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On Serre's conditions in the form ring of an ideal

By

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Introduction

Let A be a commutative noetherian ring, I an ideal contained in the radical of A and $G=G(A, I)=\bigoplus_{n\geq 0} I^n/I^{n+1}$ the form ring of A relative to I. The problem of the descent of a property from the ring G to the ring A was first tackled by Krull; he proved the following classical result: if G is a normal ring, so is A. Later on Hochster and Ratliff proved a similar result with respect to the Cohen-Macaulay (C. M.) property ([8], theorem 4.11). In the proof of the latter theorem the Rees ring $R=R(A, I)=\bigoplus_{n\in Z} I^n$ plays a fundamental role because of its close links with G and A. Indeed G is a quotient of R by a non zero-divisor, while R and A are connected by flat and local ring homomorphisms.

In this paper we prove that the following local properties of rings descend from G to A: regularity, locally complete intersection (C. I.), S_n, S_{n+1} and R_n, S_{n+1} and T_n. We show that the type of A is less than or equal to the type of G; in particular if G is a Gorenstein ring, so is A. Moreover we give an example which shows that the properties R_n and T_n do not pass from G to A without the further assumption that S_{n+1} holds.

The first section deals with basic facts on graded ring. More precisely we prove that if S is a graded ring such that for all homogeneous prime ideals $\mathfrak{p}$, the property S_n(resp. R_n, T_n, locally C. I.) holds for $S_{\mathfrak{p}}$, then the same holds for S. This kind of problem was raised by Nagata in [14] with respect to the C. M. property and investigated by several authors ([4], [8], [11], [12], [15], [22]).

In the second section we study the behaviour of the properties S_n, R_n, T_n in the passage from S/xS to S, where S is a graded ring and x a non zero-divisor belonging to the homogeneous radical of S. Precisely we prove that if S_n(resp. S_{n+1} and R_n, S_{n+1} and T_n) holds for S/xS, then the same holds for S. As corollaries of the above results we get in an unified version some known statements concerning the adjunction of an indeterminate (polynomial or power series).

Finally in the third section we prove that G is a regular (resp. locally C. I., Gorenstein) ring if and only if such is R and the type of G is equal to the type of R; moreover, if S_n(resp. S_{n+1} and R_n, S_{n+1} and T_n) holds for G, the same

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holds for R, but the converse is not necessarily true. On the other hand each of the above-mentioned properties descends from R to A.

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1. Terminology and basic results on graded rings

All the rings considered in this paper are assumed to be commutative, with 1 and noetherian.

Let $S=\bigoplus_{n\in Z} S_n$ be a graded ring; since S is noetherian, S is a finitely generated S_0-algebra (it is an easy consequence of [15], chapter II, proposition 3.2).

A homogeneous ideal $\mathfrak{m}$ of S is called *h-maximal* if $\mathfrak{m}$ is maximal among homogeneous ideals of S, that is the subring of the elements of degree 0 of $S/\mathfrak{m}$ is a field k and either $S/\mathfrak{m}=k$ or $S/\mathfrak{m}=k[T, T^{-1}]$, where T is transcendental over k. The intersection of all the h-maximal ideals of S is called *homogeneous radical* of S. If S has a unique h-maximal ideal, then S is called *h-local ring*. Clearly if S_0 is a local ring, then S is an h-local ring.

Let $\mathfrak{p}$ be a prime ideal of S. We will denote by $\mathfrak{p}'$ the greatest homogeneous ideal of S contained in $\mathfrak{p}$. Then $\mathfrak{p}'$ is again a prime ideal. Moreover, if $S_{(\mathfrak{p})}$ is the homogeneous localization of S at the multiplicative set of all the homogeneous elements of S not in $\mathfrak{p}$, then $(S_{(\mathfrak{p})}, \mathfrak{p}'S_{(\mathfrak{p})})$ is an h-local ring.

The results of this section show that several properties hold for $S_{\mathfrak{p}}$ if and only if they hold for $S_{\mathfrak{p}'}$.

Recall a few definitions, most of which can be found for example in [5], [13].

A local ring A is called *complete intersection* (C. I.) if its completion $\hat{A}$ is a homomorphic image of a regular local ring modulo a regular sequence. A ring A (not necessarily local) is called *locally C. I.* if $A_{\mathfrak{p}}$ is C. I. for all prime ideals $\mathfrak{p}$ of A.

Let $(A, \mathfrak{m}, k)$ be a local d-dimensional C. M. ring. The *(C. M.) type* of A is the number $r(A)=\dim_k \operatorname{Ext}_A^d(k, A)$. If A is a C. M. ring (not necessarily local) the *global type* of A, still denoted by $r(A)$, is the supremum of the types of the local rings $A_{\mathfrak{p}}$ as $\mathfrak{p}$ ranges through the prime ideals of A. If $r(A)=1$, then A is said to be a *Gorenstein ring*.

