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A remark on the regularity of the solution of the Dirichlet problem in a semi-infinite domain in R^2

By

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§0. We consider the Dirichlet problem

$$(0.1) \qquad \begin{cases} u_{xx}+u_{yy}=f \quad \text{in} \quad G\subset R^2, \quad f\in L^2(G) \\ u\in H_0^1(G) \end{cases}$$

in an unbounded domain which is narrow at infinity. In this note we show, following the method of K. Ibuki [2], the solution is in fact in $H^2(G)$ under some assumptions on G. As a result, we can see self-adjoint extension of the Laplacian with the Dirichlet boundary condition in such a domain is unique and coincides with the closure of the operator.

§1. We consider the problem (0.1) in a domain $G\subset R^2$ which satisfies the following conditions.

I) $G=G_0\cup\bigcup_{j=1}^{N}G_j$ where G_0 is bounded and each G_j ($j\geqq 1$) has the following form under certain orthogonal transformation of the coordinates

$$(1.1) \qquad G_j=\{(x, y)\in R^2 \mid R<x<\infty,\ a_1(x)<y<a_2(x)\}.$$

II) $a_j(x)$ ($j=1, 2$) are smooth and satisfy

$$(1.2) \qquad \begin{aligned} &\text{i)} \quad b(x)=a_2(x)-a_1(x)\to 0 \text{ as } x\to\infty \\ &\text{ii)} \quad b'(x)\leqq 0 \\ &\text{iii)} \quad a_j'(x)\to 0 \text{ as } x\to\infty \text{ and } |a_j''(x)|\leqq M \\ &\text{iv)} \quad \left|\frac{b'(x)}{b(x)}\right|\leqq M. \end{aligned}$$

We observe the problem (0.1) is solved for arbitrary $f\in L^2(G)$ by the assumptions II-i) and ii) (see F. Rellich [4]), and the solution is uniquely determined. Our main theorem is

Main theorem *Let $u\in H_0^1(G)$ be the solution of* (0.1) *in the domain G under the conditions* I *and* II, *then u is in $H^2(G)$. Here $H_0^1(G)$ and $H^2(G)$ are usual*

Sobolev spaces.

As we can localize the problem, we may assume G itself has the form (1.1) and R is sufficiently large. Then we change the variables as the following

$$\xi=\int_{R_0}^{x}\frac{1}{b(t)}dt \qquad (R_0 \text{ a fixed point})$$

(1.3)

$$\eta=\frac{1}{b(x)}(y-a_1(x)), \quad b(x)=a_2(x)-a_1(x).$$

We find G is transformed to $C_T=(T, \infty)\times(0, 1)$ $\left(T=\int_{R_0}^{R}\frac{1}{b(t)}dt\right)$, and we have

$$(1.4)\quad \begin{cases} u_x=\dfrac{1}{b(x)}(u_\xi+A_1(x)u_\eta) \\ u_y=\dfrac{1}{b(x)}u_\eta \\ u_{xx}=\dfrac{1}{b(x)^2}(u_{\xi\xi}+A_2(x)u_{\xi\eta}+A_3(x)u_{\eta\eta})+\dfrac{1}{b(x)}(B_2(x)u_\xi+B_2(x)u_\eta) \\ u_{xy}=\dfrac{1}{b(x)^2}(u_{\xi\eta}+A_4(x)u_{\eta\eta})+\dfrac{1}{b(x)}B_4(x)u_\eta \\ u_{yy}=\dfrac{1}{b(x)^2}u_{\eta\eta}, \quad dxdy=\dfrac{1}{b(x)^2}d\xi d\eta \end{cases}$$

where A_j and B_j are uniformly bounded and A_j can be made arbitrarily small if we take sufficiently large T (by virtue of (1.2)).

Using the computations above, we can see easily the followings.

