{2,b})$ for $-2\leqq b\leqq 5$.

In a short exact sequence: $0\to A\to B\to C\to 1$, we write the group composition in A as addition, and it in B and C as multiplication.

Then the following is our main result obtained in §4.

Main Theorem. *We have the following exact sequences*:

(i) $0 \longrightarrow D(\boldsymbol{Z}_{168}\oplus \boldsymbol{Z}_2) \longrightarrow \mathcal{E}(G_{2,b}) \longrightarrow \boldsymbol{Z}_2 \longrightarrow 1 \qquad (b=-1, 0, 2, 3, 5)$,

(ii) $0 \longrightarrow D(\boldsymbol{Z}_{168}\oplus \boldsymbol{Z}_6) \longrightarrow \mathcal{E}(G_{2,b}) \longrightarrow \boldsymbol{Z}_2 \longrightarrow 1 \qquad (b=1, 4)$,

(iii) $0 \longrightarrow \boldsymbol{Z}_{168}\oplus \boldsymbol{Z}_6 \longrightarrow \mathcal{E}(G_{2,-2}) \longrightarrow G \longrightarrow 1$,

$0 \longrightarrow \boldsymbol{Z}_2\oplus \boldsymbol{Z}_2 \longrightarrow G \longrightarrow \boldsymbol{Z}_2 \longrightarrow 1$,

where, for an abelian group H, $D(H)$ is a group given by the split exact se-

quence: $0\to H\to D(H)\to \boldsymbol{Z}_2\to 1$, *with the splitting action* $\boldsymbol{Z}_2$ *on* H *given by* $(-1)\cdot h=-h$ *for* $-1\in\boldsymbol{Z}_2$ *and* $h\in H$.

The paper is organized as follows. The Barcus-Barratt theorem is introduced in § 2. In § 3 some results on homotopy of $G_{2,b}$, which will be needed in § 4, are prepared. In § 4 we study the group $\mathcal{E}(G_{2,b})$ by making use of the results of the previous sections and we obtain the main theorem. In the last section, § 5, we give a proof of the lemma used in § 4.

Throughout the paper, all spaces have homotopy types of CW-complexes with base points and all (continuous) maps and homotopies preserve the base points. For given spaces X and Y, we denote by $[X, Y]$ the set of (based) homotopy classes of maps from X to Y, and by the same letter f a map f: $X\to Y$ and its homotopy class $f\in[X, Y]$. The integral coefficient of the homology is omitted: $H_i(X)=H_i(X;\boldsymbol{Z})$. $X^{(n)}$ stands for the n-skeleton of X and $\pi_i(X: p)$ the p-component of $\pi_i(X)$ and $\boldsymbol{Q}_p$ the ring of those fractions, whose denominators, in the lowest form, are prime to p.

§ 2. The theorem of Barcus-Barratt.

Let K be a simply connected CW-complex of finite dimension. Let

$$S^q \xrightarrow{\alpha} K \xrightarrow{i} K\underset{\alpha}{\cup} e^{q+1} \xrightarrow{p} S^{q+1}, \qquad q>\dim K,$$

be the sequence of induced cofiberings. The coaction

$$l:\ K\cup e^{q+1}\longrightarrow (K\cup e^{q+1})\vee S^{q+1}$$

is defined by shrinking the equator $S^q\times\{1/2\}$ of e^{q+1}. We define a map

$$\Lambda:\ \pi_{q+1}(K\cup e^{q+1})\longrightarrow [K\cup e^{q+1}, K\cup e^{q+1}] \tag{2.1}$$

by $\Lambda(\xi)=\nabla\circ(1\vee\xi)\circ l$, where ∇ is the folding map and 1 is the class of identity map of $K\cup e^{q+1}$. Furthermore, since $q>\dim K$, by the restriction Λ on $i_*\pi_{q+1}(K)$ we can define a homomorphism (cf. Lemmas 1.4 and 1.8 of [10])

$$\lambda:\ i_*\pi_{q+1}(K)\longrightarrow \mathcal{E}(K\cup e^{q+1}).$$

Since $q>\dim K$, the induced maps

(2.2) $\qquad i_*: [K, K]\longrightarrow [K, K\cup e^{q+1}]$ *and*

$\qquad p^*: [S^{q+1}, S^{q+1}]\longrightarrow [K\cup e^{q+1}, S^{q+1}]$ *are both bijective.*

So the maps Φ: $[K\cup e^{q+1}, K\cup e^{q+1}]\to[K, K]$ and Ψ: $[K\cup e^{q+1}, K\cup e^{q+1}]\to[S^{q+1}, S^{q+1}]$ can be defined by the following homotopy commutative diagram:

$$\begin{array}{ccccc} K & \xrightarrow{i} & K\cup e^{q+1} & \xrightarrow{p} & S^{q+1} \\ \downarrow{\scriptstyle \Phi(h)} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Psi(h)} \\ K & \xrightarrow{i} & K\cup e^{q+1} & \xrightarrow{p} & S^{q+1}. \end{array}$$

Since these two maps preserve composition of maps, a homomorphism

$$\phi\times\psi:\ \mathcal{E}(K\cup e^{q+1})\longrightarrow\mathcal{E}(K)\times\mathcal{E}(S^{q+1})$$

can be defined by the restriction of $\Phi\times\Psi$. Let $g:\ (S^q, s_o)\rightarrow(K, k_o)$ be a map representing $\alpha\in\pi_q(K)$. Let X^Y be the function space of maps: $(Y, y_o)\rightarrow(X, x_0)$. Then Barcus-Barratt defined in [2] a homomorphism

$$\alpha_i:\ \pi_1(X^K, i)\xrightarrow{g^*}\pi_1(X^{S^q}, i\circ g)\xrightarrow{(i\circ g)_\natural}\pi_{q+1}(X).$$

Here $X=K\cup e^{q+1}$, $g^*(F)=F\circ(g\times 1_I)$, 1_I is the identity of $I=[0, 1]$, $(i\circ g)_\natural(F)=d(F, (i\circ g)^\flat)$ where $(i\circ g)^\flat:\ S^q\times I\rightarrow K\cup e^{q+1}$ is the map defined by $(i\circ g)^\flat(s, t)=i\circ g(s)$, and $d(F, (i\circ g)^\flat)$ is the separation element of F and $(i\circ g)^\flat$ (see [2] for the definition). Then we have the following theorem due to Barcus-Barratt.

Theorem 2.1. (Theorem 6.1 of [2]) *The following sequence is exact*:

$$0\longrightarrow\Delta\longrightarrow i_*\pi_{q+1}(K)\xrightarrow{\lambda}\mathcal{E}(K\underset{\alpha}{\cup} e^{q+1})\xrightarrow{\phi\times\psi}G\longrightarrow 1.$$

Here $\Delta=i_*\pi_{q+1}(K)\cap\alpha_i\pi_1((K\cup e^{q+1})^K, i)$, *the subgroup* G *of* $\mathcal{E}(K)\times\mathcal{E}(S^{q+1})$ *is isomorphic to*

$G_1=\{h\in\mathcal{E}(K)\,|\,h_*\alpha=\varepsilon\alpha,\ \varepsilon=\pm 1,\ in\ \pi_q(K)\}$, *if* $2\alpha\neq 0$,
and G *is isomorphic to* $G_1\times\boldsymbol{Z}_2$, *if* $2\alpha=0$.

The next corollary will be used in the later section.

Corollary 2.2. (cf. Remark in p. 304 of [12]) *If* $K\cup e^{q+1}$ *has a multiplication, the homomorphism* λ *and the group* Δ *are given as follows*:

$$\lambda(\xi)=1+\xi\circ p\,,\qquad \xi\in i_*\pi_{q+1}(K)\,,$$

$$\Delta=(S\alpha)^*[SK,\ K\cup e^{q+1}]\,.$$

Therefore we have the exact sequence:

(2.3) $$0\longrightarrow H\longrightarrow\mathcal{E}(K\cup e^{q+1})\longrightarrow G\longrightarrow 1\,,$$

where G *is given in the above theorem, and* H *is given as follows*:

(2.4) $$H=i_*\pi_{q+1}(K)/(S\alpha)^*[SK,\ K\cup e^{q+1}]\,.$$

Proof. By the definition of λ, we have a homotopy commutative diagram:

$$\begin{tikzcd}
K\cup e^{q+1} \arrow[r, "l"] \arrow[d, "d"] & (K\cup e^{q+1})\vee S^{q+1} \arrow[rr, "1\vee\xi"] \arrow[dr, "j"] & & (K\cup e^{q+1})\vee(K\cup e^{q+1}) \arrow[rr, "\nabla"] \arrow[drr, "j"] & & K\cup e^{q+1} \\
(K\cup e^{q+1})\times(K\cup e^{q+1}) \arrow[rr, "1\times p"] & & (K\cup e^{q+1})\times S^{q+1} \arrow[rrr, "1\times\xi"] & & & (K\cup e^{q+1})\times(K\cup e^{q+1}), \arrow[u, "m"]
\end{tikzcd}$$

where m is a multiplication on $K\cup e^{q+1}$, j is the inclusion and d is the diagonal map. Hence we have $\lambda(\xi)=1+\xi\circ p$ and so we have

$$\lambda^{-1}(1)=p^{*-1}(0)\cap i_*\pi_{q+1}(K).$$

Consider the following commutative diagram consisting of the Puppe exact sequence:

$$\begin{array}{ccccc} [SK, K\cup e^{q+1}] & \xrightarrow{(S\alpha)^*} & [S^{q+1}, K\cup e^{q+1}] & \xrightarrow{p^*} & [K\cup e^{q+1}, K\cup e^{q+1}] \\ & & \downarrow p_* & & \downarrow p_* \\ & & [S^{q+1}, S^{q+1}] & \xrightarrow{p^*} & [K\cup e^{q+1}, S^{q+1}] \end{array}$$

where the lower p^* is bijective by (2.2). We have

$$(S\alpha)^*[SK, K\cup e^{q+1}]\subset \mathrm{Ker}\{p_*: \pi_{q+1}(K\cup e^{q+1}) \longrightarrow \pi_{q+1}(S^{q+1})\}=i_*\pi_{q+1}(K)$$

since $\pi_{q+1}(K)\xrightarrow{i_*}\pi_{q+1}(K\cup e^{q+1})\xrightarrow{p_*}\pi_{q+1}(S^{q+1})\longrightarrow\cdots$ is exact. Therefore we have

$$p^{*-1}(0)=(S\alpha)^*[SK, K\cup e^{q+1}]\subset i_*\pi_{q+1}(K).$$

So, we have $\Delta=\lambda^{-1}(1)=(S\alpha)^*[SK, K\cup e^{q+1}]$. q.e.d.

