J. Math. Kyoto Univ. (JMKYAZ)
27-3 (1987) 507-530

Variational formulas on arbitrary Riemann surfaces under pinching deformation

By

Masahiko TANIGUCHI

Introduction.

The method of orthogonal decomposition plays a crucial role in the theory of abelian differentials on Riemann surfaces. Actually, we have found its new application in deriving variational formulas on Riemann surfaces under quasiconformal deformation (cf. [1] and [3]). The argument consists of two steps. Namely, we show first certain continuity (or distortion) with respect to Dirichlet norm of the given family of differentials, by using inner orthogonality of the family, and secondly we derive variational formulas by using another orthogonality of the family to the linear operator considered in each formula.

The first step was generalized to the case of deformation by pinching a finite number of loops (cf. [5, §3], where certain continuity of square integrable harmonic differentials was treated. See also [7, Theorem 1]). The purpose of this paper is to generalize the second step to the case of pinching deformation and to give associated variational formulas for basic differentials such as period reproducers and Green's functions.

For this purpose, we give in §1 the definition of pinching deformation and a general fundamental variational formula (Theorem 1). (This formula reduces to a trivial one in case of quasiconformal deformation, but has some applications, cf. [8] which also contains a refinement of it.) By applications of Theorem 1, we have in §2 certain variational formulas for basic differentials (Theorems 2, 3 and 4). The proofs are given in §§4 and 5. The decisive parts of the proofs are Lemmas 5, 8 and 9, which can be considered as fruits of the method of orthogonal decomposition, though the proofs need certain investigation on differentials associated with pinching loops. We give in §3 the order estimate and metrical continuity of such differentials (Theorems 5 and 6, respectively). We note that Theorem 6 can be considered as a corollary of the proof of [5, Theorem 3] after applying the inverse operation of the so-called variation by reopening nodes of Schiffer-Spencer's type, and that using this operation we can also characterize the conformal topology. Appendix includes one of such characterization (cf. [7, Theorem 3]).

Received March 25, 1986

§1. A general variational formula.

Let R_0 be an arbitrary Riemann surface with a finite number of nodes $\{p_j\}_{j=1}^n$ (cf. [5, §1, 1°)]. Recall, in particular, that the universal covering surface of any component of $R_0'=R_0-\bigcup_{j=1}^n\{p_j\}$ is conformally equivalent to the unit disk). For every j, we fix a neighbourhood U_j of p_j on R_0 such that each component, say $U_{j,k}$ ($k=1,2$), of $U_j-\{p_j\}$ is conformally equivalent to $D_0=\{0<|z|<1\}$ by a conformal mapping, say $z_{j,k}(p)$. Also we suppose that $\{\bar{U}_j\}_{j=1}^n$ are mutually disjoint.

For every $t>0$, let f_t be a quasiconformal mapping of R_0' onto another union R_t' of Riemann surfaces with the complex dilatation μ_t. Further we assume that

a) the support of μ_t is contained in R_0-U, where $U=\bigcup_{j=1}^n U_j$, and

b) there is a bounded $(-1, 1)$-form μ on R_0' such that

$$\lim_{t\to 0}\|(\mu_t/t)-\mu\|_\infty = 0\,,$$

where $\|\cdot\|_\infty$ is the L^∞-norm on R_0'. For $t=0$, we denote by f_0 the identical mapping of R_0' onto itself.

Next for every fixed $t\geq 0$ and s_j with $0<s_j<1/2$ ($j=1,\cdots,n$), let $R_{t,s}$ (with $s=(s_1,\cdots,s_n)$) be the Riemann surface obtained from R_t' by deleting two punctured disks $z_{j,k,t}{}^{-1}(\{0<|z|<s_j\})$ ($k=1, 2$) and identifying the borders $B_{j,k,t,s}=z_{j,k,t}{}^{-1}(\{|z|=s_j\})$ by the mapping

$$z_{j,2,t}{}^{-1}(\eta_j\cdot s_j^2/z_{j,1,t}(p))$$

for every j, where $z_{j,k,t}=z_{j,k}\circ f_t^{-1}$ (which maps $U_{j,k,t}=f_t(U_{j,k})$ conformally onto D_0) and η_j is a constant with $|\eta_j|=1$. We denote by $C_{j,t,s}$ the loop on $R_{t,s}$ corresponding to $\{B_{j,k,t,s}\}_{k=1}^2$ and equipped with the same orientation as that of $B_{j,1,t,s}$. The parameter $s=(s_1,\cdots,s_n)$ can be considered as pinching parameters for these loops $\{C_{j,t,s}\}_{j=1}^n$, and we can construct a *canonical pinching mappings* $f_{t,s}$ of $R_{t,s}$ to R_0 as follows. Let $J_{t,s}$ be the natural embedding of $R_{t,s}''=R_{t,s}-\bigcup_{j=1}^n C_{j,t,s}$ into R_t', $V_{j,k}=z_{j,k}{}^{-1}(\{0<|z|<1/2\})$, $V_j=\bigcup_{k=1}^2 V_{j,k}$, $V=\bigcup_{j=1}^n V_j$, and $z_{j,k,t,s}=z_{j,k,t}\circ J_{t,s}$ to $U_{j,k,t,s}=J_{t,s}{}^{-1}(U_{j,k,t})$, and we set

$$\begin{aligned} f_{t,s}{}^{-1}(p) &= J_{t,s}{}^{-1}\circ f_t(p) \quad \text{on} \quad R_0'-V\,, \quad \text{and} \\ &= z_{j,k,t,s}{}^{-1}((1-2s_j)\cdot z_{j,k}(p)+s_j\cdot(z_{j,k}(p)/|z_{j,k}(p)|)) \\ &\qquad \text{on} \quad V_{j,k}\ (j=1,\cdots,n;\ k=1,\ 2)\,. \end{aligned}$$

And finally we set $f_{t,s}(C_{j,t,s})=p_j$ for every j. Then note that $f_{t,s}$ maps $R_{t,s}''$ and $V_{j,k,t,s}=J_{t,s}{}^{-1}(f_t(V_{j,k}))$ homeomorphically onto R_0' and $V_{j,k}$, respectively, for every j and k. These mappings $\{f_{t,s}\}$ are defined in a special manner on V. But the variational formula stated below does not depend on such special choice of $f_{t,s}$ on V, but only on s, f_t and U. Also note that we have obtained the following commutative diagram of mappings (, where i means the natural embedding).

$$\begin{tikzcd}
R_0 & R'_0 \arrow[l, "i"'] \arrow[d, "f_t"] & U_{j,k} \arrow[l, "i"'] \arrow[d, "f_t"] \arrow[dr, "z_{j,k}"] & \\
 & R'_t & U_{j,k,t} \arrow[l, "i"'] \arrow[r, "z_{j,k,t}"'] & D_0 \\
R_{t,s} \arrow[uu, "f_{t,s}"] & R''_{t,s} \arrow[l, "i"'] \arrow[u, "J_{t,s}"] & U_{j,k,t,s} \arrow[l, "i"'] \arrow[u, "J_{t,s}"] \arrow[ur, "z_{j,k,t,s}"'] &
\end{tikzcd}$$

Here in case that some $s_j=0$, we regard that $C_{j,t,s}$ collapses to a node $P_{j,t,s}$ of $R_{t,s}$ corresponding to p_j. Hence, in particular, $R''_{t,0}=R'_t$, $f_{t,0}$ is coincident with f_t^{-1} on R'_t, and $J_{t,0}$ is the identical mapping of R'_t.

Remark. From the construction, $R_{t,s}$ converges to R_0 in the finitely augmented Teichmüller space $\hat{T}(R^*)$ as (t, s) tends to $(0, 0)$, where we set $R^*=R_{t^*,s^*}$ with fixed positive t^* and s_j^* $(j=1, \cdots, n)$, or more percisely, $\{f_{t,s}\}$ is an admissible family of marking-preserving deformations of $R_{t,s}$ to R_0 (cf. [5, §1, 1°)]).

