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A note on elements of the Burnside ring of a finite group

By

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1. Introduction

In [2], T. tom Dieck defined the Burnside ring $A(G)$ of a compact Lie group G using a certain equivalence relation on the set of closed smooth G-manifolds (see § 2). In this paper, when G is a finite group, we prove the following:

Theorem. *Let G be a finite group. For an arbitrary element $\alpha \in A(G)$, there exists a connected closed smooth G-manifold X such that*

$$\alpha = [X] \quad \text{in} \quad A(G).$$

Throughout this paper G will be a finite group.

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2. The Burnside ring

In this section we recall some basic facts about the Burnside ring which are due to tom Dieck [2].

On the set of closed smooth G-manifolds consider the equivalence relation: $X \sim Y$ if and only if for all subgroups H of G the Euler-Characteristics $\chi(X^H)$ and $\chi(Y^H)$ are equal. Let $A(G)$ be the set of equivalence classes and let $[X] \in A(G)$ be the class of X. Disjoint union and cartesian product induce addition and multiplication, respectively, on $A(G)$. Then $A(G)$ becomes a commutative ring with identity. We call $A(G)$ the Burnside ring of G.

Let $C(G)$ be the set of conjugacy classes of subgroups of G. Denote by (H) the conjugacy class of H in G.

Proposition 2.1. *Additively, $A(G)$ is a free abelian group generated by $\{[G/H] \mid (H) \in C(G)\}$.*

Let Y be a closed smooth H-manifold; then $G \times_H Y$ is a closed smooth G-manifold. Then the assignment $Y \mapsto G \times_H Y$ induces an additive homomorphism

$$\mathrm{Ind}_H^G : A(H) \longrightarrow A(G).$$

We remark that $\mathrm{Ind}_H^G([H/H])=[G/H]$.

3. Examples

In this section we introduce some closed smooth G-manifolds and see their classes in $A(G)$.

Example 3.1. If M is a closed smooth G-manifold with trivial G-action, then

$$[M]=\chi(M)[G/G] \quad \text{in } A(G).$$

Example 3.2. Let V be an orthogonal representation space of G. We put $S(V)=\{v\in V \mid \|v\|=1\}$, $D(V)=\{v\in V \mid \|v\|\leqq 1\}$ and $\Sigma^V=D(V)/S(V)$. If U is a unitary representation space of G, then

$$[\Sigma^U]=2[G/G] \quad \text{in } A(G).$$

Example 3.3. Let V be an n-dimensional orthogonal representation space of G and let $\rho_V: G\to O(n)$ be its associated representation. We define a G-action on the $(n-1)$-dimensional real projective space $\boldsymbol{R}P^{n-1}$ by

$$g\circ[x]=[\rho_V(g)\cdot x] \quad \text{for } g\in G,\ [x]\in \boldsymbol{R}P^{n-1},$$

where $[x]$ is a point of $\boldsymbol{R}P^{n-1}$ represented by a non-zero vector x of $\boldsymbol{R}^n$. This action is well-defined and smooth. We denote this smooth G-manifold by $\boldsymbol{R}P(V)$.

Then we have

Proposition 3.4. *If U is a unitary representation space of G, then*

$$[\boldsymbol{R}P(\boldsymbol{R}^1\oplus U)]=[G/G] \quad \text{in } A(G),$$

where $\boldsymbol{R}^1$ denotes the one-dimensional trivial representation space of G.

Proof. To prove Proposition 3.4, it suffices to show that

$$\chi(\boldsymbol{R}P(\boldsymbol{R}^1\oplus U)^H)=1 \quad \text{for any subgroup } H \text{ of } G.$$