Proposition 1.1. *When we put* $\rho(\xi)=\dfrac{1}{b(x(\xi))}$ *and* $L^2(C_T)=L^2((T, \infty)\times(0, 1))$, *we find*

i) $u\in L^2(G)$ *if and only if* $\rho^{-1}u\in L^2(C_T)$

ii) $u\in H^1(G)$ *if and only if* $\rho^{-1}u\in L^2(C_T)$ *and* u_ξ, $u_\eta\in L^2(C_T)$

iii) $u\in H^2(G)$ *if and only if* $u\in H^1(C_T)$ *and* $\rho u_{\xi\xi}$, $\rho u_{\xi\eta}$, $\rho u_{\eta\eta}\in L^2(C_T)$.

Proposition 1.2.

$$(1.5)\qquad u_{xx}+u_{yy}=\rho^2(\xi)(u_{\xi\xi}+u_{\eta\eta}+A(\xi, \eta, D_\xi, D_\eta)u)$$

where A is a second order differential operator whose coefficients can be made arbitrarily small if we take sufficiently large T.

§2. In this section we consider the following boundary-value problem

$$(2.1)\qquad \begin{cases} u_{\xi\xi}+u_{\eta\eta}=\rho^{-2}(\xi)f(\xi, \eta) & \text{in } C_T=(T, \infty)\times(0, 1) \\ u=0 & \text{on } \partial C_T. \end{cases}$$

Firstly, we notice some properties of the function $\rho(\xi)=\dfrac{1}{b(x(\xi))}$. Since $\rho'(\xi)=-\dfrac{b'}{b^2}x'(\xi)=-\dfrac{b'}{b}$, we can see

(2.2) $$0\leqq\rho'(\xi)\leqq M \quad \text{and} \quad \rho'/\rho\to 0 \quad \text{as} \quad \xi\to\infty\,.$$

We set

(2.3) $$\|u\|_2^2=\|\rho^{-1}u\|^2+\|u_\xi\|^2+\|u_\eta\|^2+\|\rho u_{\xi\xi}\|^2+\|\rho u_{\xi\eta}\|^2+\|\rho u_{\eta\eta}\|^2\,.$$

Here $\|\ \|$ denotes the L^2-norm on C_T and Proposition 1.1 says the norm $\|\ \|_2$ is equivalent to the original norm in $H^2(G)$. When we expand $u=\sum_{n=1}^{\infty}u_n(\xi)\sin n\pi\eta$, we have

$$\|u\|_2^2=\sum_{n=1}^{\infty}\|\rho^{-1}u_n\|^2+\|u_n'\|^2+n^2\|u_n\|^2+\|\rho u_n''\|^2+n^2\|\rho u_n'\|^2+n^4\|\rho u_n\|^2$$

$$\sim\sum_{n=1}^{\infty}\|\rho u_n''\|^2+n^2\|\rho u_n'\|^2+n^4\|\rho u_n\|^2\,.$$

Thus we have obtained

Proposition 2.1.

(2.3) $$\|u\|_2^2\sim\sum_{n=1}^{\infty}\|\rho u_n''\|^2+n^2\|\rho u_n'\|^2+n^4\|\rho u_n\|^2\,,$$

where $\sim$ *means the equivalence of the norms and* $\|\ \|$ *denotes the norm of* $L^2(T,\infty)$.

We are going to construct the Green's function for the problem (2.1) by method of separation of the variables. When we expand $f(\xi,\eta)$ as $\sum_{n=1}^{\infty}f_n(\xi)\sin n\pi\eta$, we can see we have only to consider the problem

(2.4) $$\begin{cases} u_n''-n^2\pi^2u_n=\rho^{-2}(\xi)f_n(\xi) \\ u_n(T)=u_n(\infty)=0\,. \end{cases}$$

We set $G_n(t,s)=\dfrac{1}{2n\pi}\{e^{-n\pi(t+s-2T)}-e^{-n\pi|t-s|}\}$, which is the Green's function for the problem (2.4). Then we can show easily

Proposition 2.2.