For an ideal I of a ring A we denote by $\operatorname{gr}(I)$ the grade of I, that is the common length of all maximal regular sequences in I. If $(A, \mathfrak{m})$ is a local ring, $\operatorname{depth}(A)$ means the grade of $\mathfrak{m}$.

A ring A is called S_n *ring* if $\operatorname{depth}(A_{\mathfrak{p}})\geqq\min(n, \operatorname{ht}(\mathfrak{p}))$ for all prime ideals $\mathfrak{p}$ of A.

A ring A is called R_n (resp. T_n) *ring* if $A_{\mathfrak{p}}$ is a regular (resp. Gorenstein) ring, for all prime ideals $\mathfrak{p}$ of A such that $\operatorname{ht}(\mathfrak{p})\leqq n$.

A ring which is both S_n and T_{n-1} is called *n-Gorenstein ring*.

Lemma 1.1. *Let S be a graded ring and $\mathfrak{p}$ a non homogeneous prime ideal of S. Then:*

i) $\mathrm{ht}(\mathfrak{p})=\mathrm{ht}(\mathfrak{p}')+1$ *and* $\mathrm{depth}(S_{\mathfrak{p}})=\mathrm{depth}(S_{\mathfrak{p}'})+1$.

ii) *The ring* $S_{\mathfrak{p}}$ *is regular (resp. C.M.,* $r(S_{\mathfrak{p}})=n$, *Gorenstein) if and only if such is* $S_{\mathfrak{p}'}$.

Proof. The first part of ii), stated in a different way, is in [11], theorem 2.1 and [15], chapter III, theorem 2.3. The remaining part of the lemma is in [4], corollary 1.1.3.

Remark 1.2. From the lemma 1.1 (ii) and [7], Satz 6.16 it follows that the type of a graded C.M. ring S is the supremum of the types of the local rings $S_{\mathfrak{m}}$ as $\mathfrak{m}$ ranges through the h-maximal ideals of S.

Corollary 1.3. *Let S be a graded ring. Then:*

i) *S is an S_n ring if and only if for every homogeneous prime ideal $\mathfrak{q}$ of S,* $\mathrm{depth}(S_{\mathfrak{q}})\geqq\min(n, \mathrm{ht}(\mathfrak{q}))$.

ii) *S is an R_n(resp. T_n) ring if and only if for every homogeneous prime ideal $\mathfrak{q}$ of S such that* $\mathrm{ht}(\mathfrak{q})\leqq n$, *the ring $S_{\mathfrak{q}}$ is regular (resp. Gorenstein).*

Proof. i) Let $\mathfrak{p}$ be a non homogeneous prime ideal of S. Then, by lemma 1.1 (i), we have: $\mathrm{depth}(S_{\mathfrak{p}})=\mathrm{depth}(S_{\mathfrak{p}'})+1\geqq\min(n, \mathrm{ht}(\mathfrak{p}'))+1\geqq\min(n, \mathrm{ht}(\mathfrak{p}))$.

ii) Let $\mathfrak{p}$ be a non homogeneous prime ideal of S such that $\mathrm{ht}(\mathfrak{p})\leqq n$. Then $\mathrm{ht}(\mathfrak{p}')=\mathrm{ht}(\mathfrak{p})-1<n$, hence $S_{\mathfrak{p}'}$ is a regular (resp. Gorenstein) ring and, by lemma 1.1 (ii), $S_{\mathfrak{p}}$ is regular (resp. Gorenstein) too.

The converse of i) and ii) is trivial.

The following proposition has been inspired by the proof of the proposition 4.10 in [8].

Proposition 1.4. *Let S be a graded ring and $\mathfrak{p}$ a prime ideal of S. Then $S_{\mathfrak{p}}$ is C.I. if and only if so is $S_{\mathfrak{p}'}$.*

Proof. The condition is clearly necessary. Conversely, let X be an indeterminate of degree 0. For every prime ideal $\mathfrak{p}$ of S, the ring homomorphism $S_{\mathfrak{p}}\rightarrow S[X]_{\mathfrak{p}S[X]}$ is flat and local and the fibre over $\mathfrak{p}S_{\mathfrak{p}}$ is a field. Hence $S_{\mathfrak{p}}$ is C.I. if and only if so is $S[X]_{\mathfrak{p}S[X]}$ ([1], theorem 2).