§ 3. Some homotopy of $G_{2,b}$.

Let G_2 be the compact, exceptional Lie group of rank 2. Let $f: V_{7,2}\to BS^3$ be the classifying map of G_2. Let $\phi: V_{7,2}\to V_{7,2}\vee S^{11}$ be the map shrinking the equator $S^{10}\times\{1/2\}$ in $V_{7,2}=M^6\cup CS^{10}$. Let α be a generator of $\pi_{11}(BS^3)\cong\pi_{10}(S^3)\cong \boldsymbol{Z}_{15}$ which corresponds to 8ω under the monomorphism: $\pi_{10}(S^3)\to\pi_{10}(G_2^{(9)})$ (see Lemma 3.9). For each integer b, let $g_b: S^{11}\to BS^3$ represent $b\alpha$ and let $G_{2,b}$ be the principal S^3-bundle over $V_{7,2}$ induced by the composition

$$f_b=\nabla\circ(f\vee g_b)\circ\phi: V_{7,2}\longrightarrow V_{7,2}\vee S^{11}\longrightarrow BS^3\vee BS^3\longrightarrow BS^3.$$

For example, $G_2=G_{2,0}$.

Recall the following

Theorem 3.1. (Theorem 5.1 of [8]) *Let X be a 1-connected, finite H-complex of rank 2 such that $H_*(X; \boldsymbol{Z})$ has 2-torsion. Then X is homotopy equivalent to $G_{2,b}$ for some b. There are just 8 homotopy types of such H-complexes: $G_{2,b}$ for $-2\leqq b\leqq 5$.*

By making use of the exact sequence associated with the fibering $SU(3)\xrightarrow{i} G_2\xrightarrow{p} S^6$, one can compute $\pi_i(G_2: 2)$ (the odd primary components of $\pi_i(G_2)$ are computed by the killing-homotopy method).

Lemma 3.2. ([6]) *$\pi_i(G_2)$ for $i\leqq 14$ are as follows:*

i	1	2	3	4	5	6	7	8	9	10
$\pi_i(G_2)$	0	0	$\boldsymbol{Z}$	0	0	$\boldsymbol{Z}_3$	0	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_6$	0
gen. of 2-comp.			$i_*\iota_3$					$\langle\eta_6^2\rangle$	$\langle\eta_6^2\rangle\circ\eta_8$	

11	12	13	14
$\boldsymbol{Z}\oplus\boldsymbol{Z}_2$	0	0	$\boldsymbol{Z}_8\oplus\boldsymbol{Z}_2\oplus\boldsymbol{Z}_{21}$
$\langle 2\Delta\iota_{13}\rangle$, $i_*[\nu_5^2]$			$\langle\bar{\nu}_6+\varepsilon_6\rangle$, $i_*[\nu_5^2]\circ\nu_{11}$

where the notation $[\alpha]$ *means such an element of* $\pi_i(SU(3):2)$ *that* $q_*[\alpha]=\alpha\in\pi_i(S^5:2)$ *for the projection* $q: SU(3)\to S^5=SU(3)/SU(2)$, *and the notation* $\langle\beta\rangle$ *means such an element of* $\pi_i(G_2:2)$ *that* $p_*\langle\beta\rangle=\beta\in\pi_i(S^6:2)$ *for the projection* $p: G_2\to S^6$.

By [8, § 6] we have

$$G_{2,b}\underset{p}{\simeq}G_2 \quad \text{for} \quad p\neq 3,\ 5,$$

$$G_{2,b}\underset{3}{\simeq}\begin{cases} G_2 & (b=-1,\ 0,\ 2,\ 3,\ 5) \\ S^3\times S^{11} & (b=-2,\ 1,\ 4)\,, \end{cases}$$

$$G_{2,b}\underset{5}{\simeq}\begin{cases} G_2 & (b=-1,\ 0,\ 1,\ 2,\ 4,\ 5) \\ S^3\times S^{11} & (b=-2,\ 3)\,. \end{cases}$$

By Lemma 3.2 and by the results in Toda's book [15] these p-equivalences give

Lemma 3.3. (i) $\pi_{10}(G_{2,-2})\cong\boldsymbol{Z}_{15}$, $\pi_{10}(G_{2,b})\cong\boldsymbol{Z}_3$ $(b=1,\ 4)$, $\pi_{10}(G_{2,3})\cong\boldsymbol{Z}_5$.

(ii) $\pi_{13}(G_{2,b})\cong\boldsymbol{Z}_3$ $(b=-2,\ 1,\ 4)$.

(iii) $\pi_{14}(G_{2,b})\cong\boldsymbol{Z}_{168}\oplus\boldsymbol{Z}_6$ $(b=-2,\ 1,\ 4)$.

(iv) *The other homotopy groups* $\pi_i(G_{2,b})$ *for* $0\leqq i\leqq 14$ $(-2\leqq b\leqq 5)$ *are isomorphic to* $\pi_i(G_2)$ *given in Lemma* 3.2.

By Theorem 2.2 of [8] we have

Theorem 3.4. (i) $H^*(G_{2,b};\ \boldsymbol{Z}_2)=\boldsymbol{Z}_2[x_3]/(x_3^4)\otimes\Lambda(Sq^2x_3)$.

(ii) $H^*(G_{2,b};\ \boldsymbol{Z}_p)\cong\Lambda(x_3,\ x_{11})$ *for each prime* $p\geqq 3$, *where* $\deg x_i=i$.

Therefore $G_{2,b}$ has a cell structure:

$$G_{2,b}\simeq S^3\cup e^5\cup e^6\cup e^8\cup e^9\cup e^{11}\cup e^{14}\,.$$

Let $M^n=S^{n-1}\underset{2}{\cup} e^n$ be the mapping cone of a map: $S^{n-1}\to S^{n-1}$ of degree 2. Then we have two cofiberings:

(3.1) $$S^3\longrightarrow G_{2,b}^{(6)}\longrightarrow M^6\,,\qquad G_{2,b}^{(6)}\longrightarrow G_{2,b}^{(9)}\longrightarrow M^9\,,$$

which are equivalent to induced cofiberings by some maps $f_1: M^5 \to S^3$ and $f_2: M^8 \to G_{2,b}^{(6)}$ respectively by [3].

Lemma 3.5. (i) $[M^5, G_{2,b}^{(6)}]=[M^6, G_{2,b}^{(6)}]=[M^7, G_{2,b}^{(9)}]=0$.

(ii) $[M^9, G_{2,b}^{(9)}]\cong[M^9, G_{2,b}]\cong \boldsymbol{Z}_4$ *generated by an extension of a non-trivial element of* $\pi_8(G_{2,b})\cong \boldsymbol{Z}_2$.

(iii) $[M^{10}, G_{2,b}]\cong \boldsymbol{Z}_2$ *generated by an extension of a non-trivial element of* $\pi_9(G_{2,b}: 2)\cong \boldsymbol{Z}_2$.

Proof. (i) Consider the Puppe exact sequence

$$\begin{aligned}(3.2)\qquad \cdots \longrightarrow [S^n, G_{2,b}] \longrightarrow [S^n, G_{2,b}] \xrightarrow{p^*} [M^n, G_{2,b}]\\ \xrightarrow{i^*} [S^{n-1}, G_{2,b}] \longrightarrow [S^{n-1}, G_{2,b}] \longrightarrow \cdots.\end{aligned}$$

We have $\pi_4(G_{2,b})=\pi_5(G_{2,b})=\pi_7(G_{2,b})=0$ and $\pi_6(G_{2,b})\cong \boldsymbol{Z}_3$ by Lemmas 3.2 and 3.3. Therefore we have that $[M^5, G_{2,b}]=[M^6, G_{2,b}]=[M^7, G_{2,b}]=0$, since $[M^n, X]$ is a $\boldsymbol{Z}_4$-group by [1].

For dimensional reasons, we have $[M^5, G_{2,b}^{(6)}]=[M^6, G_{2,b}^{(6)}]=[M^7, G_{2,b}^{(9)}]=0$.

(ii) Since $\pi_8(G_{2,b})\cong \boldsymbol{Z}_2$ and $\pi_9(G_{2,b}: 2)\cong \boldsymbol{Z}_2$, by the above exact sequence for $n=9$, we have an exact sequence:

$$0 \longrightarrow \boldsymbol{Z}_2 \xrightarrow{p^*} [M^9, G_{2,b}] \xrightarrow{i^*} \boldsymbol{Z}_2 \longrightarrow 0.$$

Since $G_{2,b}$ is 2-equivalent to G_2, we may verify it in the case $b=0$. Let Ext $\langle \eta_6^2 \rangle$ be an extension of $\langle \eta_6^2 \rangle \in \pi_8(G_2)$. Since $2\, 1_{M^9}=i\circ\eta_8\circ p$ by [16], we have 2 Ext $\langle \eta_6^2 \rangle$ $=$Ext $\langle \eta_6^2 \rangle\circ i\circ\eta_8\circ p=\langle \eta_6^2 \rangle\circ\eta_8\circ p$ and so 2 Ext $\langle \eta_6^2 \rangle\neq 0$ in $[M^9, G_2]$ by Lemma 3.2 (Recall that $\langle \eta_6^2 \rangle\circ\eta_8\neq 0$ and that p^* is monic). Therefore we have $[M^9, G_{2,b}]\cong \boldsymbol{Z}_4$ generated by an extension of a non-trivial element of $\pi_8(G_{2,b})\cong \boldsymbol{Z}_2$.

(iii) By Lemmas 3.2 and 3.3 we have $\pi_9(G_{2,b}: 2)\cong \boldsymbol{Z}_2$ and $\pi_{10}(G_{2,b}: 2)=0$. Therefore we have immediately $[M^{10}, G_{2,b}]\cong\pi_9(G_{2,b}: 2)\cong \boldsymbol{Z}_2$ by (3.2). q.e.d.

Lemma 3.6. (i) $i_*: [S^3, S^3] \longrightarrow [S^3, G_{2,b}^{(6)}]$ *and*

$$i^*: [G_{2,b}^{(6)}, G_{2,b}^{(6)}] \longrightarrow [S^3, G_{2,b}^{(6)}] \text{ are both bijective}.$$

(ii) $[SG_{2,b}^{(6)}, G_{2,b}^{(9)}]=[SG_{2,b}^{(6)}, G_{2,b}]=0$.