(2.5) $$|G_n(t,s)|\leqq\frac{1}{n\pi}e^{-n\pi|t-s|}$$

(2.6) $$|G_{n,t}(t,s)|\leqq e^{-n\pi|t-s|}\,.$$

We define an operator G as the following.

$$(Gg)(\xi)=\sum_{n=1}^{\infty}G_n(\rho^{-1}g_n)(\xi)\sin n\pi\eta, \quad \text{where}$$

$$g(\xi,\eta)=\sum_{n=1}^{\infty}g_n(\xi)\sin n\pi\eta \quad \text{and}$$

$$G_n(\rho^{-1}g_n)(t)=\int_T^{\infty}G_n(t,s)\rho^{-1}(s)g_n(s)ds\,.$$

Then we have

Lemma 2.3. *If we choose sufficiently large T,*

(2.7) $$\|Gg\|_2^2 \leqq C\|g\|^2 \quad \textit{holds for all} \quad g \in L^2(C_T).$$

Proof. By definition we shall prove

$$\|\rho G_n''(\rho^{-1}g_n)\|^2 + n^2\|\rho G_n'(\rho^{-1}g_n)\|^2 + n^4\|\rho G_n(\rho^{-1}g_n)\|^2$$

$$\leqq C\|g_n\|^2 \text{ where } C \text{ is independent of } n.$$

Since $G_n''(\rho^{-1}g_n) = n^2\pi^2 G_n(\rho^{-1}g_n) + \rho^{-1}g_n$, we have only to prove

(2.8) $$n^2\|\rho G_n'(\rho^{-1}g_n)\|^2 + n^4\|\rho G_n(\rho^{-1}g_n)\|^2 \leqq C\|g_n\|^2.$$

As

$$|G_n(\rho^{-1}g_n)(t)| \leqq \int_T^\infty |G_n(t, s)|\rho^{-1}(s)|g_n(s)|ds \leqq \frac{1}{n\pi}\int_T^\infty e^{-n\pi|t-s|}\rho^{-1}(s)|g_n(s)|ds,$$

we have

$$\rho(t)|G_n(\rho^{-1}g_n)(t)| \leqq \frac{1}{n\pi}\int_T^\infty \frac{\rho(t)}{\rho(s)} e^{-n\pi|t-s|}|g_n(s)|ds.$$

Estimating both sides in well known manner, we find

(2.9) $$\|\rho G_n(\rho^{-1}g_n)\|^2 \leqq \frac{1}{n^2\pi^2} I_1 I_2 \|g_n\|^2, \quad \text{where}$$

$$I_1 = \left(\sup_{t \geq T} \int_T^\infty \frac{\rho(t)}{\rho(s)} e^{-n\pi|t-s|} ds\right)$$

$$I_2 = \left(\sup_{s \geq T} \int_T^\infty \frac{\rho(t)}{\rho(s)} e^{-n\pi|t-s|} dt\right) \quad \text{(Holmgren bound).}$$

i) Estimate of I_1.

$$\int_T^\infty \frac{\rho(t)}{\rho(s)} e^{-n\pi|t-s|} ds = \int_T^t + \int_t^\infty = I_{11} + I_{12}.$$

Since $\rho'(t) \geqq 0$, $\dfrac{\rho(t)}{\rho(s)} \leqq 1$ $(s \geqq t)$. So we can see $I_{12} \leqq \int_t^\infty e^{-n\pi(s-t)} ds = \dfrac{1}{n\pi}$. For I_{11}, integrating by parts we find

$$I_{11} \leqq \frac{1}{n\pi}\left[1 - \frac{\rho(t)}{\rho(T)} e^{-n\pi(t-T)}\right] + \frac{1}{n\pi}\int_T^t \frac{\rho(t)}{\rho(s)} \frac{\rho'(s)}{\rho(s)} e^{-n\pi(t-s)} ds.$$

If we choose sufficiently large T, we can make $|\rho'/\rho|$ arbitrarily small. We have $I_{11} \leqq \dfrac{1}{n\pi} + \dfrac{1}{2n\pi}$. Then $I_1 \leqq \dfrac{C}{n}$.

ii) Estimate of I_2.