Now suppose that a given meromorphic abelian differential $\varphi_{t,s}$ on $R_{t,s}$ varies continuously with respect to (t, s) and remains bounded in norm near pinching loops. Then, if the periods of $\varphi_{t,s}$ along pinching loops vanish constantly, we have certain variational formula for $\varphi_{t,s}$ by essentially the same argument as in the case of quasiconformal deformation (cf. [1], [3]). More precisely, we can show the following

Theorem 1. *For every $t\geq 0$ and s_j in $[0, 1/2)$ $(j=1, \cdots, n)$, let $\varphi_{t,s}$ be a meromorphic abelian differential on $R_{t,s}$ (with $R_{0,0}=R_0$) such that*

1) *$\varphi_{t,s}$ converges to $\varphi_{0,0}$ metrically on $K\cup(U\cap R'_0-V)$, which means that*

$$\lim_{|(t,s)|\to 0} \|\varphi_{t,s}\circ f_{t,s}{}^{-1}-\varphi_{0,0}\|_{K\cup(U\cap R'_0-V)}=0\,,$$

where (and in the sequel) $|(t, s)|=t+\sum_{j=1}^{n} s_j$, $\varphi\circ f$ is the pull-back of φ by f, $\|\cdot\|_E$ is the Dirichlet norm on a Borel set E, and K is a closed subset of R_0-U such that $\mu_t\equiv 0$ outside K for every t,

2) $\displaystyle\int_{C_{j,t,s}} \varphi_{t,s}=0$ *for every j and (t, s), and*

3) *there is a positive constant M such that*

$$\|\varphi_{t,s}\|_{U_{j,t,s}-N(R_{t,s})}<M \qquad \textit{for every } j \textit{ and } (t, s)\,,$$

where $U_{j,t,s}=U_{j,1,t,s}\cup C_{j,t,s}\cup U_{j,2,t,s}$ (with $U_{j,0,0}=U_j$) for every j and (t, s), and in general, $N(R)$ is the set of all nodes of a Riemann surface R with nodes.

Next let ψ be a meromorphic abelian differential on R_0 such that

A) *$\|\psi\|_{K\cup(U\cap R'_0)}$ is finite, and*

B) *the $(1, 1)$-forms $\overline{\varphi_{0,0}}\wedge\psi$ and $\omega_{t,s}=\varphi_{t,s}\circ f_{t,s}{}^{-1}\wedge{}^*\psi$ $(t\geq 0,\ s_j\in[0, 1/2);\ j=1, \cdots, n)$ are absolutely integrable on R'_0.*

Then it holds that

$$\iint_{R_0'} \omega_{t,s} = t\cdot\iint_{R_0'} \varphi_{0,0}\cdot\mu\wedge{}^*\psi + o(|(t,s)|)$$

as $|(t, s)|$ *tends to* 0.

Here and in the sequel, a differential on a surface R with nodes means one on $R-N(R)$.

To prove Theorem 1, we begin with the following

Lemma 1. *Given r with $0<r<1/4$, and let $f(z)$ be a holomorphic function on $W=\{r^2<|z|<1\}$ such that*

i) $\int_{|z|=r} f(z)dz=0$, *i.e.* $f(z)dz$ *is exact, and*

ii) $\iint_W |f(z)|^2\, dxdy \leq A^2$,

where $z=x+iy$ and A is a positive constant independent of r.

Then it holds that

$$\max_{\{|z|=r\}} |f(z)| < 3A\,.$$

Proof. Set $f(z)=\sum_{n=0}^{\infty} a_n z^n+\sum_{n=1}^{\infty} b_n z^{-n}$ on W, then i) implies that $b_1=0$. Hence it holds that

$$\begin{aligned}
&(\max_{\{|z|=r\}} |f(z)|)^2\\
&\leq (\textstyle\sum_{n=0}^{\infty} |a_n|\cdot r^n+\sum_{n=2}^{\infty} |b_n|\cdot r^{-n})^2\\
&\leq S\cdot(\textstyle\sum_{n=0}^{\infty}(2n+2)\cdot 4\cdot(2r)^{2n}+\sum_{n=2}^{\infty}(2n-2)\cdot 4\cdot(2r)^{2n-4})\\
&= S\cdot 16\cdot(1-(2r)^2)^{-2}< 30S\,,
\end{aligned}$$

where we set

$$S=\sum_{n=0}^{\infty} \frac{|a_n|^2}{2n+2}\left(\frac{1}{2}\right)^{2n+2}+\sum_{n=2}^{\infty} \frac{|b_n|^2}{2n-2}(2r^2)^{2-2n}\,.$$

On the other hand, since $2r^2<1/2$, we have

$$\begin{aligned}
(1-16r^4)\cdot S &\leq \sum_{n=0}^{\infty} \frac{|a_n|^2}{2n+2}\left[\left(\frac{1}{2}\right)^{2n+2}-(2r^2)^{2n+2}\right]\\
&\quad+\sum_{n=2}^{\infty} \frac{|b_n|^2}{2n-2}\left[(2r^2)^{2-2n}-\left(\frac{1}{2}\right)^{2-2n}\right]\\
&= (1/2\pi)\cdot\iint_{\{2r^2<|z|<1/2\}} |f(z)|^2 dxdy < A^2/2\pi\,,
\end{aligned}$$

which implies the assertion. q.e.d.

Next fix j and k arbitrarily and set $\varphi_{0,0}\circ z_{j,k}{}^{-1}(z)=a_0(z)dz$ and $\varphi_{t,s}\circ z_{j,k,t,s}{}^{-1}(z)=$

$a_{t,s}(z)dz$ for every t and s. Recall that $a_{t,s}(z)$ and $a_0(z)$ are holomorphic on $D_s=\{s<|z|<1\}$ and D_0, respectively. Denote the mapping $z_{j,k,t,s}\circ f_{t,s}{}^{-1}\circ z_{t,s}{}^{-1}$ by $F_{t,s}(z)$, i.e.

$$F_{t,s}(z)=(1-2s_j)\cdot z+s_j\cdot z/|z| \qquad \text{on} \quad D_0\,.$$

Then we have the following

Lemma 2. i) *$a_{t,s}(F_{t,s}(z))$ are uniformly bounded on $E_1=\{0<|z|<1/2\}$ for every (t, s) with a sufficiently small $|(t, s)|$.*

ii) *$a_{t,s}(F_{t,s}(z))$ converges to $a_0(z)$ locally uniformly on D_0 as $|(t, s)|$ tends to* 0.

Proof. First, when $s_j>0$, $z_{j,k,t,s}{}^{-1}$ can be extended to a conformal mapping of $U_{j,t,s}$ onto $\{s_j^2<|z|<1\}$, and we may regard that $a_{t,s}(z)$ is a holomorphic function on $\{s_j^2<|z|<1\}$. Then by the assumptions 2) and 3) in Theorem 1 and by Lemma 1, we see that $\sup_{\{|z|=s_j\}}|a_{t,s}(z)|<3\cdot M^{1/2}$ for every (t, s) with $0<s_j<1/4$.

On the other hand, by 1) in Theorem 1, we can see that $a_{t,s}(z)$ converges to $a_0(z)$ uniformly on, say $\{|z|=3/4\}$ as $|(t, s)|$ tends to 0. Hence the assertion i) follows from the maximal principle.

Next by the above assertion i) and the assumption 1) in Theorem 1, it holds that $a_{t,s}(z)$ converges to $a_0(z)$ locally uniformly on D_0 as $|(t, s)|$ tends to 0. Since $F_{t,s}$ converges to the identical mapping locally uniformly on D_0, we can show the assertion ii) by using i) and Cauchy's integral formula. q.e.d.

Proof of Theorem 1. For every (t, s), write

$$\varphi_{t,s}\circ f_{t,s}{}^{-1}=a'_{t,s}(w)dw+a''_{t,s}(w)d\bar{w}$$

with a generic local parameter w on R'_0. Then since $\varphi_{t,s}$ is a meromorphic differential on $R_{t,s}$, it holds that $a'_{t,s}(w)\cdot\mu_{t,s}(w)\equiv a''_{t,s}(w)$, where $\mu_{t,s}(w)d\bar{w}/dw$ is the complex dilatation of $f_{t,s}{}^{-1}$. Hence it holds that

$$\begin{aligned}\iint_{R'_0}\omega_{t,s}&=\iint_{R'_0}a''_{t,s}(w)d\bar{w}\wedge{}^*\psi\\&=\iint_{K\cup V}a'_{t,s}(w)\cdot\mu_{t,s}(w)d\bar{w}\wedge{}^*\psi\,.\end{aligned}$$

Since $\mu_{t,s}(w)d\bar{w}/dw\equiv\mu_t$ on K, it holds that

$$\begin{aligned}&\left|\iint_K a'_{t,s}\cdot\mu_{t,s}\,d\bar{w}\wedge{}^*\psi-\iint_K\psi_{0,0}\cdot\mu_t\wedge{}^*\psi\right|\\&\leq\|\mu_t\|_\infty\cdot\|\psi\|_K\cdot\|\varphi_{t,s}\circ f_{t,s}{}^{-1}-\varphi_{0,0}\|_K\end{aligned}$$

Here $\|\mu_t\|_\infty=O(t)$ by the assumption b) on $\{\mu_t\}$, $\|\psi\|_K$ is finite by A), and $\|\varphi_{t,s}\circ f_{t,s}{}^{-1}-\varphi_{0,0}\|_K$ converges to 0 as $|(t, s)|$ tends to 0 by 1). Hence we conclude that

$$\iint_K a'_{t,s}\cdot\mu_{t,s}\,d\bar{w}\wedge{}^*\psi=\iint_K\varphi_{0,0}\cdot\mu_t\wedge{}^*\psi+o(|(t, s)|)\,.$$