Proof. (i) For dimensional reasons we see easily that $i_*: [S^3, S^3] \to [S^3, G_{2,b}^{(6)}]$ is bijective. Consider the Puppe exact sequence associated with (3.1):

$$\longrightarrow [M^6, G_{2,b}^{(6)}] \longrightarrow [G_{2,b}^{(6)}, G_{2,b}^{(6)}] \xrightarrow{i^*} [S^3, G_{2,b}^{(6)}] \xrightarrow{f_1^*} [M^5, G_{2,b}^{(6)}] \longrightarrow.$$

By (i) of Lemma 3.5 $[M^n, G_{2,b}^{(6)}]=0$ for $n=5, 6$. Therefore i^* is bijective.

(ii) Consider the Puppe exact sequence associated with (3.1):

$$\cdots \longrightarrow [M^7, G_{2,b}^{(9)}] \longrightarrow [SG_{2,b}^{(6)}, G_{2,b}^{(9)}] \longrightarrow [S^4, G_{2,b}^{(9)}] \longrightarrow \cdots$$

Since $[M^7, G_{2,b}^{(9)}]=[S^4, G_{2,b}^{(9)}]=0$ by (i) of Lemma 3.5 and Lemmas 3.2 and 3.3 for dimensional reasons, we have

$$[SG_{2,b}^{(6)}, G_{2,b}]=[SG_{2,b}^{(6)}, G_{2,b}^{(9)}]=0. \qquad \text{q.e.d.}$$

Lemma 3.7. (i) $\pi_7(G_{2,b}^{(6)})\cong \boldsymbol{Z}_2$.

(ii) $\pi_8(G_{2,b}^{(6)})=\boldsymbol{Z}_4$.

Proof. Let F be the 3-connective fibre space over $G_{2,b}^{(6)}$. Then we have a fibering:

$$F \xrightarrow{i} G_{2,b}^{(6)} \xrightarrow{\pi} K(\boldsymbol{Z}, 3).$$

Since $H^*(G_{2,b}^{(6)}; \boldsymbol{Z}_2)=\{1, x_3, x_5=Sq^2x_3, x_3^2\}$, we see that $\pi^*: H^*(\boldsymbol{Z}, 3; \boldsymbol{Z}_2)\to H^*(G_{2,b}^{(6)}; \boldsymbol{Z}_2)$ is an epimorphism with $\operatorname{Ker}\pi^*=\sum_{i\geq 8} H^i(\boldsymbol{Z}, 3; \boldsymbol{Z}_2)$. Therefore, there exists a transgressive element $y_7\in H^7(F; \boldsymbol{Z}_2)$ whose transgression image is $\tau(y_7)=uSq^2u$ where $u\in H^3(\boldsymbol{Z}, 3; \boldsymbol{Z}_2)$ is the fundamental class. Then $\tau(Sq^1y_7)=Sq^1\tau(y_7)=Sq^1(uSq^2u)=u^3$. So there exists a transgressive element $y_8\in H^8(F; \boldsymbol{Z}_2)$ such that $\tau(y_8)=Sq^4Sq^2u$. Then $\tau(Sq^1y_8)=Sq^1\tau(y_8)=Sq^1Sq^4Sq^2u=Sq^5Sq^2u=(Sq^2u)^2$ and $\tau(Sq^2y_7)=Sq^2\tau(y_7)=Sq^2(uSq^2u)=(Sq^2u)^2$, whence $Sq^1y_8=Sq^2y_7$. Thus we have

$$H^*(F; \boldsymbol{Z}_2)=\{1, y_7, Sq^1y_7, y_8, Sq^1y_8, Sq^2Sq^1y_8, \cdots\}.$$

Take a CW-complex L with minimum cells 2-equivalent to F, and so we may take $L=((S^7\cup e^8)\vee S^8)\cup e^9\cup e^{11}\cup\cdots$. The attaching class of the 9-cell in L is $i\circ\eta_7\vee 2\iota_8$: $S^8\to(S^7\cup e^8)\vee S^8$ where $i: S^7\to S^7\cup e^8$ is the inclusion. Consider the exact sequence of the pair $(L, S^7\cup e^8\vee S^8)$:

$$\pi_9(L, S^7\cup e^8\vee S^8)\xrightarrow{\partial}\pi_8(S^7\cup e^8\vee S^8)\to\pi_8(L)\to\pi_8(L, S^7\cup e^8\vee S^8)\to,$$

where $\pi_9(L, S^7\cup e^8\vee S^8)\cong\pi_9(S^9)\cong\boldsymbol{Z}$ generated by ι_9, $\pi_8(L, S^7\cup e^8\vee S^8)=0$, $\pi_8(S^7\cup e^8\vee S^8)\cong\boldsymbol{Z}_2\oplus\boldsymbol{Z}$ generated by $i\circ\eta_7$ and ι_8, and $\partial(\iota_9)=i\circ\eta_7+2\iota_8$. Therefore $\pi_8(L)\cong\boldsymbol{Z}_4$. Also, we have immediately $\pi_7(L)\cong\boldsymbol{Z}_2$. Since $\pi_i(L:2)$ for $i=7, 8$ is isomorphic to $\pi_i(G_{2,b}^{(6)}:2)$, $\pi_7(G_{2,b}^{(6)}:2)$ and $\pi_8(G_{2,b}^{(6)}:2)$ are isomorphic to $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_4$ respectively. Since $i^*: H^*(G_{2,b}^{(6)}; \boldsymbol{Z}_p)\to H^*(S^3; \boldsymbol{Z}_p)$ is isomorphic for any odd prime p, the inclusion $i: S^3\to G_{2,b}^{(6)}$ is a p-equivalence. So $\pi_7(G_{2,b}^{(6)}:p)=\pi_8(G_{2,b}^{(6)}:p)=0$ for any odd prime p, since $\pi_7(S^3)\cong\boldsymbol{Z}_2$ and $\pi_8(S^3)\cong\boldsymbol{Z}_2$ by [15]. Therefore, we have $\pi_7(G_{2,b}^{(6)})\cong\boldsymbol{Z}_2$ and $\pi_8(G_{2,b}^{(6)})\cong\boldsymbol{Z}_4$. q.e.d.

Lemma 3.8. *Let β be a generator of $[M^9, G_{2,b}^{(9)}]\cong\boldsymbol{Z}_4$ and let $\pi: G_{2,b}^{(9)}\to M^9=G_{2,b}^{(9)}/G_{2,b}^{(6)}$ be the projection. Then we have $\pi_*\beta=2\ 1_{M^9}\in[M^9, M^9]\cong\boldsymbol{Z}_4$.*

Proof. Consider the following commutative diagram of the exact sequence:

$$\begin{array}{ccccccccc}
[S^8, G_{2,b}^{(6)}] & \xrightarrow{i_*} & [S^8, G_{2,b}^{(9)}] & \xrightarrow{\pi_*} & [S^8, M^9] & \xrightarrow{\partial} & [S^7, G_{2,b}^{(6)}] & \xrightarrow{i_*} & [S^7, G_{2,b}^{(9)}] \\
\uparrow j^* & & \uparrow j^* & & \uparrow j^* & & \uparrow j^* & & \uparrow j^* \\
[M^9, G_{2,b}^{(6)}] & \xrightarrow{i_*} & [M^9, G_{2,b}^{(9)}] & \xrightarrow{\pi_*} & [M^9, M^9] & \xrightarrow{\partial} & [M^8, G_{2,b}^{(6)}] & \xrightarrow{i_*} & [M^8, G_{2,b}^{(9)}]
\end{array}$$

where $j: S^n \to M^{n+1}$ and $i: G_{2,b}^{(6)} \to G_{2,b}^{(9)}$ are the natural inclusions. Recall that $\pi_8(G_{2,b}^{(6)}) \cong \boldsymbol{Z}_4$ and $\pi_7(G_{2,b}^{(6)}) \cong \boldsymbol{Z}_2$ by Lemma 3.7, $\pi_8(G_{2,b}^{(9)}) \cong \boldsymbol{Z}_2$ and $\pi_7(G_{2,b}^{(9)}) = 0$ by Lemmas 3.2 and 3.3 and $\pi_8(M^9) \cong \boldsymbol{Z}_2$ generated by j, by [9].

Clearly the upper π_* is trivial and so $j^*\pi_*\beta = \pi_* j^* \beta = 0$. Therefore $\pi_*\beta$ is not a generator of $[M^9, M^9] \cong \boldsymbol{Z}_4$, since $[M^9, M^9] \cong \boldsymbol{Z}_4$ generated by 1_{M^9}, by [9]. We have $\operatorname{Im}\{j^*: [M^9, G_{2,b}^{(6)}] \to [S^8, G_{2,b}^{(6)}]\} = \operatorname{Tor}(\pi_8(G_{2,b}^{(6)}), \boldsymbol{Z}_2) = \boldsymbol{Z}_2$ by (ii) of Lemma 3.7. It follows from this fact that $j^* i_* = i_* j^* = 0$ in the left diagram and that a generator $\beta \in [M^9, G_{2,b}^{(9)}]$ is not contained in $\operatorname{Im}\{i_*: [M^9, G_{2,b}^{(6)}] \to [M^9, G_{2,b}^{(9)}]\}$, since $j^*\beta$ is non-trivial by (ii) of Lemma 3.5. Therefore $\pi_*\beta$ is non-trivial and is not a generator of $[M^9, M^9]$. Thus $\pi_*\beta = 2\, 1_{M^9}$. q.e.d.

The following lemma is a summary of Lemmas 4.3, 5.2 and 5.3 in [8].

Lemma 3.9. (i) *$G_{2,b}^{(9)} \simeq G_2^{(9)}$ and $\pi_{10}(G_{2,b}^{(9)}) \cong \boldsymbol{Z}_{120}$.*

(ii) *The attaching class of the 11-cell in $G_{2,b}^{(11)} = G_{2,b}^{(9)} \cup e^{11}$ is $(1+8b)\omega$ with ω a generator of $\pi_{10}(G_{2,b}^{(9)}) \cong \boldsymbol{Z}_{120}$.*

(iii) *Let $\pi: G_2^{(9)} \to M^9 = G_2^{(9)}/G_2^{(6)}$ be the projection. Then $\pi_*(\omega) = \gamma$ is a generator of $\pi_{10}(M^9) \cong \boldsymbol{Z}_4$.*

Let $i: G_{2,b}^{(9)} \to G_{2,b}^{(11)}$ be the inclusion.

Lemma 3.10. $i_*\pi_{11}(G_{2,b}^{(9)}) \cong \boldsymbol{Z}_2$.