By (2.2), we can see easily the integral I_2 is absolutely convergent.

$$\int_T^\infty \frac{\rho(t)}{\rho(s)} e^{-n\pi|t-s|} dt = \int_T^s + \int_s^\infty = I_{21} + I_{22} .$$

As before $I_{21} \leqq \frac{1}{n\pi}(1-e^{-n\pi(s-T)})$, and by integration by parts we find

$$\begin{aligned} I_{22} &= \frac{1}{n\pi} - \frac{1}{n\pi}\int_s^\infty \frac{\rho'(t)}{\rho(s)} e^{-n\pi(t-s)} dt \\ &= \frac{1}{n\pi} - \frac{1}{n\pi}\int_s^\infty \frac{\rho(t)}{\rho(s)} \frac{\rho'(t)}{\rho(t)} e^{-n\pi(t-s)} dt . \end{aligned}$$

After the same arguments as in i), we have $I_2 \leqq \frac{C}{n}$. Replacing I_1 and I_2 in (2.9) by the estimates obtained above, we find

$$n^4 \|\rho G_n(\rho^{-1} g_n)\|^2 \leqq C \|g_n\|^2 \tag{2.10}$$

where C is independent of n.

The estimate of $G_n'(\rho^{-1}g_n)$ can be carried out just in the same way as that of $G_n(\rho^{-1}g_n)$ and we obtain

$$n^2 \|\rho G_n'(\rho^{-1} g_n)\|^2 \leqq C \|g_n\|^2 . \tag{2.11}$$

Thus we have proved the lemma.

§ **3.** In this section we prove the main theorem with the aid of the preceeding lemma. Considering Proposition 1.1 and 1.2, the theorem is equivalent to the next one.

Theorem 3.1. *Let $f \in L^2_{loc}(C_T)$ such that $\rho^{-1} f \in L^2(C_T)$, and $u \in L^2_{loc}(C_T)$, u_ξ, $u_\eta \in L^2(C_T)$ be the solution of the problem*

$$\begin{cases} u_{\xi\xi} + u_{\eta\eta} + A(\xi, \eta, D_\xi, D_\eta) u = \rho^{-2}(\xi) f(\xi, \eta) \\ u|_{\partial C_T} \text{ (the trace of } u \text{ on } \partial C_T) = 0 , \end{cases} \tag{3.1}$$

then $\|u\|_2 < \infty$.

Proof. Since the solution of (3.1) is uniquely determined, we can expect to have the solution in the form

$$u = G(\rho^{-1} g) = \sum_{n=1}^\infty G_n(\rho^{-2} g_n) \sin n\pi\eta . \tag{3.2}$$

Applying $\Delta_{\xi\eta} + A$ to the both sides of (3.2), we find

$$u_{\xi\xi} + u_{\eta\eta} + Au = \rho^{-2} g + (AG)(\rho^{-1} g) = \rho^{-2} f .$$

Therefore g must satisfies

$$\rho^{-1} g + (\rho AG)(\rho^{-1} g) = \rho^{-1} f . \tag{3.3}$$

We set $K = \rho AG$. Since A is a second order differential operator whose coefficients can be made as small as we desire when we take sufficiently large T, we can see the operator K is a bounded operator from $L^2(C_T)$ to $L^2(C_T)$ and that the operator norm of K can be made smaller than 1. So $(I+K)$ is invertible, and for any $f\in L^2_{loc}(C_T)$ such that $\rho^{-1}f\in L^2(C_T)$, $g=\rho(I+K)^{-1}(\rho^{-1}f)$ is the solution of (3.3). We set $u=G(\rho^{-1}g)$, then u is the solution of (3.1) and we find

$$\|u\|_2\leqq C\|g\|\leqq C(1-\|K\|)^{-1}\|\rho^{-1}f\|$$

which proves the theorem.

Last of all, we should mention the uniqueness of self-adjoint extension of the Laplacian with the Dirichlet boundary condition (see H. Tamura [5] and F. Asakura [1]) follows from the preceeding theorem. Proof of the theorem is carried out in the same way as in S. Mizohata [3], Chap. III, § 16.

Theorem 3.2. *We consider the Laplacian as a symmetric operator from* $C^2(G)\cap C^0_0(G)$ *to* $L^2(G)$, *then the closure of the operator is a strictly self-adjoint operator with the domain* $H^2(G)\cap H^1_0(G)$. *So self-adjoint extension is unique in this case.*

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