Let S^1 be the circle group consisting of complex numbers of absolute value 1. Then we define an S^1-action on $\boldsymbol{R}P(\boldsymbol{R}^1\oplus U)$ by

$$z\circ[t, u]=[t, z\cdot u] \quad \text{for } z\in S^1,\ [t, u]\in \boldsymbol{R}P(\boldsymbol{R}^1\oplus U),$$

where $[t, u]$ is a point of $\boldsymbol{R}P(\boldsymbol{R}^1\oplus U)$ represented by a non-zero vector $(t, u)\in \boldsymbol{R}^1\oplus U$. Then $\boldsymbol{R}P(\boldsymbol{R}^1\oplus U)$ becomes an $S^1\times G$-manifold. Let $H\,(=\{1\}\times H\subset S^1\times G)$ be an arbitrary subgroup of G. Then $\boldsymbol{R}P(\boldsymbol{R}^1\oplus U)^H$ is an S^1-submanifold and

$$\begin{aligned}(\boldsymbol{R}P(\boldsymbol{R}^1\oplus U)^H)^{S^1}&=\boldsymbol{R}P(\boldsymbol{R}^1\oplus U)^{S^1\times H}\\&=(\boldsymbol{R}P(\boldsymbol{R}^1\oplus U)^{S^1})^H\\&=\boldsymbol{R}P(\boldsymbol{R}^1)^H\\&=\boldsymbol{R}P(\boldsymbol{R}^1).\end{aligned}$$

It follows from Bredon [1; III. 7.10] that we have

$$\chi(\boldsymbol{R}P(\boldsymbol{R}^1\oplus U)^H)=\chi(\boldsymbol{R}P(\boldsymbol{R}^1))=1.$$

This completes the proof.

4. A generalized equivariant connected sum

In this section we introduce the notion of a generalized equivariant connected sum. (Compare Sebastiani [3].)

Let X be a smooth G-manifold with G-invariant Riemannian metric. We denote the isotropy subgroup of G at $x\in X$ by G_x and the orbit of x under G by $G(x)$, which is G-diffeomorphic to G/G_x. We regard T_xX, the tangent space of X at x, as an orthogonal representation space of G_x.

Definition 4.1. Let H be a subgroup of G and V an orthogonal representation space of H. Then we say that (M, m) satisfies Condition (G, H, V) if and only if

(i) M is a closed smooth G-manifold with G-invariant Riemannian metric and $m\in M$,

(ii) $G_m=H$,

(iii) $T_mM\cong V$ as orthogonal representation spaces of H.

Suppose that (M_1, m_1) and (M_2, m_2) satisfy Condition (G, H, V). Then we give a definition of the generalized equivariant connected sum $M_1\#_VM_2$. By the differentiable slice theorem (see Bredon [1; VI]), there are open G-embeddings

$$\phi_i: G\times_HV\longrightarrow M_i \qquad \text{for } i=1, 2$$

such that $\phi_i([e, 0])=m_i$. Now we obtain $M_1\#_VM_2$ from the disjoint union

$$(M_1-G(m_1))\amalg(M_2-G(m_2))$$

by identifying $\phi_1([g, tv])$ with $\phi_2([g, (1-t)v])$ for $g\in G$, $v\in S(V)$, $0<t<1$. It is clear that $M_1\#_VM_2$ is a closed smooth G-manifold. Obviously, $M_1\#_VM_2$ depends on the choice of m_1, m_2, ϕ_1 and ϕ_2, but the next proposition indicates that $[M_1\#_VM_2]\in A(G)$ is independent of the choice of them.

Proposition 4.2. *If (M_1, m_1) and (M_2, m_2) satisfy Condition (G, H, V), then*

$$[M_1\#_VM_2]=[M_1]+[M_2]-\mathrm{Ind}_H^G([\Sigma^V]) \qquad \textit{in } A(G).$$

Proof. We shall show that

$$\chi((M_1 \sharp_V M_2)^K)=\chi(M_1^K)+\chi(M_2^K)-\chi((G\times_H \Sigma^V)^K)$$

for any subgroup K of G. We identify $M_i-G(m_i)$ with its image in $M_1\sharp_V M_2$. Since $M_i-\phi_i((G\times_H D(V))$ is a G-deformation retract of $M_i-G(m_i)$, we have

$$\chi((M_i-G(m_i))^K)=\chi(M_i^K)+\chi((G\times_H S(V))^K)-\chi((G\times_H D(V))^K)$$

for $i=1, 2$. Clearly

$$\chi((G\times_H \Sigma^V)^K)=2\chi((G\times_H D(V))^K)-\chi((G\times_H S(V))^K).$$

Since $(M_1-G(m_1))\cap(M_2-G(m_2))$ is G-homotopy equivalent to $G\times_H S(V)$, we have

$$\begin{aligned}\chi((M_1\sharp_V M_2)^K)&=\chi((M_1-G(m_1))^K)+\chi((M_2-G(m_2))^K)-\chi((G\times_H S(V))^K)\\&=\chi(M_1^K)+\chi(M_2^K)-\chi((G\times_H \Sigma^V)^K).\end{aligned}$$

This completes the proof.