Proof. Consider the exact sequence of the pair $(G_{2,b}^{(11)}, G_{2,b}^{(9)})$:

$$\cdots \longrightarrow \pi_{11}(G_{2,b}^{(9)}) \xrightarrow{i_*} \pi_{11}(G_{2,b}^{(11)}) \longrightarrow \pi_{11}(G_{2,b}^{(11)}, G_{2,b}^{(9)}) \xrightarrow{\partial} \pi_{10}(G_{2,b}^{(9)}) \longrightarrow \cdots,$$

where $\pi_{11}(G_{2,b}^{(11)}) \cong \pi_{11}(G_{2,b}) = \boldsymbol{Z} \oplus \boldsymbol{Z}_2$ by Lemmas 3.2 and 3.3, $\pi_{11}(G_{2,b}^{(11)}, G_{2,b}^{(9)}) \cong \pi_{11}(S^{11}) \cong \boldsymbol{Z}$ by the Blakers-Massey theorem, and $\pi_{10}(G_{2,b}^{(9)}) \cong \boldsymbol{Z}_{120}$ by (i) of Lemma 3.9. Then we have immediately

$$i_*\pi_{11}(G_{2,b}^{(9)}) \cong \boldsymbol{Z}_2 .$$

q.e.d.

Lemma 3.11. *For ω a generator of $\pi_{10}(G_{2,b}^{(9)})$, the homomorphism*

$$(S\omega)^*: [SG_{2,b}^{(9)}, G_{2,b}] \longrightarrow [S^{11}, G_{2,b}]$$

is trivial for $-2 \leqq b \leqq 5$.

Proof. Since the suspension homomorphism: $\pi_{10}(M^9) \to \pi_{11}(M^{10})$ is clearly isomorphic, we have $\pi \circ S\omega = \gamma$ by (iii) of Lemma 3.9, where $\pi: SG_{2,b}^{(9)} \to M^{10}$ is the projection and γ is a generator of $\pi_{11}(M^{10})$. We have the following commutative diagram:

(3.3)
$$\begin{tikzcd}
{} \arrow[r] & {[M^{10}, G_{2,b}]} \arrow[r, "\pi^*"] \arrow[dr, "\gamma^*"'] & {[SG_{2,b}^{(9)}, G_{2,b}]} \arrow[r] \arrow[d, "(S\omega)^*"] & {[SG_{2,b}^{(6)}, G_{2,b}]} \\
 & & {[S^{11}, G_{2,b}]} &
\end{tikzcd}$$

where the horizontal sequence is the Puppe exact sequence associated with the cofibering (3.1).

Since $[SG_{2,b}^{(6)}, G_{2,b}]=0$ by (ii) of Lemma 3.6, we see that

(3.4) $\pi^*: [M^{10}, G_{2,b}] \longrightarrow [SG_{2,b}^{(9)}, G_{2,b}]$ *is epimorphic.*

Next we will show that

(3.5) $$\gamma^*=0: [M^{10}, G_{2,b}] \longrightarrow [S^{11}, G_{2,b}].$$

As $[M^n, X]$ is a $\boldsymbol{Z}_4$-group and $G_{2,b}$ is 2-equivalent to G_2, it is sufficient to show it for the case $b=0$. By Lemma 3.2 and (iii) of Lemma 3.5 we have $[M^{10}, G_2] \cong \boldsymbol{Z}_2$ generated by an extension $\mathrm{Ext}\{\langle\eta_6^2\rangle\circ\eta_8\}$ of $\langle\eta_6^2\rangle\circ\eta_8\in\pi_9(G_2)$. Since γ is a coextension of $\eta_{10}\in\pi_{11}(S^{10})$, we have

$$\mathrm{Ext}\{\langle\eta_6^2\rangle\circ\eta_8\}\circ\gamma\in\{\langle\eta_6^2\rangle\circ\eta_8, 2\iota_9, \eta_9\}.$$

By (5.4) and (5.5) of [15] and Lemma 3.2 we have

$$\begin{aligned}\{\langle\eta_6^2\rangle\circ\eta_8, 2\iota_9, \eta_9\} &\supset \langle\eta_6^2\rangle\circ\{\eta_8, 2\iota_9, \eta_9\}\\ &=\langle\eta_6^2\rangle\circ\{S^5\nu', -S^5\nu'\}\\ &=\langle\eta_6^2\rangle\circ\{2\nu_8, -2\nu_8\}=0\end{aligned}$$

$$\text{modulo } \langle\eta_6^2\rangle\circ\eta_8\circ\pi_{11}(S^9)+\pi_{10}(G_2)\circ\eta_{10}=\{\langle\eta_6^2\rangle\circ\eta_8^3\}=\{\langle\eta_6^2\rangle\circ 4\nu_8\}=0.$$

Therefore we have $\mathrm{Ext}\{\langle\eta_6^2\rangle\circ\eta_8\}\circ\gamma=0$, and so (3.5) was proved. By (3.4) and (3.5) and by the commutativity of (3.3) we have

$$\mathrm{Im}\,(S\omega)^*=\mathrm{Im}\,(S\omega)^*\circ\pi^*=\mathrm{Im}\,\gamma^*=0. \qquad \text{q.e.d.}$$

Lemma 3.12. (i) *For* $b=-1, 0, 2, 3, 5$, $\pi_{13}(G_{2,b}^{(11)})\cong\boldsymbol{Z}$ *generated by the attaching class of the* 14-*cell in* $G_{2,b}=G_{2,b}^{(11)}\cup e^{14}$.

(ii) *For* $b=-2, 1, 4$, *the short exact sequence*

$$0 \longrightarrow \pi_{14}(G_{2,b}, G_{2,b}^{(11)}) \xrightarrow{\partial} \pi_{13}(G_{2,b}^{(11)}) \longrightarrow \pi_{13}(G_{2,b}) \longrightarrow 0$$

has a splitting homomorphism $\rho: \pi_{13}(G_{2,b})\to\pi_{13}(G_{2,b}^{(11)})$, *where* $\pi_{14}(G_{2,b}, G_{2,b}^{(11)})\cong\boldsymbol{Z}$ and $\pi_{13}(G_{2,b})\cong\boldsymbol{Z}_3$, *and so we have*

$$\pi_{13}(G_{2,b}^{(11)})\cong\rho\pi_{13}(G_{2,b})\oplus\partial\pi_{14}(G_{2,b}, G_{2,b}^{(11)})=\boldsymbol{Z}_3\oplus\boldsymbol{Z}.$$

Here the free part of $\pi_{13}(G_{2,b}^{(11)})$ *is generated by the attaching class of the* 14-*cell in* $G_{2,b}=G_{2,b}^{(11)}\cup e^{14}$.

(iii) *Let* $i: G_{2,b}^{(11)}\to G_{2,b}$ *be the inclusion. Then the homomorphism* $i_*: \pi_{14}(G_{2,b}^{(11)})\to\pi_{14}(G_{2,b})$ *is epimorphic for* $-2\leqq b\leqq 5$.

Proof. Consider the exact sequence of the pair $(G_{2,b}, G_{2,b}^{(11)})$:

(3.6) $$\cdots \longrightarrow \pi_{14}(G_{2,b}^{(11)}) \xrightarrow{i_*} \pi_{14}(G_{2,b}) \xrightarrow{j_*} \pi_{14}(G_{2,b}, G_{2,b}^{(11)}) \xrightarrow{\partial} \pi_{13}(G_{2,b}^{(11)}) \longrightarrow \pi_{13}(G_{2,b}) \longrightarrow \cdots,$$

where $\pi_{14}(G_{2,b}, G_{2,b}^{(11)}) \cong \pi_{14}(S^{14}) \cong \boldsymbol{Z}$ by the Blakers-Massey theorem. Recall from Lemmas 3.2 and 3.3 we have $\pi_{14}(G_{2,b}) \cong \boldsymbol{Z}_{168} \oplus \boldsymbol{Z}_2$ and $\pi_{13}(G_{2,b})=0$ for $b=-1, 0, 2, 3, 5$. Therefore we see that $\pi_{13}(G_{2,b}^{(11)}) \cong \boldsymbol{Z}$ generated by the attaching class of the 14-cell in $G_{2,b}=G_{2,b}^{(11)} \cup e^{14}$. For $b=-2, 1, 4$, we have $\pi_{13}(G_{2,b}) \cong \boldsymbol{Z}_3$ by (ii) of Lemma 3.3 and $G_{2,b}^{(11)}$ is 3-equivalent to $S^3 \vee S^{11}$ by [8, § 6], where $\pi_{13}(S^3 \vee S^{11}) \cong \pi_{13}(S^3) \oplus \pi_{13}(S^{11}) \oplus \pi_{14}(S^3 \times S^{11}, S^3 \vee S^{11}) = \boldsymbol{Z}_{12} \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}$ by [15]. Therefore we see that $\pi_{13}(G_{2,b}^{(11)}) \cong \boldsymbol{Z}_3 \oplus \boldsymbol{Z}$, and so by (3.6) we have the split exact sequence in the lemma and the desired results.

It follows from the above argument that the homomorphism $j_*: \pi_{14}(G_{2,b}) \to \pi_{14}(G_{2,b}, G_{2,b}^{(11)})$ is trivial, and so by (3.6) we have (iii). q.e.d.

Lemma 3.13. *Let f be the attaching class of the 14-cell in $G_{2,b}=G_{2,b}^{(11)} \cup e^{14}$. Then the homomorphism*

$$(Sf)^*: [SG_{2,b}^{(11)}, G_{2,b}] \longrightarrow [S^{14}, G_{2,b}]$$

is trivial for $-2 \leqq b \leqq 5$.

Proof. Let $\pi: SG_{2,b}^{(9)} \to SG_{2,b}^{(9)}/SG_{2,b}^{(6)}=M^{10}$ be the projection. Let γ be a generator of $\pi_{11}(M^{10})$. Then by (iii) of Lemma 3.9 we have a commutative diagram

$$\begin{tikzcd}
\pi_{14}(SG_{2,b}^{(9)}) \arrow[r, "i_*"] & \pi_{14}(SG_{2,b}^{(11)}) \arrow[r] & \pi_{14}(SG_{2,b}^{(11)}, SG_{2,b}^{(9)}) \arrow[r, "\partial"] \arrow[dl, "\cong"'] & \pi_{13}(SG_{2,b}^{(9)}) \arrow[d, "\pi_*"] \\
& \pi_{14}(S^{12}) & \pi_{13}(S^{11}) \arrow[l, "S", "\cong"'] \arrow[ur, "(S\omega)_*"] \arrow[r, "\gamma_*"] & \pi_{13}(M^{10}),
\end{tikzcd}$$

where the horizontal sequence is exact. Since we can see that $\pi_*(S\omega)_*\eta_{11}^2 = \gamma_*\eta_{11}^2$ is a generator of $\pi_{13}(M^{10}) \cong \boldsymbol{Z}_2$ by [9], the homomorphism $\partial: \pi_{14}(SG_{2,b}^{(11)}, SG_{2,b}^{(9)}) \to \pi_{13}(SG_{2,b}^{(9)})$ is monomorphic, and so $i_*: \pi_{14}(SG_{2,b}^{(9)}) \to \pi_{14}(SG_{2,b}^{(11)})$ is epimorphic. Therefore for $Sf \in \pi_{14}(SG_{2,b}^{(11)})$ there exists an element $\bar{f} \in \pi_{14}(SG_{2,b}^{(9)})$ such that $Sf = i_*\bar{f}$.