Now we assume that $\mathfrak{p}$ is non homogeneous. We can replace S by the homogeneous localization S^* of $S[X]$ at $\mathfrak{p}S[X]$. In fact, since $\mathrm{ht}(\mathfrak{p}'S[X])=\mathrm{ht}(\mathfrak{p}')=\mathrm{ht}(\mathfrak{p})-1=\mathrm{ht}(\mathfrak{p}S[X])-1$, $\mathfrak{p}'S^*$ is the only h-maximal ideal of S^*, thus the hypothesis on $\mathfrak{p}'$ still holds for $\mathfrak{p}'S^*$. From now on S, $\mathfrak{p}$, $\mathfrak{p}'$ will respectively mean S^*, $\mathfrak{p}^*$, $\mathfrak{p}'S^*$.

Let $\hat{S}_0$ be the completion of S_0 and $\bar{S}$ the h-local ring $S\otimes_{S_0}\hat{S}_0$. Since $S/\mathfrak{p}'=k[T, T^{-1}]$, we have $\bar{S}/\mathfrak{p}'\bar{S}\cong S/\mathfrak{p}'\otimes_{S_0}\hat{S}_0\cong k[T, T^{-1}]$. Thus $\mathfrak{p}'\bar{S}$ is the h-maximal ideal of $\bar{S}$ and the canonical ring homomorphism $S_{\mathfrak{p}'}\rightarrow\bar{S}_{\mathfrak{p}'\bar{S}}$ is flat and local and its fibre over $\mathfrak{p}'S_{\mathfrak{p}'}$ is a field. Then $\bar{S}_{\mathfrak{p}'\bar{S}}$ is C.I. ([1], loc. cit.). The ring homomorphism $S_{\mathfrak{p}}\rightarrow\bar{S}_{\mathfrak{p}\bar{S}}$ is flat and local too, hence if $\bar{S}_{\mathfrak{p}\bar{S}}$ is C.I., so is $S_{\mathfrak{p}}$. Therefore replacing S by $\bar{S}$ we can assume that S_0 is a complete local ring with infinite residue field k and S is a homomorphic image of a regular ring. Then the C. I.-locus of S is an open set ([5], IV, 19.3.3). Let I be the defining radical ideal of the non C. I.-locus. If I is a proper ideal, then it suffices to show that I is homogeneous to get a contradiction.

For every unit x in S_0 we have an S_0-automorphism of S which takes each form F of degree d to $x^d F$. Let $\sum_{i=-p}^{q} F_i \in I$ (where each F_i is a form of degree i). We choose units $x_{-p}, \cdots, x_q$ in S_0 with distinct images in k (k is infinite). For all j, $-p \leqq j \leqq q$, we have $\sum_{i=-p}^{q} x_j^i F_i \in I$, because I is invariant under every automorphism on S. Now $\det(x_j^i) = \pm(\prod_{j=-p}^{q} x_j^{-p}) \cdot (\prod_{i<j}(x_i - x_j))$ is a unit in S_0. Therefore each $F_i \in I$.

Remark. Since the considered properties are stable under localization, the results of lemma 1.1 and proposition 1.4 can be expressed as follows:

A graded ring S is regular (resp. locally C. I., Gorenstein, C. M.) if and only if so is $S_{\mathfrak{m}}$ for every h-maximal ideal $\mathfrak{m}$ of S.

2. Relations among properties of a graded ring S and a quotient ring S/xS

Lemma 2.1. *Let I be an ideal of a ring A and x an element of A such that $J=(I, x)$ is a proper ideal. If $\mathfrak{p}$ is a minimal prime ideal over J, then* $\operatorname{depth}(A_{\mathfrak{p}}) \leqq \operatorname{gr}(IA_{\mathfrak{p}})+1$.

Proof. Since $JA_{\mathfrak{p}}$ is $\mathfrak{p}A_{\mathfrak{p}}$-primary, we have $\operatorname{depth}(A_{\mathfrak{p}}) = \operatorname{gr}(JA_{\mathfrak{p}})$. On the other hand $\operatorname{gr}(JA_{\mathfrak{p}}) \leqq \operatorname{gr}(IA_{\mathfrak{p}})+1$ ([9], theorem 127).