Suppose that (M, m) satisfies Condition (G, H, V) and (N, n) satisfies Condition (H, H, V). Then $(G\times_H N, [e, n])$ satisfies Condition (G, H, V) and we can construct $M\sharp_V(G\times_H N)$.

Corollary 4.3.

$$[M\sharp_V(G\times_H N)]=[M]+\mathrm{Ind}_H^G([N]-[\Sigma^V]) \qquad \text{in } A(G).$$

5. Proof of Theorem

For a non-zero integer k, we put

$$N(k)=\begin{cases} \boldsymbol{C}P_1^2\sharp\boldsymbol{C}P_2^2\sharp\cdots\sharp\boldsymbol{C}P_k^2 & \text{if } k>0\\ \boldsymbol{R}P_1^4\sharp\boldsymbol{R}P_2^4\sharp\cdots\sharp\boldsymbol{R}P_{-k}^4 & \text{if } k<0,\end{cases}$$

where $\boldsymbol{C}P_i^2$, $\boldsymbol{R}P_i^4$ $(1\leqq i\leqq |k|)$ are copies of $\boldsymbol{C}P^2$, the complex projective space, and $\boldsymbol{R}P^4$, and $\sharp$ means the ordinary connected sum. It is easy to see that

Lemma 5.1. $\chi(N(k))=k+2$.

Proof of Theorem. Let $\alpha\in A(G)$ be an arbitrary element. Then, by Proposition 2.1, there exist $a_i\in\boldsymbol{Z}-\{0\}$ and $(H_i)\in C(G)$ $(1\leqq i\leqq k)$ such that

$$\alpha=\sum_{i=1}^k a_i[G/H_i] \qquad \text{in } A(G).$$

Let U be the complex regular representation space of G. Then there are $x_i\in\Sigma^U$ $(1\leqq i\leqq k)$ with isotropy group H_i. We put $U_i=T_{x_i}\Sigma^U$. Then U_i is a unitary representation space of H_i, given by restricting the G-action on U to the H_i-action. We put $M=T^4\times\Sigma^U$ and $m_i=(t, x_i)\in M$ for $1\leqq i\leqq k$, where the G-action on T^4, the 4-dimensional torus, is trivial and $t\in T^4$. Then (M, m_i) satisfies Condition $(G, H_i, \boldsymbol{R}^4\oplus U_i)$.

On the other hand, we consider an H_i-manifold $N_i=N(a_i)\times \boldsymbol{RP}(\boldsymbol{R}^1\oplus U_i)$ and $n_i=(s_i, [1, 0])\in N_i$, where the H_i-action on $N(a_i)$ is trivial and $s_i\in N(a_i)$. Then (N_i, n_i) satisfies Condition $(H_i, H_i, \boldsymbol{R}^4\oplus U_i)$.

Now we can construct

$$X=M\#_{R^4\oplus U_1}(G\times_{H_1}N_1)\#_{R^4\oplus U_2}\cdots\#_{R^4\oplus U_k}(G\times_{H_k}N_k)\,.$$

Using Proposition 3.4, Corollary 4.3 and Lemma 5.1, we have

$$\begin{aligned}
[X]&=[M]+\sum_{i=1}^{k}\mathrm{Ind}_{H_i}^{G}([N_i]-[\Sigma^{R^4\oplus U_i}])\\
&=[T^4]\cdot[\Sigma^U]+\sum_{i=1}^{k}\mathrm{Ind}_{H_i}^{G}([N(a_i)]\cdot[\boldsymbol{RP}(\boldsymbol{R}^1\oplus U_i)]-[\Sigma^{R^4\oplus U_i}])\\
&=\sum_{i=1}^{k}\mathrm{Ind}_{H_i}^{G}(a_i[H_i/H_i])\\
&=\sum_{i=1}^{k}a_i[G/H_i]\\
&=\alpha\,.
\end{aligned}$$

Moreover it is clear that X is connected. Hence X has our required properties. This completes the proof of Theorem.

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