Consider the Puppe exact sequence associated with cofibering (3.1):

$$\cdots \longrightarrow [M^{10}, G_{2,b}] \xrightarrow{\pi^*} [SG_{2,b}^{(9)}, G_{2,b}] \longrightarrow [SG_{2,b}^{(6)}, G_{2,b}] \longrightarrow \cdots.$$

Since $[SG_{2,b}^{(6)}, G_{2,b}]=0$ by (ii) of Lemma 3.6, $\pi^*: [M^{10}, G_{2,b}] \to [SG_{2,b}^{(9)}, G_{2,b}]$ is epimorphic. Also we have $\pi \circ \bar{f} \in \pi_{14}(M^{10})=0$ by [9]. Therefore we have

$$\operatorname{Im}(Sf)^* = \operatorname{Im}(i \circ \bar{f})^* \subset \operatorname{Im} \bar{f}^* = \operatorname{Im} \bar{f}^* \circ \pi^* = 0. \qquad \text{q.e.d.}$$

§ 4. Self-homotopy equivalences of $G_{2,b}^{(k)}$.

In this section, we study the group $\mathcal{E}(G_{2,b}^{(k)})$ for the k-skeleton $G_{2,b}^{(k)}$ of $G_{2,b}$ for $-2 \leqq b \leqq 5$ by making use of the results of the previous section, and obtain our main result of this paper.

Lemma 4.1. (i) $\mathcal{E}(G^{(k)}_{2,b})\cong \boldsymbol{Z}_2$ *for* $k=3, 6$.

(ii) $\mathcal{E}(G^{(9)}_{2,b})\cong \boldsymbol{Z}_2\oplus \boldsymbol{Z}_2\oplus \boldsymbol{Z}_2$.

Proof. (i) Clearly, $\mathcal{E}(G^{(3)}_{2,b})=\mathcal{E}(S^3)\cong \boldsymbol{Z}_2$. Let $i: S^3\to G^{(6)}_{2,b}$ be the inclusion. Then, by (i) of Lemma 3.6 the composition: $[S^3, S^3]\xrightarrow{i_*}[S^3, G^{(6)}_{2,b}]\xrightarrow{i^{*-1}}[G^{(6)}_{2,b}, G^{(6)}_{2,b}]$ is a bijection which preserves the composition. Hence we have $\mathcal{E}(G^{(6)}_{2,b})\cong\mathcal{E}(S^3)\cong \boldsymbol{Z}_2$.

(ii) Let p be an odd prime and let $\boldsymbol{P}$ be the set of all primes. Let $\sigma_1=-\iota_3\times-\iota_{11}: S^3\times S^{11}\to S^3\times S^{11}$ and let σ_2 be the inversion of G_2, then the localizations $(\sigma_1)_{(p)}$ and $(\sigma_2)_{P-\{p\}}$ are two side inversions of H-complexes $(S^3\times S^{11})_{(p)}$ and $(G_2)_{P-\{p\}}$ such that $(\sigma_1)_{(p)}\circ(\sigma_1)_{(p)}=1$ and $(\sigma_2)_{P-\{p\}}\circ(\sigma_2)_{P-\{p\}}=1$, respectively. Furthermore the localization $(\sigma_1)_Q$ is homotopic to $(\sigma_2)_Q$.

Therefore $G_{2,b}$ has the two side inversion σ as a pull-back of $(\sigma_1)_{(p)}$ and $(\sigma_2)_{P-(p)}$ such that $\sigma\circ\sigma=1$ by [5].

Let $\sigma^{(k)}: G^{(k)}_{2,b}\to G^{(k)}_{2,b}$ be the restriction of σ for $k=3, 6, 9, 11$. Then we define two maps

$$\lambda \text{ and } \bar{\lambda}: [M^9, G^{(9)}_{2,b}] \longrightarrow [G^{(9)}_{2,b}, G^{(9)}_{2,b}]$$

by $\lambda(\xi)=\nabla\circ(1\vee\xi)\circ l$ and $\bar{\lambda}(\xi)=\nabla\circ(\sigma^{(9)}\vee\xi)\circ l$ respectively, where $l: G^{(9)}_{2,b}\to G^{(9)}_{2,b}\vee M^9$ is the map shrinking $M^8\times\{1/2\}$ in $G^{(9)}_{2,b}=G^{(6)}_{2,b}\cup CM^8$. Since $i_*: [G^{(6)}_{2,b}, G^{(6)}_{2,b}]\to[G^{(6)}_{2,b}, G^{(9)}_{2,b}]$ is bijective for dimensional reasons, we can define a homomorphism

$$\phi: \mathcal{E}(G^{(9)}_{2,b}) \longrightarrow \mathcal{E}(G^{(6)}_{2,b})$$

by the restriction of the composition $[G^{(9)}_{2,b}, G^{(9)}_{2,b}]\xrightarrow{i^*}[G^{(6)}_{2,b}, G^{(9)}_{2,b}]\underset{\cong}{\xrightarrow{i_*^{-1}}}[G^{(6)}_{2,b}, G^{(6)}_{2,b}]$. By (i) we have $\mathcal{E}(G^{(6)}_{2,b})\cong \boldsymbol{Z}_2$ generated by $\sigma^{(6)}$. If $h\in[G^{(9)}_{2,b}, G^{(9)}_{2,b}]$ satisfies $i_*^{-1}i^*(h)=1$ or $i_*^{-1}i^*(h)=\sigma^{(6)}$, then there exists an element $\xi\in[M^9, G^{(9)}_{2,b}]$ such that $\lambda(\xi)=h$ or $\bar{\lambda}(\xi)=h$ by p. 326 of [11]. Therefore we have

$$\{\lambda(\xi), \bar{\lambda}(\xi), \xi\in[M^9, G^{(9)}_{2,b}]\}\supset\mathcal{E}(G^{(9)}_{2,b}), \tag{4.1}$$

since $i_*^{-1}i^*(\gamma)=1$ or $\sigma^{(6)}$ for any $\gamma\in\mathcal{E}(G^{(9)}_{2,b})$.

We have the following homotopy commutative diagram for any $\xi\in[M^9, G^{(9)}_{2,b}]$ by the definition of λ:

$$\begin{tikzcd}
G^{(9)}_{2,b} \arrow[r, "l"] \arrow[d, "d"] & G^{(9)}_{2,b}\vee M^9 \arrow[r, "1\vee\xi"] \arrow[d, "j"] & G^{(9)}_{2,b}\vee G^{(9)}_{2,b} \arrow[r, "\nabla"] \arrow[d, "j"] & G^{(9)}_{2,b} \arrow[d, "j"] \\
G^{(9)}_{2,b}\times G^{(9)}_{2,b} \arrow[r, "1\times\pi"] & G^{(9)}_{2,b}\times M^9 \arrow[r, "j\times j\circ\xi"] & G_{2,b}\times G_{2,b} \arrow[r, "m"] & G_{2,b}
\end{tikzcd}$$

where m is a multiplication on $G_{2,b}$. Let $j+\pi^*$ and $\sigma\circ j+\pi^*: [M^9, G_{2,b}]\to[G^{(9)}_{2,b}, G_{2,b}]$ be the maps defined by $(j+\pi^*)(\xi)=j+\pi^*(\xi)$ and $(\sigma\circ j+\pi^*)(\xi)$

$=\sigma\circ j+\pi^*(\xi)$ respectively, where $+$ is the multiplication induced by a multiplication on $G_{2,b}$. Then we have the following commutative diagram:

(4.2)
$$\begin{tikzcd}
{[G^{(9)}_{2,b}, G^{(9)}_{2,b}]} \arrow[d, "j_*"] & {[M^9, G^{(9)}_{2,b}]} \arrow[l, "\lambda"'] \arrow[r, "\bar{\lambda}"] \arrow[d, "j_*"] & {[G^{(9)}_{2,b}, G^{(9)}_{2,b}]} \arrow[d, "j_*"] \\
{[G^{(9)}_{2,b}, G_{2,b}]} & {[M^9, G_{2,b}]} \arrow[l, "j+\pi^*"'] \arrow[r, "\sigma\circ j+\pi^*"] & {[G^{(9)}_{2,b}, G_{2,b}]}
\end{tikzcd}$$

where all j_* are bijective for dimensional reasons. If $\lambda(\xi_1)=\lambda(\xi_2)$, then we have $(j+\pi^*)j_*(\xi_1)=(j+\pi^*)j_*(\xi_2)$, and so $\pi^*j_*(\xi_1)=\pi^*j_*(\xi_2)$. We have $\xi_1=\xi_2$, since π^*: $[M^9, G_{2,b}]\to[G^{(9)}_{2,b}, G_{2,b}]$ is monomorphic by (ii) of Lemma 3.6 and since j_* is bijective. Thus we have that λ is injective. Quite similarly we have that $\bar{\lambda}$ is injective. Let β be a generator of $[M^9, G^{(9)}_{2,b}]\cong \boldsymbol{Z}_4$ and $0\leqq t\leqq 3$. Then by the definition of λ, we have that $\lambda(t\beta)|G^{(6)}_{2,b}$ is the inclusion j: $G^{(6)}_{2,b}\to G^{(9)}_{2,b}$. Also, by Lemma 3.8 we have the following homotopy commutative diagram:

$$\begin{tikzcd}
G^{(9)}_{2,b} \arrow[r, "l"] \arrow[d, "\pi"] & G^{(9)}_{2,b}\vee M^9 \arrow[r, "1\vee t\beta"] \arrow[d, "\pi\vee 1"] & G^{(9)}_{2,b}\vee G^{(9)}_{2,b} \arrow[r, "\nabla"] \arrow[d, "\pi\vee\pi"] & G^{(9)}_{2,b} \arrow[d, "\pi"] \\
M^9 \arrow[r, "l"] & M^9\vee M^9 \arrow[r, "1\vee 2t1"] & M^9\vee M^9 \arrow[r, "\nabla"] & M^9 .
\end{tikzcd}$$

j_*: $H_i(G^{(6)}_{2,b})\to H_i(G^{(9)}_{2,b})$ is isomorphic for $0\leqq i\leqq 7$ and π_*: $H_i(G^{(9)}_{2,b})\to H_i(M^9)$ is isomorphic for $i\geqq 8$. Therefore by the above diagram, $\lambda(t\beta)_*$: $H_i(G^{(9)}_{2,b})\to H_i(G^{(9)}_{2,b})$ is isomorphic for all i and so we have