Proposition 2.2. *Let S be a graded ring, x a non zero-divisor (not necessarily homogeneous) belonging to the homogeneous radical of S, $T=S/xS$. If T is an S_n ring, then S is too.*

Proof. By corollary 1.3 (i) it is enough to show: $\operatorname{depth}(S_{\mathfrak{q}}) \geqq \min(n, \operatorname{ht}(\mathfrak{q}))$ for every homogeneous prime ideal $\mathfrak{q}$ of S. If $x \in \mathfrak{q}$, we have: $\operatorname{depth}(S_{\mathfrak{q}}) = \operatorname{depth}(T_{\bar{\mathfrak{q}}})+1 \geqq \min(n, \operatorname{ht}(\bar{\mathfrak{q}}))+1 = \min(n, \operatorname{ht}(\mathfrak{q})-1)+1 \geqq \min(n, \operatorname{ht}(\mathfrak{q}))$ where $\bar{\mathfrak{q}} = \mathfrak{q}/xS$. Now we suppose that $x \notin \mathfrak{q}$. Then $(\mathfrak{q}, x) \neq S$. Let $\mathfrak{p}$ be a minimal prime over $(\mathfrak{q}, x)$. From lemma 2.1 and our hypothesis on T it follows: $\operatorname{depth}(S_{\mathfrak{q}}) \geqq \operatorname{gr}(\mathfrak{q}S_{\mathfrak{p}}) \geqq \operatorname{depth}(S_{\mathfrak{p}})-1 = \operatorname{depth}(T_{\bar{\mathfrak{p}}}) \geqq \min(n, \operatorname{ht}(\bar{\mathfrak{p}})) \geqq \min(n, \operatorname{ht}(\mathfrak{q}))$.

If S/xS is an R_n ring, then S is not necessarily an R_n ring ([5], IV, 5.12.6).

Proposition 2.3. *Let S be a graded ring, x a non zero-divisor (not necessarily homogeneous) belonging to the homogeneous radical of S, $T=S/xS$. If T is an S_{n+1} and R_n (resp. S_{n+1} and T_n) ring, so is S.*

Proof. By proposition 2.2, S is an S_{n+1} ring. It suffices to prove that $S_{\mathfrak{q}}$ is a regular (resp. Gorenstein) ring, for every homogeneous prime ideal $\mathfrak{q}$ of S such that $\operatorname{ht}(\mathfrak{q}) \leqq n$ (corollary 1.3 (ii)). If $x \in \mathfrak{q}$ ($\mathfrak{q}$ not necessarily homogeneous) and $\operatorname{ht}(\mathfrak{q}) \leqq n+1$, we put $\bar{\mathfrak{q}} = \mathfrak{q}/xS$, then $\operatorname{ht}(\bar{\mathfrak{q}}) \leqq n$ and $T_{\bar{\mathfrak{q}}}$ is regular (resp. Gorenstein). Since $T_{\bar{\mathfrak{q}}}=S_{\mathfrak{q}}/xS_{\mathfrak{q}}$, from [5], 0_{IV}, 17.1.8 (resp. [3], corollary 2.6), we get $S_{\mathfrak{q}}$ is regular (resp. Gorenstein). If $x\notin\mathfrak{q}$ ($\mathfrak{q}$ homogeneous) and $\mathrm{ht}(\mathfrak{q})\leqq n$, let $\mathfrak{p}$ be a minimal prime ideal over $(\mathfrak{q}, x)$ and $\bar{\mathfrak{p}}=\mathfrak{p}/xS$. We have: $\mathrm{depth}(T_{\bar{\mathfrak{p}}})\geqq \min(n+1, \mathrm{ht}(\bar{\mathfrak{p}}))$. If $\mathrm{ht}(\bar{\mathfrak{p}})\geqq n+1$ using lemma 2.1 we get the following contradiction:

$$n+1\leqq \mathrm{depth}(T_{\bar{\mathfrak{p}}})=\mathrm{depth}(S_{\mathfrak{p}})-1\leqq \mathrm{gr}(\mathfrak{q}S_{\mathfrak{p}})\leqq \mathrm{gr}(\mathfrak{q}S_{\mathfrak{q}})\leqq \mathrm{ht}(\mathfrak{q})\leqq n\,.$$

Then $\mathrm{ht}(\bar{\mathfrak{p}})<n+1$ and $\mathrm{ht}(\mathfrak{p})\leqq n+1$. Since $x\in\mathfrak{p}$, from the first part of the proof if follows that $S_{\mathfrak{p}}$, and hence $S_{\mathfrak{q}}$, is regular (resp. Gorenstein).

Corollary 2.4. *If T is a reduced (resp. normal, C.M., Gorenstein, regular) ring, so is S.*

Proof. A ring is reduced (resp. normal, C.M., Gorenstein, regular) if and only if it is S_1 and R_0 (resp. S_2 and R_1, S_n for all n, T_n for all n, R_n for all n).

Remark. Propositions 2.2 and 2.3 yield the result that polynomial adjunction preserves the following properties: S_n, S_{n+1} and R_n, n-Gorenstein, and hence reducedness, normality, C.M., Gorenstein, regularity. Thus we get in an unified version several known statements concerning the adjunction of an indeterminate (polynomial or power series).