And by B) and b) (on $\{\mu_t\}$), we have

$$(*)\qquad \iint_K a'_{t,s}\cdot\mu_{t,s}\,d\bar{w}\wedge{}^*\psi = t\cdot\iint_K \varphi_{0,0}\cdot\mu\wedge{}^*\psi+o(|(t,s)|)\,.$$

Next by a simple computation, we can see that

$$(F_{t,s})_{\bar{z}}(z)\left(\equiv\frac{\partial}{\partial\bar{z}}F_{t,s}(z)\right) = -\frac{s_j}{2}\left(\frac{z}{|z|}\right)^2\cdot\frac{1}{|z|}\,,$$

hence $|(F_{t,s})_{\bar{z}}| = s_j/(2|z|)$ on $z_{j,k}(V_{j,k})=D_0$. So by Lemma 2-i) we have

$$\begin{aligned}|a'_{t,s}(z)\cdot\mu_{t,s}(z)| &= |a''_{t,s}(z)|\\ &= |a_{t,s}(F_{t,s}(z))\cdot(F_{t,s})_{\bar{z}}(z)| = s_j\cdot\tilde{M}\cdot|z|^{-1}\end{aligned}$$

on every $z_{j,k}(V_{j,k})$ with a suitable constant $\tilde{M}$ for every (t, s) with a sufficiently small $|(t, s)|$. Hence by Lemma 2-ii) and Lebesgue's convergence theorem, we have

$$\lim_{s_j\to 0}(1/s_j)\cdot\iint_{V_j} a'_{t,s}\cdot\mu_{t,s}\,d\bar{z}\wedge{}^*\psi = \iint_{V_j} a_0(z)\,b(z)\,\frac{z^2}{2|z|^3}dz\wedge d\bar{z}\,,$$

where ${}^*\psi=b(z)dz$, for $|b(z)|$ is bounded on every $V_{j,k}$, as is seen by A). Since both $a_0(z)$ and $b(z)$ have removable singularities at the origin, the integral on the right hand side is equal to 0 by Cauchy's theorem, i.e. we conclude that

$$(**)\qquad \iint_V a'_{t,s}\cdot\mu_{t,s}\,d\bar{w}\wedge{}^*\psi = o(|(t,s)|)\,.$$

Thus the assertion follows from (*) and (**). q.e.d.

Remark. From the above proof, we can see that B) in Theorem 1 may be replaced by the following weaker condition

B′) $\overline{\varphi_{0,0}}\wedge\psi$ and every $\omega_{t,s}$ are absolutely integrable on R_0-U.

§2. Variational formulas for basic differentials.

A simple closed curve d on an arbitrary Riemann surface S is called *essentially trivial* if d is dividing and a component of $S-d$ is a *parabolic part* (i.e. a subregion of type SO_{HB}) of S. Two essentially non-trivial curves d_1 and d_2 are called *equivalent* if either $d_1=\pm d_2$, or they are disjoint and bound a parabolic part (, i.e. there is a parabolic part G such that the interior of $\bar{G}$ is G and the relative boundary of G is $\bigcup_{j=1}^{2} d_j$ as a points set in S). A set $\{d_j\}_{j=1}^{K}(K\geq 0)$ of mutually disjoint simple closed curves is called *free* if no subset of $\{d_j\}_{j=1}^{K}$ bounds a parabolic part, and is called *essentially free* if there is a free subset $\{d_{j'}\}_{j'=1}^{H}$ of $\{d_j\}_{j=1}^{K}$ such that every d_j is either essentially trivial or equivalent to one of $\{d_{j'}\}_{j'=1}^{H}$.

Next we recall definitions of basic differentials on a Riemann surface R with (a finite number of) nodes.

i) *Period reproducers* (cf. [5, §1, 2°)]): For every 1-cycle d on $R'=R-N(R)$, we denote by $\sigma(d, R)$ the period reproducer for d on R. And we set $\theta(d, R)=\sigma(d, R)+i\cdot{}^*\sigma(d, R)$.

ii) *Green's functions* (cf. [7, §1]). When a point or a puncture q is given on a component S of R' which admits Green's functions (i.e. $S\notin O_G$), then Green's function $g(p; q)$ on R with the pole q is, by definition, equal to usual Green's function on $S\cup\{q\}$ with the pole q and vanishes identically on $R'-S$. When two points or punctures q_1 and q_2 are given on a component S of R' belonging to O_G, then (indefinite) Green's function $g(p; q_1, q_2)$ on R with the ordered pair of poles q_1 and q_2 is, by definition, equal to a harmonic function $g(p; q_1, q_2)$ on $S-\{q_1, q_2\}$ defined in [5, §1, 1°)] and vanishes identically on $R'-S$. Recall that such a function $g(p; q_1, q_2)$ on S is determined only up to additive constants.

In both cases, we set

$$\phi(q, R)=dg(\cdot\,; q)+i\cdot{}^*dg(\cdot\,; q)\,, \quad \text{and}$$

$$\phi(q_1, q_2; R)=dg(\cdot\,; q_1, q_2)+i\cdot{}^*dg(\cdot\,; q_1, q_2)\,, \quad \text{respectively.}$$

Now returning to the situation in §1, we say that R_0 is *essentially free* if so is the set $\{C_{j,t^*,s^*}\}_{j=1}^{n}$ on R_{t^*,s^*} for some (, hence every) positive t^* and s_j^* $(j=1, \cdots, n)$. And in this section, we always assume that R_0 *is essentially free.* (A reason for this restriction will be found in §3, Example.) Also, for every j, we denote by C_j the simple closed curve $-\partial V_{j,1}$ on R_0, and say that C_j is *essentially trivial* if so is C_{j,t^*,s^*} (which is freely homotopic to $f_{t^*,s^*}{}^{-1}(C_j)$). Similarly, we say that two curves C_{j_1} and C_{j_2} are *equivalent* if so are C_{j_1,t^*,s^*} and C_{j_2,t^*,s^*}, and that a subset of $\{C_j\}_{j=1}^{n}$ is *free* if so is the corresponding subset of $\{C_{j,t^*,s^*}\}_{j=1}^{n}$. Recall that the assumption on R_0 implies the existence of a maximal free subset of $\{C_j\}_{j=1}^{n}$ (i.e. a free subset such that every C_j is either essentially trivial or equivalent to one of elements).

In the sequel of this paper, we assume that *every C_j with $1\leq j\leq m$ is essentially non-trivial, while every C_j with $m+1\leq j\leq n$ is essentially trivial, and that $\{C_j\}_{j=1}^{H}$ $(H\leq m)$ is a maximal free subset of $\{C_j\}_{j=1}^{n}$.*

To state variational formulas, we should define differentials associated to some of pinching loops. For every j with $1\leq j\leq m$, a differential $\phi(C_j, R_0)$ is defined as follows. First let $\{S_k\}_{k=0}^{N(=N(j))}$ be a set of components of R_0' uniquely determined by the conditions; (i) S_{k-1} and S_k are connected by a single node, say p_k, of R_0 for every k with $1\leq k\leq N$, (ii) $S_k\in O_G$ for every k with $1\leq k\leq N-1$, and (iii) $\{p_k\}_{k=1}^{N}$ corresponds to the set of all C_h equivalent to C_j. Then since C_j is essentially non-trivial, we can see either that $S_0=S_N$, or that $S_0\neq S_N$ and none of S_0 and S_N belongs to O_G. We denote by $q_{k,1}$ and $q_{k,2}$ the punctures of S_k corresponding to p_k and p_{k+1}, respectively, for every k except for $q_{0,1}$ and $q_{N,2}$ which are undefined. Here we may assume that the puncture of R_0' bounded by $-C_j$ is one of $\{q_{k,1}\}_{k=1}^{N}$. Now when none of S_0 and S_N belongs to O_G, then we set

$$\phi(C_j, R_0)=\frac{1}{2\pi i}\cdot(\phi(q_{N,1}, R_0)-\phi(q_{0,2}, R_0)) \quad \text{on} \quad S_0\cup S_N\,.$$

If not, then $S_0=S_N$ and we set

$$\phi(C_j, R_0)=\frac{1}{2\pi i}\cdot\phi(q_{N,1}, q_{0,2}; R_0) \quad \text{on} \quad S_0=S_N .$$

On every other S_k ($k=1, \cdots, N-1$), we set

$$\phi(C_j, R_0)=\frac{1}{2\pi i}\cdot\phi(q_{k,1}, q_{k,2}; R_0) \quad \text{on} \quad S_k .$$

Finally, setting $\phi(C_j, R_0)\equiv 0$ on $R_0'-\bigcup_{k=0}^{N} S_k$, we have a holomorphic differential $\phi(C_j, R_0)$ on R_0.