(4.3) $$\lambda(t\beta)\in\mathcal{E}(G^{(9)}_{2,b}) .$$

We have $\sigma^{(9)}{}_*(\beta)=-\beta$, since j_*: $[M^9, G^{(9)}_{2,b}]\to[M^9, G_{2,b}]$ is bijective and $\sigma_*=-1$: $[M^9, G_{2,b}]\to[M^9, G_{2,b}]$. So the diagram

$$\begin{tikzcd}
G^{(9)}_{2,b} \arrow[r, "l"] & G^{(9)}_{2,b}\vee M^9 \arrow[r, "1\vee(-t\beta)"] \arrow[dr, "\sigma^{(9)}\vee t\beta"'] & G^{(9)}_{2,b}\vee G^{(9)}_{2,b} \arrow[r, "\nabla"] \arrow[d, "\sigma^{(9)}\vee\sigma^{(9)}"] & G^{(9)}_{2,b} \arrow[d, "\sigma^{(9)}"] \\
 & & G^{(9)}_{2,b}\vee G^{(9)}_{2,b} \arrow[r, "\nabla"] & G^{(9)}_{2,b}
\end{tikzcd}$$

homotopy commutes and leads us that $\bar{\lambda}(t\beta)=\sigma^{(9)}\circ\lambda(-t\beta)$. Hence we have

(4.4) $$\bar{\lambda}(t\beta)\in\mathcal{E}(G^{(9)}_{2,b})$$

by the fact that $\sigma^{(9)}\in\mathcal{E}(G^{(9)}_{2,b})$ and $\lambda(-t\beta)\in\mathcal{E}(G^{(9)}_{2,b})$. Thus by (4.1), (4.3) and (4.4) we have

(4.5) $\mathcal{E}(G_{2,b}^{(9)})=\{\lambda(t\beta),\ \bar{\lambda}(t\beta)\,;\ 0\leqq t\leqq 3\}$ *as a set*

with β *a generator of* $[M^9,\ G_{2,b}^{(9)}]\cong \boldsymbol{Z}_4$.

Remark that the self homotopy equivalences f_t and $\bar{f}_t$ of $G_{2,b}^{(9)}$ for $0\leqq t\leqq 3$ have been defined in (5.1) of [8] and $f_t=\lambda(t\beta)$ and $\bar{f}_t=\bar{\lambda}(t\beta)$. It is also shown in p. 623 of [8] that $f_{t*}(\omega)=\omega+t\beta_*\gamma$ and $\bar{f}_{t*}(\omega)=-\omega+t\beta_*\gamma$ where ω is a generator of $\pi_{10}(G_{2,b}^{(9)})\cong \boldsymbol{Z}_{120}$, $\gamma=\pi_*\omega$ is a generator of $\pi_{10}(M^9)$ (see Lemma 3.9) and $\beta_*\gamma=\pm 30\omega$ By taking a suitable generator β such that $\beta_*\gamma=30\omega$, we have

(4.6) $\lambda(t\beta)_*(\omega)=(1+30t)\omega$ *and* $\bar{\lambda}(t\beta)(\omega)=(-1+30t)\omega$ *for* $t=0,\ 1,\ 2,\ 3$.

It follows from this that the natural homomorphism

$$\mathcal{E}(G_{2,b}^{(9)})\longrightarrow \operatorname{Aut}\pi_{10}(G_{2,b}^{(9)})$$

is monomorphic by (4.5). This fact and the equality

$$\lambda(t\beta)_*\lambda(t\beta)_*(\omega)=(1+30t)^2\omega=\omega$$

lead to a conclusion that $\lambda(t\beta)^2=1$. By a similar calculation we have $\bar{\lambda}(t\beta)^2=1$. Thus we have $\mathcal{E}(G_{2,b}^{(9)})\cong \boldsymbol{Z}_2\oplus \boldsymbol{Z}_2\oplus \boldsymbol{Z}_2$. q.e.d.

By Lemma 4.1 and Theorem 2.1, we have

Lemma 4.2. (i) $\mathcal{E}(G_{2,b}^{(11)})\cong \boldsymbol{Z}_2\oplus \boldsymbol{Z}_2$ *for* $-1\leqq b\leqq 5$.

(ii) *There is an exact sequence*:

$$0\longrightarrow \boldsymbol{Z}_2\oplus \boldsymbol{Z}_2\longrightarrow \mathcal{E}(G_{2,-2}^{(11)})\longrightarrow \boldsymbol{Z}_2\longrightarrow 1\,.$$

Proof. We apply Theorem 2.1 to the cell structure

$$G_{2,b}^{(11)}=G_{2,b}^{(9)}\cup e^{11}$$

in which the attaching class of the 11-cell is $(1+8b)\omega$ by (ii) of Lemma 3.9.

First we compute the group

$$G_1=\{h\in\mathcal{E}(G_{2,b}^{(9)}),\ h_*(1+8b)\omega=\varepsilon(1+8b)\omega,\ \varepsilon=\pm 1\}$$

given in Theorem 2.1, since $2(1+8b)\omega\neq 0$. By (4.6) the conditions

$$\lambda(t\beta)_*(1+8b)\omega=\varepsilon(1+8b)\omega \quad\text{and}\quad \bar{\lambda}(t\beta)_*(1+8b)\omega=\varepsilon(1+8b)\omega$$

are equivalent to

$$(1+8b)(1+30t)\omega=\varepsilon(1+8b)\omega \quad\text{and}\quad (1+8b)(-1+30t)\omega=\varepsilon(1+8b)\omega$$

respectively. For $-1\leqq b\leqq 5$, we have easily

$$(1+8b)(1+30t)\omega=(1+8b)\omega \qquad \text{if and only if } t=0\,;$$

$$(1+8b)(1+30t)\omega\neq-(1+8b)\omega \qquad \text{if } t=0,\ 1,\ 2,\ 3\,;$$

$$(1+8b)(-1+30t)\omega=-(1+8b)\omega \qquad \text{if and only if } t=0\,;$$

$$(1+8b)(-1+30t)\omega \neq (1+8b)\omega \qquad \text{if } t=0,\ 1,\ 2,\ 3.$$

Therefore, if a non-trivial element $h \in \mathcal{E}(G_{2,b}^{(9)})$ satisfies the condition $h_*(1+8b)\omega = \varepsilon(1+8b)\omega$ with $\varepsilon=\pm 1$, then $h=\bar{\lambda}(0)=\sigma^{(9)}$. Thus we have

(4.7) $$G_1 \cong \boldsymbol{Z}_2 \text{ generated by } \sigma^{(9)}.$$

For $b=-2$, we have the following

$$(1+8b)(1+30t)\omega=(1+8b)\omega \qquad \text{if and only if } t=0;$$

$$(1+8b)(1+30t)\omega=-(1+8b)\omega \qquad \text{if and only if } t=1;$$

$$(1+8b)(-1+30t)\omega=-(1+8b)\omega \qquad \text{if and only if } t=0;$$

$$(1+8b)(-1+30t)\omega=(1+8b)\omega \qquad \text{if and only if } t=3.$$

Therefore by the definition of G_1 and by Lemma 4.1 we have

$$G_1=\{\lambda(0)=1,\ \lambda(\beta),\ \bar{\lambda}(0)=\sigma^{(9)},\ \bar{\lambda}(3\beta)\} \cong \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2.$$

Next we consider the homomorphism

(4.8) $$\lambda:\ i_*\pi_{11}(G_{2,b}^{(9)}) \longrightarrow \mathcal{E}(G_{2,b}^{(11)})$$

given in Theorem 2.1, where $i_*\pi_{11}(G_{2,b}^{(9)}) \cong \boldsymbol{Z}_2$ by Lemma 3.10. This homomorphism λ is the restriction of the map Λ defined by (2.1).

Let $j+\pi^*:[S^{11}, G_{2,b}] \to [G_{2,b}^{(11)}, G_{2,b}]$ be a map defined by $(j+\pi^*)(\xi)=j+\pi^*(\xi)$ where $+$ is a multiplication induced by a multiplication on $G_{2,b}$. Then we have a commutative diagram by the similar way to that in (4.2)

$$\begin{array}{ccc} [S^{11}, G_{2,b}^{(11)}] & \xrightarrow{\ \Lambda\ } & [G_{2,b}^{(11)}, G_{2,b}^{(11)}] \\ \downarrow j_* & & \downarrow j_* \\ [S^{11}, G_{2,b}] & \xrightarrow{\ j+\pi^*\ } & [G_{2,b}^{(11)}, G_{2,b}] \end{array}$$

where both j_* are bijective for dimensional reasons, and hence we have

$$j_*\Lambda^{-1}(1)=\pi^{*-1}(0)=(1+8b)(S\omega)^*[SG_{2,b}^{(9)}, G_{2,b}]$$

from the Puppe exact sequence:

$$\cdots \longrightarrow [SG_{2,b}^{(9)}, G_{2,b}] \xrightarrow{(1+8b)(S\omega)^*} [S^{11}, G_{2,b}] \xrightarrow{\ \pi^*\ } [G_{2,b}^{(11)}, G_{2,b}] \longrightarrow \cdots.$$

Therefore $\Lambda^{-1}(1)=0$ by Lemma 3.11, since j_* is bijective. By this fact, (4.7), (4.8) and Theorem 2.1 we have a short exact sequence:

$$0 \longrightarrow \boldsymbol{Z}_2 \longrightarrow \mathcal{E}(G_{2,b}^{(11)}) \longrightarrow G_1 \longrightarrow 1,$$

where G_1 is isomorphic to $\boldsymbol{Z}_2$ generated by $\sigma^{(9)}$ for $-1 \leqq b \leqq 5$ and G_1 is isomorphic to $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ generated by $\sigma^{(9)}$ and $\lambda(\beta)$ for $b=-2$. In the above sequence the subgroup $\boldsymbol{Z}_2$ generated by $\sigma^{(9)}$ of G_1 splits, since the splitting homomorphism ρ: $\boldsymbol{Z}_2 \to \mathcal{E}(G_{2,b}^{(11)})$ can be defined by $\rho(\sigma^{(9)})=\sigma^{(11)}$. Hence we have $\mathcal{E}(G_{2,b}^{(11)})=\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$

for $-1\leqq b\leqq 5$ and there is an exact sequence

$$0 \longrightarrow Z_2\oplus Z_2 \longrightarrow \mathcal{E}(G^{(11)}_{2,-2}) \longrightarrow Z_2 \longrightarrow 1. \qquad \text{q.e.d.}$$

Let $\chi\colon \mathcal{E}(G^{(11)}_{2,b}) \longrightarrow \mathrm{Aut}\,\pi_{13}(G^{(11)}_{2,b})$ be the natural homomorphism.