Remark. The propositions 2.2 and 2.3 hold in particular when S is a trivially graded ring and x belongs to the radical of S. Hence the first of these results is a slight improvement of [5], IV, 5.12.4 and [10], proposition 1.8, the latter (with respect to the T_n property) of [19], proposition 3.

3. Relations among properties of a ring A, the form ring and the Rees ring of A with respect to an ideal I

Let A be a ring and $I=(a_1, \cdots, a_r)$ an ideal of A.

Denote by $R=R(A, I)=\bigoplus_{n\in Z} I^n$ (where $I^n=A$ for $n\leqq 0$) the Rees ring and by $G=G(A, I)=\bigoplus_{n\geqq 0} I^n/I^{n+1}$ the form ring of A with respect to I. Then R is the subring of $A[T, T^{-1}]$ consisting of all finite sums $\sum_{i=-p}^{q} c_iT^i$ with $c_i\in I^i$. It results $R=A[a_1T, \cdots, a_rT, T^{-1}]$, thus R is a noetherian graded ring. If we put $u=T^{-1}$, the element u is a non zero-divisor in R.

If J is an ideal of A, we denote by J^* the homogeneous ideal $JA[T, u]\cap R$; it is clear that $J^*=\{\sum_{i=-p}^{q} c_iT^i/c_i\in I^i\cap J\}$.

Lemma 3.1. *Let I, J be ideals of a ring A, $\mathfrak{p}$ a prime ideal of A. Then:*

a) *$R(A/J, I+J/J)\cong R(A, I)/J^*$* ([18], lemma 1.1).

b) *The ideal $\mathfrak{p}^*$ of $R(A, I)$ is prime and* $\mathrm{ht}(\mathfrak{p}^*)=\mathrm{ht}(\mathfrak{p})$ ([18], theorm 1.5 and [16], remark 3.7).

c) *If $\mathfrak{p}\supset I$, then $(\mathfrak{p}^*, u)$ is a prime ideal of $R(A, I)$ and* $\mathrm{ht}(\mathfrak{p}^*, u)=\mathrm{ht}(\mathfrak{p}^*)+1$ ([17], remark 2.2.6 (ii)).

d) *Let* $\mathfrak{P}$ *be a homogeneous prime ideal of* $R(A, I)$. *If* $u \notin \mathfrak{P}$, *then* $\mathfrak{P} = (\mathfrak{P} \cap A)^*$ ([17] remark 2.2.5 (i)).

e) *The h-maximal ideals of* $R(A, I)$ *are in one-to-one correspondence with the maximal ideals of* A: *precisely they are* $(\mathfrak{m}^*, u)$ *if* $\mathfrak{m} \supset I$ *or* $\mathfrak{m}^*$ *if* $\mathfrak{m} \not\supset I$.

f) $G(A, I) \cong R(A, I)/uR(A, I)$ ([18], theorem 2.1).

Proof of e). If $\mathfrak{m} \not\supset I$, then $\mathfrak{m}$ and I are comaximal. Thus $R(A, I)/\mathfrak{m}^* = R(A/\mathfrak{m}, I+\mathfrak{m}/\mathfrak{m}) = A/\mathfrak{m}[T, T^{-1}]$ hence $\mathfrak{m}^*$ is h-maximal. Otherwise if $\mathfrak{m} \supset I$, the components of degree $i \neq 0$ of $(\mathfrak{m}^*, u)$ are equal to I^i so $R(A, I)/(\mathfrak{m}^*, u) = A/\mathfrak{m}$ and $(\mathfrak{m}^*, u)$ is h-maximal. Conversely, let $\mathfrak{M}$ be an h-maximal ideal of $R(A, I)$. Then $\mathfrak{M} \cap A = \mathfrak{m}$ is maximal in A. If $\mathfrak{m} \not\supset I$, then $\mathfrak{m}R(A, I) = \mathfrak{m}^*$ (for $\mathfrak{m} \cap I^i = \mathfrak{m}I^i$), so $\mathfrak{M} \supset \mathfrak{m}^*$. If $\mathfrak{m} \supset I$, then $\mathfrak{M} \subset (\mathfrak{m}^*, u)$. Hence $\mathfrak{M} = \mathfrak{m}^*$ or $\mathfrak{M} = (\mathfrak{m}^*, u)$.