We call this $\phi(C_j, R_0)$ *the associated differential for* C_j *on* R_0. Note that $\int_{C_j}\phi(C_j, R_0)=1$ for every j and that, if C_{j_1} and C_{j_2} are equivalent, then $\phi(C_{j_1}, R_0)\equiv\phi(C_{j_2}, R_0)$ or $\equiv-\phi(C_{j_2}, R_0)$.

Now we will state several variational formulas for basic differentials, where and in the sequel, we use the same notation for a 1-cycle on R_0' and the corresponding one on any $R_{t,s}$. Also, denoting by $\{C_{j(k)}\}_{k=1}^{N(=N(j))}$ the set of all C_h equivalent to C_j, we set

$$s(j)=\prod_{k=1}^{N} s_{j(k)}$$

and regard that $\log(1/s(j))=+\infty$ and $1/\log(1/s(j))=0$ when $s(j)=0$. (Recall that $\{C_j\}_{j=1}^{m}=\bigcup_{j=1}^{H}\{C_{j(k)}\}_{k=1}^{N(j)}$.)

Theorem 2. *Let d and d' be 1-cycles on R_0', then it holds that*

$$\begin{aligned}&\int_{d'}\sigma(d, R_{t,s})-\int_{d'}\sigma(d, R_0)\\&=t\cdot\mathrm{Re}\iint_{R_0'}\theta(d, R_0)\cdot\mu\wedge{}^*\theta(d', R_0)\\&\quad+\sum_{j=1}^{H}\frac{\pi}{\log(1/s(j))}\cdot\int_{d}\phi(C_j, R_0)\cdot\int_{d'}\phi(C_j, R_0)+o(\|(t, s)\|)\end{aligned}$$

as $|(t, s)|$ *tends to* 0, *where and in the sequel, we set*

$$\|(t, s)\|=|(t, s)|+\sum_{j=1}^{H}\frac{1}{\log(1/s(j))}$$

Remark. Write $\theta=a(w)dw$, $\mu=\mu(w)d\bar{w}/dw$ and $\theta'=b(w)dw$ with a generic local parameter $w=u+iv$ on R_0', and we have

$$\mathrm{Re}\iint_{R_0'}\theta\cdot\mu\wedge{}^*\theta'=2\iint_{R_0'}\mathrm{Re}\,[a(w)\cdot\mu(w)\cdot b(w)]\,dudv ,$$

which is sometimes written as $2\cdot\mathrm{Re}\iint_{R_0'}\theta'\theta\mu$.

Recalling that $\int_d \sigma(d, R) = \|\sigma(d, R)\|_{R'}^2$ is equal to the extremal length $\lambda(d, R)$ of the homology class of d on R' by Accola's theorem, we have by Theorem 2 the following

Corollary 1. *For every* 1*-cycle* d *on* R_0', *it holds that*

$$\begin{aligned}
&\lambda(d, R_{t,s}) - \lambda(d, R_0) \\
&\quad = t\cdot \mathrm{Re}\iint_{R_0'} \theta(d, R_0)\cdot \mu \wedge {}^*\theta(d, R_0) \\
&\qquad + \sum_{j=1}^{H} \frac{\pi}{\log(1/s(j))}\Big[\int_d \phi(C_j, R_0)\Big]^2 + o(\|(t, s)\|)
\end{aligned}$$

as $|(t, s)|$ *tends to* 0.

Next fix a point q on a component S of R_0', and assume that $S \notin O_G$ and that $q \in S - U$. For every (t, s), we set $g_{t,s}(p) = g(p, q_{t,s})$, where $q_{t,s} = f_{t,s}{}^{-1}(q)$ (, hence $q_{0,0} = q$). Then it is seen that $g_{t,s} \not\equiv 0$ for every t and s. Also for every j with $1 \le j \le H$, we set

$$G_j(p) = g(p, q_{N,1}) - g(p, q_{0,2}) \quad \text{on} \quad S,$$

where $q_{N,1}$ and $q_{0,2}$ are the same as in the definition of $\phi(C_j, R_0)$ (, hence G_j may vanish identically on S). Then we have the following

Theorem 3. *Let* d *be a* 1*-cycle on* $R_0' - \{q\}$, *and suppose that there is a neighbourhood* U_q *of* q *on* R_0' *such that* $\mu_t \equiv 0$ *on* U_q *for every* t. *Then it holds that*

$$\begin{aligned}
&\int_d {}^*dg_{t,s} - \int_d {}^*dg_{0,0} \\
&\quad = t\cdot \mathrm{Re}\iint_{R_0'} -i\cdot\phi(q, R_0)\cdot \mu \wedge {}^*\theta(d, R_0) \\
&\qquad - \sum_{j=1}^{H} \frac{\pi}{\log(1/s(j))}\cdot G_j(q)\cdot \int_d \phi(C_j, R_0) + o(\|(t, s)\|)
\end{aligned}$$

as $|(t, s)|$ *tends to* 0.

Finally, fix two distinct points q and q' on a component S of R_0'. And we also assume that $S \notin O_G$ and that $q, q' \in S - U$. Then we have the following

Theorem 4. *Suppose that there are neighbourhoods* U_q *and* $U_{q'}$, *respectively, of* q *and* q' *in* S *such that* $\mu_t \equiv 0$ *on* $U_q \cup U_{q'}$ *for every* t. *Then it holds that*

$$\begin{aligned}
&g_{t,s}(f_{t,s}{}^{-1}(q')) - g_{0,0}(q') \\
&\quad = -(t/2\pi)\cdot \mathrm{Re}\iint_{R_0'} i\cdot\phi(q, R_0)\cdot \mu \wedge \cdot\phi(q', R_0) \\
&\qquad - \sum_{j=1}^{H} \frac{1}{2\cdot\log(1/s(j))}\, G_j(q)\cdot G_j(q') + o(\|(t, s)\|)
\end{aligned}$$

as $|(t, s)|$ *tends to* 0.

The proofs of Theorems will be given in §5 and 6.

Remark. When $q=q'$ in Theorem 4, the right hand side of the formula gives that for so-called Robin's constants.

In all Theorems, if we set $s=0$, then all formulas reduce to well-known ones under quasiconformal deformation (cf. [1] and [3]).

The case that $t=0$ and $n=1$ can be considered as a natural generalization of Schiffer-Spencer's variation, and choosing a suitable U, we can derive a sharper formulas (see [7] and [8, §2]).

§3. Properties of associated differentials.

We can define the associated differential $\phi(C_j, R_{t,s})$ on every $R_{t,s}$ for every essentially non-trivial C_j (i.e. $j=1, \cdots, m$) as follows; when $\sigma(C_j, R_{t,s}) \not\equiv 0$, then we set

$$\phi(C_j, R_{t,s}) = \|\sigma(C_j, R_{t,s})\|_{R'_{t,s}}^{-2} \cdot \theta(C_j, R_{t,s}) .$$

When $\sigma(C_j, R_{t,s}) \equiv 0$, then since C_j is essentially non-trivial, C_j should be equivalent to some other loop, say $C_{j'}$, on $R_{t,s}$ with $s_{j'}=0$. And we set $\phi(C_j, R_{t,s}) = \phi(C_{j'}, R_{t,s})$ or $= -\phi(C_{j'}, R_{t,s})$ so that $\int_{C_j} \phi(C_j, R_{t,s}) = 1$, where $\phi(C_{j'}, R_{t,s})$ is defined in the same manner as in the definition of $\phi(C_j, R_0)$ in §2. First we show the following

Theorem 5. *There is a constant C depending only on U and R_0 such that it holds that*

$$\frac{2}{\pi} \cdot \log(1/s(j)) + C \geq \|\phi(C_j, R_{t,s})\|_{R'_{t,s}}^2 \geq \frac{2}{\pi} \cdot \log(1/s(j))$$

for every j and (t, s) with a sufficiently small $|(t, s)|$.

Hence in particular, it holds that

$$\|\sigma(C_j, R_{t,s})\|_{R'_{t,s}}^2 = \frac{\pi}{\log(1/s(j))} + o(\|(t, s)\|)$$

as $|(t, s)|$ *tends to* 0.

Proof. Fix j and (t, s), and assume that $\sigma(C_j, R_{t,s}) \not\equiv 0$, for otherwise the assertion clearly holds.