Lemma 4.3. *For* $b=-2, 1, 4$, $\mathrm{Im}\{\chi\colon \mathcal{E}(G^{(11)}_{2,b})\to\mathrm{Aut}\,\pi_{13}(G^{(11)}_{2,b})\}$ *is contained in* $Z_2\oplus Z_2$ *generated by* $-1\oplus 1$ *and* $1\oplus -1: \pi_{13}(G^{(11)}_{2,b})\cong Z_3\oplus Z\to Z_3\oplus Z\cong \pi_{13}(G^{(11)}_{2,b})$, *where the isomorphism* : $\pi_{13}(G^{(11)}_{2,b})\cong Z_3\oplus Z$ *is the one given in* (ii) *of* Lemma 3.12. *Specifically, for the attaching class* f *of the* 14*-cell in* $G_{2,b}=G^{(11)}_{2,b}\cup e^{14}$ *and for any element* h *of* $\mathcal{E}(G^{(11)}_{2,b})$,

$$h_*f=\varepsilon f, \qquad \varepsilon=\pm 1.$$

(A proof will be given in § 5.).

Now, we apply Corollary 2.2 to the cell structure

$$G_{2,b}=G^{(11)}_{2,b}\cup e^{14}$$

and we have the following main result.

Theorem 4.4. *Let* $G_{2,b}$ *be an H-complex of type* (3, 11) *in* Theorem 3.1. *Then we have the following exact sequences*:

(i) $0 \longrightarrow D(Z_{168}\oplus Z_2) \longrightarrow \mathcal{E}(G_{2,b}) \longrightarrow Z_2 \longrightarrow 1 \qquad (b=-1, 0, 2, 3, 5)$,

(ii) $0 \longrightarrow D(Z_{168}\oplus Z_6) \longrightarrow \mathcal{E}(G_{2,b}) \longrightarrow Z_2 \longrightarrow 1 \qquad (b=1, 4)$,

(iii) $0 \longrightarrow Z_{168}\oplus Z_6 \longrightarrow \mathcal{E}(G_{2,-2}) \longrightarrow \mathcal{E}(G^{(11)}_{2,-2}) \longrightarrow 1$.

Here the group $\mathcal{E}(G^{(11)}_{2,-2})$ *is given in* (ii) *of* Lemma 4.2 *and for an abelian group* H, $D(H)$ *is a group given by the following split exact sequence*: $0\to H\to D(H)\to Z_2\to 1$, *where the splitting action* Z_2 *on* H *is given by* $(-1)\cdot h=-h$ *for* $-1\in Z_2$ *and* $h\in H$.

Proof. Since $G_{2,b}$ has a multiplication, we can apply Corollary 2.2 to the cell structure $G_{2,b}=G^{(11)}_{2,b}\cup e^{14}$.

First let H be the group given in (2.4): $H=i_*\pi_{14}(G^{(11)}_{2,b})/(Sf)^*[SG^{(11)}_{2,b}, G_{2,b}]$, where f is the attaching class of the 14-cell in $G^{(11)}_{2,b}\cup e^{14}$. Then we have

$$H\cong\pi_{14}(G_{2,b}) \tag{4.9}$$

by (iii) of Lemma 3.12 and Lemma 3.13.

Next we compute the group

$$G_1=\{h\in\mathcal{E}(G^{(11)}_{2,b})\mid h_*f=\varepsilon f,\ \varepsilon=\pm 1 \text{ in } \pi_{13}(G^{(11)}_{2,b})\}$$

given in Theorem 2.1, since $2f\neq 0$ by Lemma 3.12.

For $b=-1, 0, 2, 3, 5$, f is a generator of $\pi_{13}(G^{(11)}_{2,b})\cong Z$ by (i) of Lemma 3.12 and $\mathrm{Aut}\,\pi_{13}(G^{(11)}_{2,b})\cong\mathrm{Aut}\,Z\cong Z_2$ generated by -1. Therefore for any element $h\in\mathcal{E}(G^{(11)}_{2,b})$ we have

$$h_* f = f \text{ or } -f.$$

Hence we have $G_1 \cong \mathcal{E}(G_{2,b}^{(11)})$.

For $b=-2$, 1, 4, by (ii) of Lemma 3.12 and Lemma 4.3, we have the same results $G_1 \cong \mathcal{E}(G_{2,b}^{(11)})$.

Thus, by this results, (4.9) and (2.3), we have the exact sequence

$$(4.10) \qquad 0 \longrightarrow \pi_{14}(G_{2,b}) \xrightarrow{\lambda} \mathcal{E}(G_{2,b}) \longrightarrow \mathcal{E}(G_{2,b}^{(11)}) \longrightarrow 1.$$

The subgroup $\boldsymbol{Z}_2$ generated by $\sigma^{(11)}$ of $\mathcal{E}(G_{2,b}^{(11)})$ splits, since a splitting homomorphism $\rho: \boldsymbol{Z}_2 \to \mathcal{E}(G_{2,b})$ can be defined by $\rho(\sigma^{(11)}) = \sigma$. We can easily see that $\sigma^* = 1: H^{14}(G_{2,b}; \boldsymbol{Z}_p) \to H^{14}(G_{2,b}; \boldsymbol{Z}_p)$ for any odd prime p by the ring structure of $H^*(G_{2,b}; \boldsymbol{Z}_p)$ and by the fact that

$$\sigma^* = -1: H^i(G_{2,b}; \boldsymbol{Z}_p) \longrightarrow H^i(G_{2,b}; \boldsymbol{Z}_p) \qquad \text{for } i=3, 11.$$

Also $\sigma_* = -1: \pi_{14}(G_{2,b}) \to \pi_{14}(G_{2,b})$. Therefore we have a homotopy commutative diagram:

$$\begin{array}{ccccccc}
G_{2,b} & \xrightarrow{l} & G_{2,b} \vee S^{14} & \xrightarrow{1 \vee \xi} & G_{2,b} \vee G_{2,b} & \xrightarrow{\nabla} & G_{2,b} \\
\downarrow \sigma & & \downarrow \sigma \vee 1 & & \downarrow \sigma \vee \sigma & & \downarrow \sigma \\
G_{2,b} & \xrightarrow{l} & G_{2,b} \vee S^{14} & \xrightarrow{1 \vee (-\xi)} & G_{2,b} \vee G_{2,b} & \xrightarrow{\nabla} & G_{2,b}
\end{array}$$

From this diagram we see that the splitting action is given by $\sigma \cdot \xi = -\xi$. Hence the desired results are obtained by (4.10) and Lemmas 3.2, 3.3 and 4.2. q.e.d.

§5. A proof of Lemma 4.3.

Let $G_{2,b(3)}^{(11)}$ be the localization of $G_{2,b}^{(11)}$ at 3, and let $l: \mathcal{E}(G_{2,b}^{(11)}) \to \mathcal{E}(G_{2,b(3)}^{(11)})$ be the natural homomorphism defined by the localization $l(h) = h_{(3)}: G_{2,b(3)}^{(11)} \to G_{2,b(3)}^{(11)}$ (see [5] and [7]). First we show the lemma for the case $b=-2$. Consider the following exact sequence of the pair $(G_{2,-2}^{(11)}, G_{2,-2}^{(9)})$:

$$\cdots \longrightarrow \pi_{11}(G_{2,-2}^{(9)}) \longrightarrow \pi_{11}(G_{2,-2}^{(11)}) \longrightarrow \pi_{11}(G_{2,-2}^{(11)}, G_{2,-2}^{(9)}) \xrightarrow{\partial} \pi_{10}(G_{2,-2}^{(9)}) \longrightarrow \cdots,$$

where $\pi_{11}(G_{2,-2}^{(11)}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}_2$ by Lemmas 3.2 and 3.3, $\pi_{11}(G_{2,-2}^{(11)}, G_{2,-2}^{(9)}) \cong \pi_{11}(S^{11}) \cong \boldsymbol{Z}$ generated by ι_{11}, $\pi_{10}(G_{2,-2}^{(9)}) \cong \boldsymbol{Z}_{120}$ generated by ω and $\partial(\iota_{11}) = -15\omega$ by Lemma 3.9. There exists a coextension $\widetilde{8\iota_{11}}: S^{11} \to G_{2,b}^{(11)}$ of $8\iota_{11}: S^{11} \to S^{11}$, and so we can define a 3-equivalence $q: S^3 \vee S^{11} \to G_{2,-2}^{(11)}$ by $q = \nabla \circ (i \vee \widetilde{8\iota_{11}})$ where $i: S^3 \to G_{2,-2}^{(11)}$ be the inclusion. Then we have a commutative diagram:

$$(5.1) \qquad \begin{array}{ccccc}
\mathcal{E}(G_{2,-2}^{(11)}) & \xrightarrow{l} & \mathcal{E}(G_{2,-2(3)}^{(11)}) & \xleftarrow[\cong]{q'} & \mathcal{E}(S_{(3)}^3 \vee S_{(3)}^{11}) \\
\downarrow \chi & & \downarrow \chi & & \downarrow \chi \\
\mathrm{Aut}\, \pi_{13}(G_{2,-2}^{(11)}) & \xrightarrow{l'} & \mathrm{Aut}\, \pi_{13}(G_{2,-2(3)}^{(11)}) & \xleftarrow[\cong]{q''} & \mathrm{Aut}\, \pi_{13}(S_{(3)}^3 \vee S_{(3)}^{11}),
\end{array}$$

where $l': \mathrm{Aut}\, \pi_{13}(G_{2,-2}^{(11)}) \to \mathrm{Aut}\, \pi_{13}(G_{2,-2(3)}^{(11)})$ is the canonical homomorphism defined by $l'(h)=h\otimes 1:\ \pi_{13}(G^{(11)}_{2,-2(3)})\cong\pi_{13}(G^{(11)}_{2,-2})\otimes\boldsymbol{Q}_3\to\pi_{13}(G^{(11)}_{2,-2})\otimes\boldsymbol{Q}_3\cong\pi_{13}(G^{(11)}_{2,-2(3)})$ and q' and q'' are isomorphisms induced by the 3-equivalence q.