Lemma 3.2. *Let* I *be an ideal of a ring* A, $\mathfrak{p}$ *a prime ideal of* A *and* $R = R(A, I)$. *Then*:

i) $R_{\mathfrak{p}^*} = A[u]_{\mathfrak{p}A[u]}$.

ii) *The ring homomorphism* $A_{\mathfrak{p}} \to R_{\mathfrak{p}^*}$, *induced by the canonical ring homomorphism* $A \to A[u]$, *is flat and local, moreover the fibre over* $\mathfrak{p}A_{\mathfrak{p}}$ *is a field.*

Proof. i) We have $A[u] \subset R$ and $\mathfrak{p}^* \cap A[u] = \mathfrak{p}A[u]$, hence $A[u]_{\mathfrak{p}A[u]} \subset R_{\mathfrak{p}^*}$. Now let $f/g \in R_{\mathfrak{p}^*}$ and let d be the greatest degree of the homogeneous components of f and g. Since $u \notin \mathfrak{p}^*$ we have $f/g = (u^d f)/(u^d g) \in A[u]_{\mathfrak{p}A[u]}$.

ii) It follows easily from (i).

Lemma 3.3. *Let* I *be an ideal of a ring* A *and* $\mathfrak{M}$ *a proper homogeneous ideal of* $R = R(A, I)$ *such that* $u \in \mathfrak{M}$. *Then* $u \notin \mathfrak{M}^2$. *Moreover if* $\mathfrak{M}$ *is maximal and* $J = \mathfrak{M}R_{\mathfrak{M}}$, *then* $u \in J - J^2$.

Proof. If $u \in \mathfrak{M}^2$, then $u = \sum_{j=1} a_j b_j$ where a_j and b_j are homogeneous elements of $\mathfrak{M}$ with $\deg a_j + \deg b_j = -1$. We may assume $\deg a_j < 0$ for all j, so $a_j \in uR$; it follows that $u(1 - \sum_{j=1}^{n} a'_j b_j) = 0$ $(a'_j \in R)$. But u is a non zero-divisor in R, hence $1 = \sum_{j=1}^{n} a'_j b_j \in \mathfrak{M}$ and we get a contradiction. Since, if $\mathfrak{M}$ is maximal, $\mathfrak{M}^2$ is the contraction of J^2, the remaining part follows.

From now on let A denote a ring and I an ideal contained in the radical of A unless otherwise specified.

Theorem 3.4. *Let* $R = R(A, I)$ *and* $G = G(A, I)$. *Then*:

i) *G is a regular* (*resp. locally C.I.*) *ring if and only if so is R.*

ii) *If R is a regular* (*resp. locally C.I.*) *ring, so is A.*

Proof. i) Since I is contained in the radical of A, each h-maximal ideal $\mathfrak{M}$ of R contains u (lemma 3.1 (e)), hence by lemma 1.1 (ii) (resp. proposition 1.4) it suffices to prove that $R_{\mathfrak{M}}$ is a regular (resp. C.I.) ring if and only if so is $G_{\mathfrak{N}}$ for all h-maximal ideal $\mathfrak{M}$ of R and $\mathfrak{N} = \mathfrak{M}/uR$. Our thesis follows from [5], 0_{IV}, 17.1.8 and lemma 3.3 (resp. [6], theorem 3.5.1 and corollary 3.4.2).

ii) For every maximal ideal $\mathfrak{m}$ of A, $R_{\mathfrak{m}^*}$ is a regular (resp. C.I.) local ring.

Our thesis follows by applying [13], (21, D), theorem 51 (resp. [1], theorem 2) to the flat and local ring homomorphism of lemma 3.2.

Remark 3.5. In order to prove (ii) of theorem 3.4, we do not need to assume I to be contained in the radical of A. Even throughout the rest of this paper we do not need such assumption to descend from R to A; nevertheless, as it allows us to pass from G to R, we will keep it for the sake of simplicity. The following example shows that if I is an ideal not contained in the radical of A, it can happen that G is regular, while A is not even C.M.

Example 3.6. Let $B=k[X, Y]$, $J=(X^2, XY)$, $\mathfrak{M}=(X, 1-Y)$, $A=B/J$ and $I=\mathfrak{M}/J$. Using some results of [23] it is not difficult to show that $G(A, I)\cong k[Y]$.

Theorem 3.7. *Let $R=R(A, I)$ and $G=G(A, I)$. We assume that G is a C.M. ring. Then:*

$$r(G)=r(R)\geqq r(A).$$

Proof. The rings R and A are C.M. too ([8], theorem 4.11). Since $G=R/uR$ and u is a non zero-divisor belonging to each h-maximal ideal $\mathfrak{M}$ of R, we get $r(R_{\mathfrak{M}})=r(G_{\mathfrak{N}})$ where $\mathfrak{N}=\mathfrak{M}/uR$ ([7], (1.22)), hence by remark 1.2, $r(R)=r(G)$.

Now let $\mathfrak{m}$ be a maximal ideal of A. We get $r(A_{\mathfrak{m}})=r(R_{\mathfrak{m}^*})$ by applying [7], Satz 1.24 to the flat and local ring homomorphism of lemma 3.2. As $r(R_{\mathfrak{m}^*})\leqq r(R)$ our thesis follows.