Let $\{S_k\}_{k=0}^N$ and $\{q_{k,j}\}_{k=0,j=1}^{N}{}^{2}$ be as in the definition of $\phi(C_j, R_0)$. Let $S_{k,t,s}$ be the component of $R_{t,s}(C_j) = R_{t,s} - \bigcup_{h=1}^{N} V_{j(h),t,s}$ corresponding to S_k, where $V_{j(h),t,s} = V_{j(h),1,t,s} \cup C_{j(h),t,s} \cup V_{j(h),2,t,s}$, and $\sigma_{k,t,s}$ be the reproducing differential on $S_{k,t,s}$ for a loop d_k freely homotopic to the border of $S_{k,t,s}$ which corresponds to $q_{k,2-\min(1,k)}$, for every k. We define a holomorphic differential $\alpha_{t,s}$ on $R_{t,s}(C_j)$ by setting $\alpha_{t,s} = (\sigma_{k,t,s} + i^*\sigma_{k,t,s})/\|\sigma_{k,t,s}\|_{S_{k,t,s}}^2$ on every $S_{k,t,s}$ except for the case (E) that $S_{0,t,s}$ coincides with $S_{N,t,s}$ and admits Green's functions; in that case, we set $|\alpha_{t,s}|=|\sigma_{0,t,s}+i^*\sigma_{0,t,s}|/||\sigma_{0,t,s}||^2_{S_{0,t,s}}+|\sigma_{N,t,s}+i^*\sigma_{N,t,s}|/||\sigma_{N,t,s}||^2_{S_{0,t,s}}$ on $S_{0,t,s}=S_{N,t,s}$ so that $\int_{d_0}|\alpha_{t,s}|\geq 1$, and $\int_{d_N}|\alpha_{t,s}|\geq 1$.

Next letting

$$\begin{aligned}\beta(z) &= 1/(2\pi\cdot z) \quad \text{on} \quad \{0<|z|<1/2\}\ , \quad \text{and} \\ &= ((1/2\pi)+(1/\log 2))/z \quad \text{on} \quad \{1/2\leq|z|<1\}\ ,\end{aligned}$$

we denote by $\beta_{t,s}$ the pull-back of the differential $\beta(z)dz$ onto all components of $U_{t,s}(C_j)=\bigcup_{h=1}^{N} U_{j(h),t,s}$. Finally set

$$\rho_{t,s}=\rho_{t,s}(w)|dw|=|\alpha_{t,s}|+|\beta_{t,s}| \quad \text{on} \quad R_{t,s}\,,$$

where we regard that $\alpha_{t,s}$ and $\beta_{t,s}$ are equal to 0 on $R_{t,s}-R_{t,s}(C_j)$ and $R_{t,s}-U_{t,s}(C_j)$, respectively.

Then $\rho_{t,s}(w)|dw|$ is an admissible density for the homology class of C_j on $R'_{t,s}$. In fact, let a 1-cycle c' on $R'_{t,s}$ homologous to C_j be given. If c' contains an arc I connecting a point of $R_{t,s}-R_{t,s}(C_j)$ and one of $R_{t,s}-U_{t,s}(C_j)$, then it holds that $\int_{c'}\rho_{t,s}\geq\int_{1/2}^{1}(1/\log 2)\dfrac{dr}{r}\geq 1$. And if not, c' is a union of curves contained in $R_{t,s}(C_j)$ and ones in $U_{t,s}(C_j)$. If the latter contains a non-trivial curve c'_1 on $U_{t,s}(C_j)$, then it holds that $\int_{c'}\rho_{t,s}\geq\int_{c'_1}|\beta_{t,s}|\geq 1$. If not, the 1-cycle $c'_2=c'\cap R_{t,s}(C_j)$ is homologous to C_j on $R'_{t,s}$. And then we can find a component, say $\hat{S}$, of $R_{t,s}(C_j)$ such that $\int_{c'}\rho_{t,s}\geq\int_{c'\cap\hat{S}}|\alpha_{t,s}|\geq 1$. Thus we conclude that $\rho_{t,s}$ is admissible.

Hence by Accola's theorem, we have

$$||\sigma(C_j, R_{t,s})||^2_{R'_{t,s}}\geq 1/\iint_{R'_{t,s}}\rho_{t,s}(w)^2\,dudv\,,$$

where $w=u+iv$, and a simple computation gives that

$$\begin{aligned}2\iint_{R'_{t,s}}\rho_{t,s}(w)^2\,dudv &\leq ||\alpha_{t,s}||^2_{R_{t,s}(C_j)'}+||\beta_{t,s}||^2_{R_{t,s}-R_{t,s}(C_j)} \\ &\quad +2(||\alpha_{t,s}||^2_{R_{t,s}(C_j)\cap U_{t,s}(C_j)}+||\beta_{t,s}||^2_{R_{t,s}(C_j)\cap U_{t,s}(C_j)}) \\ &\leq\frac{2}{\pi}\cdot\log(1/(2^N\cdot s(j)))+16\pi N\cdot\left(\frac{1}{2\pi}+\frac{1}{\log 2}\right)^2\cdot\log 2+3||\alpha_{t,s}||^2_{R_{t,s}(C_j)'}\,.\end{aligned}$$

Here note that $||\alpha_{t,s}||^2_{S_{k,t,s}}$ is equal to $2/||\sigma_{k,t,s}||^2_{S_{k,t,s}}$ except for the above case (E); in that case, $||\alpha_{t,s}||^2_{S_{0,t,s}}\leq 4(||\sigma_{0,t,s}||^2_{S_{0,t,s}}+||\sigma_{N,t,s}||^2_{S_{0,t,s}})/||\sigma_{0,t,s}||^2_{S_{0,t,s}}\cdot||\sigma_{N,t,s}||^2_{S_{0,t,s}}$. Since $S_{k,t,s}$ converges to the corresponding component $S_{k,0,0}$ of $R_0(C_j)=R'_0-\bigcup_{h=1}^{N}(V_{j(h),1}\cup V_{j(h),2})$ for every k in the sense of the conformal topology, $\sigma_{k,t,s}$ converges to $\sigma_{k,0,0}$ (which does not vanish identically on $S_{k,0,0}$) strongly metrically ([5, proposition 4]).

And since metrical convergence implies convergence of periods (cf. Remark in §5), we can see that $||\alpha_{t,s}||^2_{S_{k,t,s}}$ are bounded near (0, 0) for every k. (The case (E) can be treated by the same argument.) Hence we can conclude that $||\alpha_{t,s}||^2_{R_{t,s}(C_j)'}$ are bounded near (0, 0).

Thus we can find a constant C depending only on R_0 and U such that, for every (t, s) with a sufficiently small (t, s), it holds that

$$||\phi(C_j, R_{t,s})||^2_{R'_{t,s}} \leq \frac{2}{\pi} \cdot \log(1/s(j)) + C .$$

Finally, considering $|\phi(C_j, R_{t,s})|$ on $U_{t,s}(C_j)$, we can see that $(1/2)\cdot||\phi(C_j, R_{t,s})||^2_{R_{t,s}}$ is not less than the sum $(1/2\pi)\cdot 2\cdot\log(1/s(j))$ of the moduli of $U_{j(h),t,s}$ ($h=1, \cdots, N$). q.e.d.

Next we can show the following

Theorem 6. *For every j, $\phi(C_j, R_{t,s})$ converges to $\phi(C_j, R_0)$ strongly metrically (with respect to $\{f_{t,s}\}$) as $|(t, s)|$ tends to 0, i.e. for every neighborhood W of $N(R_0)$, it holds that*

$$\lim_{|(t,s)|\to 0} ||\phi(C_j, R_{t,s})\circ f_{t,s}^{-1} - \phi(C_j, R_0)||_{R_0 - W} = 0 .$$

Corollary 2. *For every j and 1-cycle d on R'_0, it holds that*

$$\lim_{|(t,s)|\to 0} \int_d \phi(C_j, R_{t,s}) = \int_d \phi(C_j, R_0) .$$

Example. Without freeness of R_0, the associated differentials do not necessarily converge. Here we give a simple example.

Let $R_{a,b,c}$ be the triply connected region $\{z\in \boldsymbol{C}: a<|z|<1/c, |z-3|>b\}$ for every sufficiently small non-negative a, b and c. Set $C_1=\{|z|=1\}$, $C_2=\{|z-3|=1\}$ and $C_3=\{|z|=5\}$. Then we can regard that $R_{a,b,c}$ converges to some R_0 with three nodes in the sense of the conformal topology as $a+b+c$ tends to 0. Note that one of component of $R_0-N(R_0)$ is conformally equivalent to $S_0=\boldsymbol{C}-\{0, 3\}$.