By (ii) of Lemma 3.12 $\pi_{13}(G^{(11)}_{2,-2})\cong\boldsymbol{Z}_3\oplus\boldsymbol{Z}$ and $\pi_{13}(G^{(11)}_{2,-2(3)})\cong\boldsymbol{Z}_3\oplus\boldsymbol{Q}_3$, and so Aut $\pi_{13}(G^{(11)}_{2,-2})$ is isomorphic to a group of matrices:

$$\left\{\begin{pmatrix} a & c \\ 0 & b\end{pmatrix} \,\middle|\, a\in\operatorname{Aut}\boldsymbol{Z}_3,\quad b\in\operatorname{Aut}\boldsymbol{Z},\quad c\in\operatorname{Hom}(\boldsymbol{Z},\boldsymbol{Z}_3)\right\}$$

where Aut $\boldsymbol{Z}_3\cong\boldsymbol{Z}_2$ generated by -1, Aut $\boldsymbol{Z}=\boldsymbol{Z}_2$ generated by -1 and Hom $(\boldsymbol{Z},\boldsymbol{Z}_3)$ $=\boldsymbol{Z}_3$ generated by the mod 3 reduction. Therefore we see that l' is monomorphic. By (ii) of Lemma 4.2 we have

$$\mathcal{E}(G^{(11)}_{2,-2})=\{1,\ \lambda(\xi),\ \sigma^{(11)},\ h,\ \lambda(\xi)\sigma^{(11)},\ \lambda(\xi)h,\ \sigma^{(11)}h,\ \lambda(\xi)\sigma^{(11)}h\}. \tag{5.2}$$

Here $\lambda(\xi)$, $\xi\neq 0$ is given in (4.8) and h is an element satisfying the following homotopy commutative diagram:

$$\begin{tikzcd}
S^{10} \arrow[r, "-15\omega"] \arrow[d, "-1"] & G^{(9)}_{2,-2} \arrow[r] \arrow[d, "\lambda(\beta)"] & G^{(11)}_{2,-2} \arrow[r] \arrow[d, "h"] & S^{11} \arrow[d, "-1"] \\
S^{10} \arrow[r, "-15\omega"] & G^{(9)}_{2,-2} \arrow[r] & G^{(11)}_{2,-2} \arrow[r] & S^{11},
\end{tikzcd} \tag{5.3}$$

where $\lambda(\beta)$ is given in (4.5).

Since $2\xi=0$, we have a homotopy commutative diagram:

$$\begin{tikzcd}
G^{(11)}_{2,-2} \arrow[r, "l"] \arrow[d, "J"] & G^{(11)}_{2,-2}\vee S^{11} \arrow[r, "1\vee\xi"] \arrow[d, "J"] & G^{(11)}_{2,-2}\vee G^{(11)}_{2,-2} \arrow[r, "\nabla"] \arrow[d, "J"] & G^{(11)}_{2,-2} \arrow[d, "J"] \\
G^{(11)}_{2,-2(3)} \arrow[r, "l_{(3)}"] & (G^{(11)}_{2,-2}\vee S^{11})_{(3)} \arrow[r, "(1\vee\xi)_{(3)}"] & (G^{(11)}_{2,-2}\vee G^{(11)}_{2,-2})_{(3)} \arrow[r, "\nabla_{(3)}"] & G^{(11)}_{2,-2(3)} \\
 & G^{(11)}_{2,-2(3)}\vee S^{11}_{(3)} \arrow[u, "\simeq"] \arrow[r, "1\vee\xi_{(3)}"] & G^{(11)}_{2,-2(3)}\vee G^{(11)}_{2,-2(3)} \arrow[u, "\simeq"] &
\end{tikzcd}$$

where $\xi_{(3)}=0\in[S^{11}_{(3)},G^{(11)}_{2,-2(3)}]\cong\pi_{11}(G^{(11)}_{2,-2(3)})$ Hence we have $l(\lambda(\xi))=1$. Therefore $l'\chi(\lambda(\xi))=\chi l(\lambda(\xi))=1$ and so we have $\chi(\lambda(\xi))=1:\ \pi_{13}(G^{(11)}_{2,-2})\to\pi_{13}(G^{(11)}_{2,-2})$. Since $\sigma^{(11)}$ is the restriction of $\sigma:\ G_{2,-2}\to G_{2,-2}$, we have

$$\begin{tikzcd}
H_{14}(G_{2,-2}) \arrow[r, "\cong"'] \arrow[d, "\sigma_*"] & H_{14}(G_{2,-2},G^{(11)}_{2,-2}) \arrow[d, "\sigma_*"] & \pi_{14}(G_{2,-2},G^{(11)}_{2,-2}) \arrow[l, "\Xi", "\cong"'] \arrow[r, "\partial"] \arrow[d, "\sigma_*"] & \pi_{13}(G^{(11)}_{2,-2}) \arrow[r] \arrow[d, "\sigma^{(11)}{}_*"] & \pi_{13}(G_{2,-2}) \arrow[d, "\sigma_*"] \\
H_{14}(G_{2,-2}) \arrow[r, "\cong"'] & H_{14}(G_{2,-2},G^{(11)}_{2,-2}) & \pi_{14}(G_{2,-2},G^{(11)}_{2,-2}) \arrow[l, "\Xi", "\cong"'] \arrow[r, "\partial"] & \pi_{13}(G^{(11)}_{2,-2}) \arrow[r] & \pi_{13}(G_{2,-2})
\end{tikzcd}$$

where $\sigma_*=1:\ H_{14}(G_{2,-2})\to H_{14}(G_{2,-2})$ by the ring structure of $H^*(G_{2,-2};\boldsymbol{Z}_p)$ (p: odd prime), $\Xi:\ \pi_{14}(G_{2,-2},G^{(11)}_{2,-2})\to H_{14}(G_{2,-2},G^{(11)}_{2,-2})$ is the Hurewicz isomorphism and $\pi_{13}(G^{(11)}_{2,-2})\cong\rho\pi_{13}(G_{2,-2})\oplus\partial\pi_{14}(G_{2,-2},G^{(11)}_{2,-2})\cong\boldsymbol{Z}_3\oplus\boldsymbol{Z}$ by (ii) of Lemma 3.12. Therefore, $\sigma_*=1:\ \pi_{14}(G_{2,-2},G^{(11)}_{2,-2})\to\pi_{14}(G_{2,-2},G^{(11)}_{2,-2})$. Since $i_*:\ \pi_{13}(S^3:3)\to$

$\pi_{13}(G_{2,-2}:3)$ is isomorphic and since $\sigma^{(3)}{}_*=-1:\pi_{13}(S^3)\to\pi_{13}(S^3)$, $\sigma_*=-1:\pi_{13}(G_{2,-2})\to\pi_{13}(G_{2,-2})$. Thus we have

$$\sigma^{(11)}{}_*=-1\oplus 1:\ \boldsymbol{Z}_3\oplus\boldsymbol{Z}\longrightarrow\boldsymbol{Z}_3\oplus\boldsymbol{Z}\,.$$

By the definition of q and (5.3), we can see that as an element of $\pi_3(G_{2,-2}^{(11)})\oplus\pi_{11}(G_{2,-2}^{(11)})$

$$h\circ q=q\circ(\iota_3\vee-\iota_{11})\ \text{modulo}\ \boldsymbol{Z}_2\subset\pi_{11}(G_{2,-2}^{(11)})\cong\boldsymbol{Z}\oplus\boldsymbol{Z}_2\,,$$

since $h\,|\,G_{2,-2}^{(6)}$ is the inclusion. Therefore we have

$$q'^{-1}l(h)=\iota_{3(3)}\vee(-\iota_{11})_{(3)}\ \text{in (5.1)}\,.$$

It follows from this that

$$q''^{-1}l'\chi(h)=\chi q'^{-1}l(h)=(\iota_{3(3)}\vee(-\iota_{11})_{(3)})_*=1\oplus-1:$$
$$\pi_{13}(S^3_{(3)}\vee S^{11}_{(3)})=(\pi_{13}(S^3)\otimes\boldsymbol{Q}_3)\oplus(\pi_{14}(S^3\times S^{11},\ S^3\vee S^{11})\otimes\boldsymbol{Q}_3)$$
$$\cong\boldsymbol{Z}_3\oplus\boldsymbol{Q}_3\longrightarrow\boldsymbol{Z}_3\oplus\boldsymbol{Q}_3\cong\pi_{13}(S^3_{(3)}\vee S^{11}_{(3)})\,.$$

Therefore by considering a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{14}(G_{2,-2(3)},\ G_{2,-2(3)}^{(11)}) & \xrightarrow{\partial} & \pi_{13}(G_{2,-2(3)}^{(11)}) & \longrightarrow & \pi_{13}(G_{2,-2(3)}) & \longrightarrow & 0\\
 & & \uparrow \bar{q}_{(3)*} & & \uparrow q_{(3)*} & & \uparrow i_{(3)*} & & \\
0 & \longrightarrow & \pi_{14}((S^3\times S^{11})_{(3)},\ (S^3\vee S^{11})_{(3)}) & \xrightarrow{\partial} & \pi_{13}((S^3\vee S^{11})_{(3)}) & \longrightarrow & \pi_{13}(S^3_{(3)}) & \longrightarrow & 0
\end{array}$$

where $\bar{q}=i+\widetilde{8\iota_{11}}:\ S^3\times S^{11}\to G_{2,-2}$ and all vertical homomorphisms are isomorphic, we have $\chi(h)=1\oplus-1:\ \pi_{13}(G_{2,-2}^{(11)})\cong\boldsymbol{Z}_3\oplus\boldsymbol{Z}\to\boldsymbol{Z}_3\oplus\boldsymbol{Z}\cong\pi_{13}(G_{2,-2}^{(11)})$, since $q''^{-1}l'$ is monomorphic. The other elements of $\mathcal{E}(G_{2,-2}^{(11)})$ are given by the composition of $\lambda(\xi)$, $\sigma^{(11)}$ and h by (5.2). Hence we complete the proof for the case $b=-2$.

The proof for the other $b\neq-2$ is given more easily by a similar way.

The latter half of Lemma 4.3 is obtained immediately from (ii) of Lemma 3.12. q.e.d.

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