Corollary 3.8. i) *The ring G is Gorenstein if and only if so is R.*

ii) *If R is a Gorenstein ring, so is A.*

Remark. In [2] it is proved, with different methods, that $r(G)\geqq r(A)$ if G is a C.M. ring which is a flat A/I-module (theorem 1.1) or if $(A, \mathfrak{m})$ is a local ring and I is an $\mathfrak{m}$-primary ideal of A (proposition 1.3).

Theorem 3.9. *Let $R=R(A, I)$ and $G=G(A, I)$. Then:*

i) *If G is an S_n ring, so is R.*

ii) *If R is an S_n ring, so is A.*

Proof. i) It follows trivially from proposition 2.2.

ii) Let $\mathfrak{p}$ be a prime ideal of A. Then $\mathfrak{p}^*$ is a prime ideal of R and $\mathrm{ht}(\mathfrak{p})=\mathrm{ht}(\mathfrak{p}^*)$ (lemma 3.1 (b)). Since the fibre over $\mathfrak{p}A_{\mathfrak{p}}$ of the flat and local ring homomorphism $A_{\mathfrak{p}}\to R_{\mathfrak{p}^*}$ is a field (lemma 3.2 (ii)), we have $\mathrm{depth}(A_{\mathfrak{p}})=\mathrm{depth}(R_{\mathfrak{p}^*})$ ([13], (21.C), corollary 1). Therefore A is an S_n ring.

The converse of theorem 3.9 is not true. However we remark that if A is an S_1 ring, so is R. This is an easy consequence of [18], theorem 1.5.

Example 3.10. ([20]) Let $B=k[X, Y, Z, W]_{(X,Y,Z,W)}$, $J=(Z^2-W^5, Y^2-XZ)$, $A=B/J$. Then A is a 2-dimensional C.M. ring. If $\mathfrak{p}=(Y, Z, W)A$ then $G(A, \mathfrak{p})\cong k[X]_{(X)}[Y, Z, W]/(Z^2, XZ, Y^2Z, Y^4)$ is not an S_1 ring. Hence $R(A, \mathfrak{p})$ is S_1 but not S_2.

If $G(A, I)$ is an R_n (resp. T_n) ring, in general it is not true that A and $R(A, I)$ are R_n (resp. T_n) ring.

Example 3.11. Let $B=k[X, Y, Z]_{(X,Y,Z)}$, $N=(XY^2, XYZ, X^2Z^2)B$, $J=(X-Z)B$, $A=B/N$, $I=J+N/N$. The ideal I is generated by a non zero-divisor of A, thus $G=G(A, I)\cong A/I[T]\cong k[Y, Z]_{(Y,Z)}[T]/(ZY^2, YZ^2, Z^4)$ and $R=R(A, I)\cong k[X, Y, Z]_{(X,Y,Z)}[T, U]/(XY^2, XYZ, X^2Z^2, X-Z-TU)$. Then G is an R_0 ring, but A and R are not even T_0.

In the theorem 3.13 we will give two sufficient condition in order that the R_n (resp. T_n) property descends from G to R.

Lemma 3.12. *Let $R=R(A, I)$, $\mathfrak{p}$ a prime ideal of A. If there exists a maximal ideal $\mathfrak{m}$ of A such that $\mathfrak{m}\supset\mathfrak{p}$ and $\mathrm{ht}(\mathfrak{m}/\mathfrak{p})=\mathrm{ht}(\mathfrak{m})-\mathrm{ht}(\mathfrak{p})$, then $\mathrm{ht}(\mathfrak{p}^*, u)=\mathrm{ht}(\mathfrak{p}^*)+1$.*

Proof. Since $R(A, I)_{(A-\mathfrak{m})}=R(A_{\mathfrak{m}}, IA_{\mathfrak{m}})$, we may assume A local. By [21], proposition 15 and lemma 9, we get:

$$\mathrm{ht}(\mathfrak{p}^*)=\mathrm{ht}(\mathfrak{p})=\mathrm{ht}((\mathfrak{p}^*, u)/uR)=\mathrm{ht}(\mathfrak{p}^*, u)-1.$$

Theorem 3.13. *Let $R=R(A, I)$ and $G=G(A, I)$.*

i) *If R is R_n (resp. T_n) ring, so is A.*

ii) *Assume G is an R_n (resp. T_n) ring and, moreover, either of the following conditions holds:*

a) *G is an S_{n+1} ring;*

b) *For every prime ideal $\mathfrak{p}$ of A there exists a maximal ideal $\mathfrak{m}$ of A such that $\mathfrak{m}\supset\mathfrak{p}$ and $\mathrm{ht}(\mathfrak{m}/\mathfrak{p})=\mathrm{ht}(\mathfrak{m})-\mathrm{ht}(\mathfrak{p})$.*

Then R is an R_n (resp. T_n) ring.