Now consider $\phi(C_3, R_{a,b,c})=||\sigma(C_3, R_{a,b,c})||^{-2}_{R_{a,b,c}}\cdot\theta(C_3, R_{a,b,c})$. When a tends to 0 first and then b and c tend to 0, $\phi(C_3, R_{a,b,c})$ converges to $\phi(C_2, R_0)$ which corresponds to $(1/2\pi i)\cdot\dfrac{dz}{z-3}$ on S_0. On the other hand, when b tends to 0 first and then a and c tend to 0, $\phi(C_3, R_{a,b,c})$ converges to $\phi(C_1, R_0)$ which corresponds to $(1/2\pi i)\dfrac{dz}{z}$ on S_0.

Proof of Theorem 6 is essentially the same as that of [7, proposition 3], but for the sake of convenience we give an outline of it.

Fix j in the sequel of this section. We may assume that $\{C_{j(k)}\}_{k=1}^{K_s(\leq N)}$ be the set of all C_h equivalent to C_j such that $s_h>0$. Then for every (t, s) and k ($\leq K_s$), we can consider the characteristic ring domain $W_{j(k),t,s}$ of $\phi(C_j, R_{t,s})$ (which is equal to $\phi(C_{j(k)}, R_{t,s})$ or $-\phi(C_{j(k)}, R_{t,s})$) for $C_{j(k)}$ on $R_{t,s}$ (cf. [4, §2]). Let $C(j(k), t, s)$ be the center trajectory of $W_{j(k),t,s}$ for every k. Then we can construct another Riemann surface $R^{\sharp}_{t,s}$ with nodes from $R_{t,s}$ as follows; first cut $R_{t,s}$ along $\bigcup_{k=1}^{K_s} C(j(k), t, s)$ and patch a once punctured disk along each border so that $\phi(C_j, R_{t,s})$ restricted on $R_{t,s}-\bigcup_{k=1}^{K_s} C(j(k), t, s)$ can be extended to a holomorphic differential, say ϕ, on the resulting surface(s). Next fill two punctures corresponding to the same $C(j(k), t, s)$ by a single point, we obtain a Riemann surface $R^{\sharp}_{t,s}$ with nodes (which is homeomorphic to $R_{t,s'}$ with s' obtained from s by replacing every $s_{j(k)}$ ($k=1, \cdots, K_s$) by 0). Then we can see that ϕ should be coincident with the associated differential $\phi(C_j, R^{\sharp}_{t,s})$ for C_j on $R^{\sharp}_{t,s}$ which is defined again in the same manner as $\phi(C_j, R_0)$.

Now fix a neighbourhood W of $N(R_0)$ arbitrarily. Here for every (t, s) with a sufficiently small $|(t, s)|$, it holds that $W_{j(k),t,s}$ contains $C_{j(k),t,s}$, that is, $f_{t,s}(W_{j(k),t,s})$ contains $p_{j(k)}$ ($k=1, \cdots, K_s$), where $p_{j(k)}$ is the node of R_0 corresponding to $C_{j(k)}$. (The assertion can be shown by the same argument as in the proof of [7, Theorem 3], i.e. by applying [6, Proposition 2] to a height function u such that $du \equiv \mathrm{Im}\, \phi(C_j, R_{t,s})$ on $z_{j(k),h,t,s}^{-1}(\{\varepsilon<|z|<1/2\})$ ($h=1, 2$) with a sufficiently small positive ε.) Since we can regard $W_{j(k),t,s}$ as a neighbourhood of $C(j(k), t, s)$ also in $R^{\sharp}_{t,s}$ ($k=1, \cdots, K_s$), we may regard that $f_{t,s}^{-1}(R_0')$ is a subsurface of $R^{\sharp}_{t,s}$ such that each component of $R^{\sharp}_{t,s}-\overline{f_{t,s}^{-1}(R_0')}$ is conformally equivalent to a once punctured disk. Hence similarly as in the proof of [7, Proposition 2], we can construct an admissible family $\{(h_{t,s}; R^{\sharp}_{t,s}, R_0)\}_{|(t,s)|<\eta}$ of deformations of $R^{\sharp}_{t,s}$ (with the natural markings) to R_0, where η is sufficiently small, such that, for every (t, s), it holds that

1) $f_{t,s} \equiv h_{t,s}$ on $f_{t,s}^{-1}(R_0-W)$, and
2) $h_{t,s}^{-1}$ is conformal on $V-N(R_0)$ with a suitably fixed neighbourhood V of $\{p_{j(k)}\}_{k=1}^{N}$.

Hence we conclude by [7, Theorem 1] (which remains valid for any admissible family with a vector valued parameter) and the following Proposition that

$$\lim_{|(t,s)|\to 0} \|\phi(C_j, R^{\sharp}_{t,s})\circ h_{t,s}^{-1}-\phi(C_j, R_0)\|_{R_0-W} = 0\,,$$

which implies that

$$\lim_{|(t,s)|\to 0} \|\phi(C_j, R_{t,s})\circ f_{t,s}^{-1}-\phi(C_j, R_0)\|_{R_0-W} = 0\,.$$

And since W is arbitrary, we have the assertion of Theorem 6.

Proposition. *Let $\{(f_u; R_u, R_0)\}$ be an admissible family of marking-preserving deformations with a vector valued parameter u, and two punctures q_1 and q_2 be given on a component S of $R_0-N(R_0)$. Suppose that S and the component of $R_u-N(R_u)$ containing $f_u^{-1}(S)$ belong to O_G and that there is a neighbourhood V of $\{q_1, q_2\}$ on $R_0-N(R_0)$ such that f_u^{-1} is conformal on V for every u.*

Then $\phi(f_u^{-1}(q_1), f_u^{-1}(q_2); R_u)$ converges to $\phi(q_1, q_2; R_0)$ strongly metrically with respect to $\{f_u\}$ as the norm of u tends to 0.

Proof of this Proposition is given by the same argument as in that of [5, Theorem 3], if we know that

$$\limsup_{|u|\to 0} ||\phi(f_u^{-1}(q_1), f_u^{-1}(q_2); R_u)||_{R_u - f_u^{-1}(V)} < +\infty ,$$

which, in turn, can be seen as in the proof of [7, Lemma 2] by using the following Lemma 3 instead of [7, Lemma 1].

Lemma 3. *Let R and $\tilde{R}$ be Riemann surfaces belonging to O_G. Fix two points q_1 and q_2 on R and a real number E so large that each component of the open set $D_E = \{p \in R: |g(p)| > E\}$ is simply connected and relatively compact in $R \cup \{q_1, q_2\}$, where $g(p) = g(p; q_1, q_2)$ is (an indefinite) Green's function on R* (cf. §2). *Then there is an absolute constant A_0 such that for every K-quasiconformal mapping f from D_E into $\tilde{R}$, it holds that*

$$\sup_{p\in\tilde{R}-f(D_E)} \tilde{g} - \inf_{p\in\tilde{R}-f(D_E)} \tilde{g} \leq 2\pi/||\sigma(d, \tilde{s})||^2_{\tilde{S}}$$

where $\tilde{g}(p) = g(p; f(q_1), f(q_2))$ is (an indefinite) Green's function on $\tilde{R}$, $\tilde{S} = \tilde{R} - f(\{p \in R;\ |g(p)| \geq E + KA_0\})$ and d is the dividing cycle on $\tilde{S}$ corresponding to the relative boundary component $f(\{g = E + KA_0\})$ of $\tilde{S}$ on $\tilde{R}$.

Proof. First, by [6, Proposition 2], we can show (cf. the proof of [7, Lemma 1]) that there is an absolute constant A_0 such that

$$\sup_{p\in\tilde{R}-f(D_E)} \tilde{g} \leq \inf_{p\in f(\{g \geq E+KA_0\})} \tilde{g} = a_1 , \quad \text{and}$$

$$\inf_{p\in R-f(D_E)} \tilde{g} \geq \sup_{p\in f(\{g\leq -(E+KA_0)\})} \tilde{g} = a_2 .$$

for every K-quasiconformal mapping f from D_E into $\tilde{R}$.

On the other hand, since $\tilde{R} \in O_G$, we can see that the modulus of $\tilde{S}$ is equal to $1/||\sigma(d, \tilde{S})||^2_{\tilde{S}}$. And since $\tilde{D} = \{p \in \tilde{R};\ a_2 < \tilde{g} < a_1\}$ is contained in $\tilde{S}$, this modulus is not less than the modulus $(a_1 - a_2)/2\pi$ of $\tilde{D}$, which implies the assertion. q.e.d.

Remark. We can see by [6, Theorem 1] that the modulus of $W_{j(k),t,s}$ tends to $+\infty$ as $|(t, s)|$ tends to 0 for every k. Also existence of a family $\{(h_{t,s};\ R^{\sharp}_{t,s}, R_0)\}$ implies convergence of $R^{\sharp}_{t,s}$ to R_0 in the sense of the conformal topology.