Proof. i) For every prime ideal $\mathfrak{p}$ of A such that $\mathrm{ht}(\mathfrak{p})\leqq n$, we have $\mathrm{ht}(\mathfrak{p}^*)\leqq n$, so $R_{\mathfrak{p}^*}$ is a regular (resp. Gorenstein) ring. Our thesis follows by applying [13], (21.D), theorem 51 (resp. [24], part I, n. 2, theorem 1(2)) to the flat and local ring homomorphism of lemma 3.2.

ii) If the condition (a) holds, our thesis follows from proposition 2.3. Now we assume the condition (b). By corollary 1.3 (ii) it suffices to prove that for every homogeneous prime ideal $\mathfrak{P}$ of R with $\mathrm{ht}(\mathfrak{P})\leqq n$, the ring $R_{\mathfrak{P}}$ is regular (resp. Gorenstein). If $u\in\mathfrak{P}$ and $\mathrm{ht}(\mathfrak{P})\leqq n+1$, as in proposition 2.3, we can prove that $R_{\mathfrak{P}}$ is regular (resp. Gorenstein). If $u\notin\mathfrak{P}$ and $\mathrm{ht}(\mathfrak{P})\leqq n$, by lemma 3.12, there exists a homogeneous prime ideal $\mathfrak{Q}$ of R such that $\mathfrak{Q}\supset(\mathfrak{P}, u)$ and $\mathrm{ht}(\mathfrak{Q})\leqq n+1$. From what already shown $R_{\mathfrak{Q}}$, and hence $R_{\mathfrak{P}}$, is a regular (resp. Gorenstein) ring.

Remark. The condition (b) of the theorem 3.13 is satisfied if A is a C.M. ring. The example 3.10 shows that it is independent from the condition (a).

Corollary 3.14. *If G is a reduced (resp. normal, n-Gorenstein) ring, so are R and A.*

Remark 3.15. If A is an R_0 ring, so is R. This fact follows easily from [18], theorem 1.5, [13], (21.D), theorem 51 and lemma 3.2. The following example exhibit a regular ring A which has a Rees ring R not R_1.

Let $A=k[X, Y]_{(X,Y)}$, $I=(X^2, Y^2)A$. Then $R(A, I)\cong A[T_1, T_2, U]/(X^2-T_1U, Y^2-T_2U)$ localized at (x, y, u) is not regular. Moreover $G(A, I)\cong A/I[T_1, T_2]$ is not even R_0.

Remark 3.16. Let $\mathcal{P}$ be one of the following properties: regularity, locally C.I., type less than or equal to n, Gorenstein, S_n, S_{n+1} and R_n, S_{n+1} and T_n. Everything proved in this paper with regard to the descent of $\mathcal{P}$ from $G=G(A, I)$ to A under the assumption "I contained in the radical of A" holds also under the assumption "A is a graded ring and I is contained in the homogeneous radical of A" (I not necessarily homogeneous). In fact we assume that G has $\mathcal{P}$.

1) In order to prove that A has $\mathcal{P}$ it suffices to prove that $A_{\mathfrak{m}}$ has $\mathcal{P}$ for every h-maximal ideal $\mathfrak{m}$ of A (see section 1). For every such $\mathfrak{m}$ let $G_{(A-\mathfrak{m})}$ the localization of G with respect to the multiplicative set $A-\mathfrak{m}$. Since $I\subset\mathfrak{m}$ and $G_{(A-\mathfrak{m})}=G(A_{\mathfrak{m}}, IA_{\mathfrak{m}})$ has $\mathcal{P}$, from what we have already proved in the case "I contained in the radical of A", it follows that $A_{\mathfrak{m}}$ has $\mathcal{P}$.

2) Let $R=R(A, I)$. For every homogeneous prime ideal $\mathfrak{P}$ of R containing u, $R_{\mathfrak{p}}$ has $\mathcal{P}$. On the other hand a homogeneous prime ideal of R not containing u is of the kind $\mathfrak{p}^*$ with $\mathfrak{p}$ prime ideal of A. From 1) it follows that $A_{\mathfrak{p}}$ has $\mathcal{P}$. The existence of a flat and local ring homomorphism $A_{\mathfrak{p}}\to R_{\mathfrak{p}^*}$ having as fibre over $\mathfrak{p}A_{\mathfrak{p}}$ a field, assures that $R_{\mathfrak{p}^*}$ has $\mathcal{P}$ ([1], theorem 2, [7], Satz 1.24, [13], (21.D), theorem 51). Hence R has $\mathcal{P}$ too.

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