Moreover, we can show that, in general, two conditions such as above implies convergence of $R_{t,s}$ to R_0 in the sense of the conformal topology (cf. [7, Theorem 3]). We will give such a kind of characterization of the conformal topology in Appendix.

§4. Proof of Theorem 2.

Let $X_{t,s} = (x_{i,j;t,s})$ be the $H \times H$ matrix with the (i, j)-th component $x_{i,j;t,s} = \int_{C_i} \phi(C_j, R_{t,s})$ and $Y_{t,s} = (y_{j;t,s})$ be the H-dimensional vector with the j-th component $y_{j;t,s} = \int_{C_j} \sigma(d, R_{t,s})$ for every (t, s). By Corollary 2, $x_{i,j;t,s}$ converges to $x_{i,j;0,0} = \int_{C_i} \phi(C_j, R_0)$ for every i and j as $|(t, s)|$ tends to 0. And since every C_j corresponds to a node of R_0, $x_{i,j;0,0}=\delta_{ij}$ (Kronecker's delta). Hence $x_{i,j;t,s}=\delta_{ij}+o(1)$ as $|(t, s)|$ tends to 0. In particular, $X_{t,s}$ is non-singular for every (t, s) with a sufficiently small $|(t, s)|$, hence there is a unique solution $A_{t,s}=(a_{j;t,s})$ of the equation $Y_{t,s}=X_{t,s}\cdot A_{t,s}$, namely,

$$\int_{C_j}\sigma(d, R_{t,s})=\sum_{k=1}^{H}a_{k;t,s}\cdot\int_{C_j}\phi(C_k, R_{t,s}) \quad (j=1, \cdots, H),$$

for every such (t, s). In the sequel, we consider only such (t, s). Also we note the following

Lemma 4. *For every j, it holds that*

1) $$a_{j;t,s}=O(\|\sigma(C_j, R_{t,s})\|_{R'_{t,s}}), \quad \text{and}$$

2) $$a_{j;t,s}=\|\sigma(C_j, R_{t,s})\|^2_{R'_{t,s}}\cdot\int_d\phi(C_j, R_0)+o(\|(t, s)\|)$$
$$=O(\|(t, s)\|)$$

as $|(t, s)|$ *tends to* 0.

Proof. First note that $a_{j;t,s}=y_{j;t,s}+o(\sum_{k=1}^{H}|y_{j;t,s}|)$ as $|(t, s)|$ tends to 0. And since

*) $$y_{j;t,s}=\int_d\sigma(C_j, R_{t,s})=\|\sigma(C_j, R_{t,s})\|^2_{R'_{t,s}}\cdot\int_d\phi(C_j, R_{t,s}),$$

which is $O(\|(t, s)\|)$ as $|(t, s)|$ tends to 0 by Corollary 2 and Theorem 5, we conclude the assertion 2) again by Corollary 2.

Next to show 1), fix an integer j_0 in $[1, H]$. When $\sigma(C_{j_0}, R_{t,s})\equiv 0$, then $x_{j_0,j;t,s}=\delta_{j_0 j}$ and $y_{j_0;t,s}=0$, hence by Cramer's rule, we see that $a_{j_0;t,s}=0$. When $\sigma(C_{j_0}, R_{t,s})\not\equiv 0$, then a rough estimation gives that $|x_{i,j_0;t,s}|\leq\|\sigma(C_i, R_{t,s})\|_{R'_{t,s}}/\|\sigma(C_{j_0}, R_{t,s})\|_{R'_{t,s}}$ for every i. And the above *) implies that $y_{i;t,s}=O(\|\sigma(C_i, R_{t,s})\|_{R'_{t,s}})$ for every i. Hence again by Cramer's rule, we can show that $a_{j_0;t,s}=O(\|\sigma(C_{j_0}, R_{t,s})\|_{R'_{t,s}})$. q.e.d.

Now for every (t, s) as above, we set

$$\varphi_{t,s}=\theta(d, R_{t,s})-\sum_{j=1}^{H}a_{j;t,s}\cdot\phi(C_j, R_{t,s}).$$

Then from the definition, we can see that $\int_{C_{j,t,s}}\varphi_{t,s}=0$ for every j, i.e. $\{\varphi_{t,s}\}$ satisfies the condition 2) in Theorem 1. Since $\theta(d, R_{t,s})$ and $\sum_{j=1}^{H}a_{j;t,s}\cdot\phi(C_j, R_{t,s})$ converges to $\theta(d, R_0)$ and 0, respectively, strongly metrically (with respect to $\{f_{t,s}\}$) by [5, Proposition 4] and by Lemma 4 and Theorem 6, $\varphi_{t,s}$ converges to $\varphi_{0,0}=\theta(d, R_0)$ strongly metrically as $|(t, s)|$ tends to 0. In particular, $\{\varphi_{t,s}\}$ satisfies also the condition 1) in Theorem 1. And since $\|\sigma(d, R_{t,s})\|^2_{R'_{t,s}}$ converges to $\|\sigma(d, R_0)\|^2_{R'_0}$

(which can be shown as before by [5, Proposition 4] and Remark in §5) and $\sum_{j=1}^{H}|a_{j:t,s}|\cdot\|\phi(C_j, R_{t,s})\|_{R'_{t,s}}=O(1)$ by Lemma 4-1), $\|\varphi_{t,s}\|_{R'_{t,s}}=O(1)$ as $|(t, s)|$ tends to 0, which implies that $\{\varphi_{t,s}\}$ satisfies the condition 3) in Theorem 1.

Next we set $\psi=\theta(d', R_0)$, then it is clear that $\|\psi\|_{R'_0}<+\infty$ and $\overline{\varphi_{0,0}}\wedge\psi$ is absolutely integrable on R'_0. And as in the proof of Theorem 1, we can see that

$$|\omega_{t,s}\circ z_{j,k}^{-1}|/|dz\wedge d\bar{z}|\leq L\cdot|s_j|/|z| \quad \text{on} \quad z_{j,k}(V_{j,k})=\{0<|z|<1/2\}$$

for every j, k and (t, s) with a sufficiently small $|(t, s)|$, where L is a suitable constant and $\omega_{t,s}=\varphi_{t,s}\circ f_{t,s}^{-1}\wedge{}^*\psi$. In particular, $\omega_{t,s}$ is absolutely integrable on V. Since $\omega_{t,s}$ is clearly absolutely integrable on R'_0-V, we conclude the absolute integrability of $\omega_{t,s}$ on the whole R'_0. Thus we have shown that ψ satisfies the conditions A) and B) in Theorem 1 (cf. Remark in §1).

Now apply Theorem 1 to these $\{\varphi_{t,s}\}$ and ψ, and we have

$$\iint_{R'_0}\omega_{t,s}=t\cdot\iint_{R'_0}\theta(d, R_0)\cdot\mu\wedge{}^*\theta(d', R_0)+o(|(t, s)|)$$

as $|(t, s)|$ tends to 0. Also we can show that

$$\sharp)\qquad \int_{d'}\varphi_{t,s}-\int_{d'}\varphi_{0,0}=\mathrm{Re}\iint_{R'_0}\omega_{t,s}$$

for every (t, s). Hence we conclude by Lemma 4-2) and Corollary 2 that

$$\begin{aligned}
&\int_{d'}\sigma(d, R_{t,s})-\int_{d'}\sigma(d, R_0)\\
&=\sum_{j=1}^{H}a_{j:t,s}\cdot\int_{d'}\phi(C_j, R_{t,s})+t\cdot\mathrm{Re}\iint_{R'_0}\theta(d, R_0)\cdot\mu\wedge{}^*\theta(d', R_0)+o(|(t, s)|)\\
&=\sum_{j=1}^{H}\|\sigma(C_j, R_{t,s})\|^2_{R'_{t,s}}\cdot\int_{d}\phi(C_j, R_0)\cdot\int_{d'}\phi(C_j, R_{t,s})\\
&\quad+t\cdot\mathrm{Re}\iint_{R'_0}\theta(d, R_0)\cdot\mu\wedge{}^*\theta(d', R_0)+o(\|(t, s)\|)
\end{aligned}$$

as $|(t, s)|$ tends to 0. Thus the desired formula follows by Theorem 5 and Corollary 2.

Finally, the equation $\sharp$) follows from the following

Lemma 5. *For every (t, s), it holds that*

$$1)\qquad \int_{d'}\mathrm{Re}\,\varphi_{t,s}=\iint_{R'_0}\mathrm{Re}\,\varphi_{t,s}\circ f_{t,s}^{-1}\wedge{}^*\sigma(d', R_0)\,,\quad \textit{and}$$

$$2)\qquad \iint_{R'_0}\mathrm{Im}\,(\varphi_{t,s}\circ f_{t,s}^{-1}-\varphi_{0,0})\wedge\sigma(d', R_0)=0$$

In fact, by this lemma, we have

$$\int_{d'} \varphi_{t,s} - \int_{d'} \varphi_{0,0}$$
$$= \iint_{R_0'} \mathrm{Re}\, \varphi_{t,s} \circ f_{t,s}{}^{-1} \wedge {}^*\sigma(d', R_0) - \iint_{R_0'} \mathrm{Re}\, \varphi_{0,0} \wedge {}^*\sigma(d', R_0)$$
$$= \iint_{R_0'} \mathrm{Re}\, (\varphi_{t,s} \circ f_{t,s}{}^{-1} - \varphi_{0,0}) \wedge \mathrm{Im}\, \psi + \iint_{R_0'} \mathrm{Im}\, (\varphi_{t,s} \circ f_{t,s}{}^{-1} - \varphi_{0,0}) \wedge \mathrm{Re}\, \psi$$
$$= \mathrm{Re} \iint_{R_0'} (\varphi_{t,s} \circ f_{t,s}{}^{-1} - \varphi_{0,0}) \wedge {}^*\psi = \mathrm{Re} \iint_{R_0'} \varphi_{t,s} \circ f_{t,s}{}^{-1} \wedge {}^*\psi\,.$$

Proof of Lemma 5. For every positive δ ($<1/2$), define a Dirichlet function $e_\delta(p)$ on R_0' by setting

$$e_\delta(p) = 1 \quad \text{on} \quad R_0' - V\,, \quad \text{and}$$
$$= \max\{1 - (\log 2 \cdot |z_{j,k}(p)|)/\log 2\delta,\ 0\} \quad \text{on} \quad V_{j,k}$$

for every j and k. Then

$$F_\delta(\varphi_{t,s}) = e_\delta \cdot \varphi_{t,s} \circ f_{t,s}{}^{-1} + H_{t,s} \circ f_{t,s}{}^{-1} \cdot de_\delta$$

is a square integrable closed differential on R_0' for every (t, s), where $H_{t,s}(p)$ is a holomorphic function on $\bigcup_{j=1}^{n} (U_{j,t,s} - C_{j,t,s})$ such that $dH_{t,s} \equiv \varphi_{t,s}$ (cf. [5, §2, 3°)]). Moreover, since $\mathrm{Im}\, F_\delta(\varphi_{t,s}) - I_{f_{t,s}}({}^*\sigma(d', R_{t,s}))$ belongs to $\Gamma_{e0}(R_0)$ (cf. [5, §1, 2°)]) from the definition, and since ${}^*\sigma(d', R_0) = I_{f_{t,s}}({}^*\sigma(d', R_{t,s}))$ by [5, Lemmas 4 and 7 -i)], where $I_{f_{t,s}}$ is defined in [5, §2, 3°)], it holds that $\mathrm{Im}\, F_\delta(\varphi_{t,s}) - {}^*\sigma(d', R_0) \in \Gamma_{e0}(R_0)$. Hence we have

$$2') \qquad (\mathrm{Im}\, (F_\delta(\varphi_{t,s}) - \varphi_{0,0})\,,\ {}^*\sigma(d', R_0))_{R_0'} = 0$$

Also since $\mathrm{Re}\, \varphi_{t,s}$ and ${}^*\sigma(d', R_{t,s})$ belong to $\Gamma_h(R_{t,s}, R_0)$ which is orthogonal to ${}^*\Gamma_N(R_{t,s}, R_0)$, we have by [5, Lemma 7-ii)]

$$(\mathrm{Re}\, \varphi_{t,s},\ \sigma(d', R_{t,s}))_{R_{t,s}'} \left(= \int_d \mathrm{Re}\, \varphi_{t,s}\right)$$
$$= (H_{f_{t,s}}(I_{f_{t,s}}(\mathrm{Re}\, \varphi_{t,s}))\,,\ -{}^*(H_{f_{t,s}}(I_{f_{t,s}}({}^*\sigma(d', R_{t,s})))))_{R_{t,s}'}\,.$$

Hence by [5, Lemma 5] we have

$$1') \qquad \int_d \mathrm{Re}\, \varphi_{t,s} = (I_{f_{t,s}}(\mathrm{Re}\, \varphi_{t,s})\,,\ -{}^*I_{f_{t,s}}({}^*\sigma(d', R_{t,s})))_{R_0'}$$
$$= (I_{f_{t,s}}(\mathrm{Re}\, \varphi_{t,s}),\ \sigma(d', R_0))_{R_0'} = (F_\delta(\mathrm{Re}\, \varphi_{t,s}),\ \sigma(d', R_0))_{R_0'}\,.$$

On the other hand, by Lemma 2, we can shoose $\{H_{t,s}\}$ so that $H_{t,s} \circ f_{t,s}{}^{-1}$ are uniformly bounded on every $V_{j,k}$. Hence as before, letting δ become 0, we can show the assertions 1) and 2) from 1′) and 2′), respectively, by Lebesque's convergence theorem. q.e.d.

§5. Proofs of Theorems 3 and 4.

First fix an integer j in $[1, H]$, and let $\{R^{\sharp}_{t,s}\}$ and $\{h_{t,s}\}$ be as in the proof of Theorem 6. Then since $S \notin 0_G$, the component $S_{t,s}$ of $R^{\sharp}_{t,s}-N(R^{\sharp}_{t,s})$ containing $h_{t,s}{}^{-1}(S)$ also admits Green's functions for every (t, s) (for which $h_{t,s}$ can be defined). Hence when $\phi(C_j, R_0)=(1/2\pi i)\cdot(\phi(q_1, S)-\phi(q_2, S))$ on S with suitable punctures q_1 and q_2 on R'_0 (, but not necessarily on S), $\phi(C_j, R^{\sharp}_{t,s})$ should be equal to $(1/2\pi i)$ $\cdot(\phi(h_{t,s}{}^{-1}(q_1), S_{t,s})-\phi(h_{t,s}{}^{-1}(q_2), S_{t,s}))$ on $S_{t,s}$. We set $G_{j,t,s}(p)=g(p, h_{t,s}{}^{-1}(q_1))-g(p, h_{t,s}{}^{-1}(q_2))$ on $S_{t,s}$. Then $G_{j,t,s}\circ h_{t,s}{}^{-1}$ converges to G_j locally uniformly on S ([7, Corollary 1]). Also we can show the following generalization of [7, §4 (13)].

Lemma 6. *For every (t, s) such as above, it holds that*

$$\int_{C_j} {}^*dg_{t,s} = -\|\sigma(C_j, R_{t,s})\|^2_{R'_{t,s}}\cdot G_{j,t,s}\circ h_{t,s}{}^{-1}(q)\,.$$

Proof. If $\sigma(C_j, R_{t,s})\equiv 0$ on the component $T_{t,s}$ of $R'_{t,s}$ containing $f_{t,s}{}^{-1}(S)$, then we can see that both sides are equal to 0.

Suppose that $\sigma(C_j, R_{t,s})\not\equiv 0$ on $T_{t,s}$, and take suitable compact regular trajectory $C(j, t, s)$ of $\phi(C_j, R_{t,s})$ freely homotopic to C_j on $T_{t,s}-f_{t,s}{}^{-1}(q)$. (For example, take one of $\{C(j(k), t, s)\}_{k=1}^{N}$ appeared in the proof of Theorem 6.) Since ${}^*\sigma(C_j, R_{t,s})$ is exact on $T_{t,s}-C(j, t, s)$, there is a harmonic function $u_{j,t,s}$ on $T_{t,s}-C(j, t, s)$ such that $du_{j,t,s}\equiv {}^*\sigma(C_j, R_{t,s})$ and $u_{j,t,s}$ coincides with a Dirichlet potential on $T_{t,s}$ outside some compact neighbourhood of $C(j, t, s)$. Note that $u_{j,t,s}$ is a constant on each border of $T_{t,s}-C(j, t, s)$. Denote these two borders by d_1 and d_2 so that d_1 has the same orientation as C_j and let $u_{j,t,s}\equiv M_k$ on d_k $(k=1, 2)$. Then we can see that $M_1-M_2=-1$.

Now apply [7, Lemma 4] to $u_{j,t,s}$ and ${}^*dg_{t,s}$ on each component of $T_{t,s}-\{g_{t,s}\geq M\}\cup C(j, t, s)$ with a sufficiently large M, and we have

$$\begin{aligned}-\int_{C_j} {}^*dg_{t,s}+2\pi\cdot u_{j,t,s}\circ f_{t,s}{}^{-1}(q)\\ =({}^*\sigma(C_j, R_{t,s}), dg_{t,s})_{T_{t,s}(M)}\,,\end{aligned}$$

where $T_{t,s}(M)=T_{t,s}-\{g_{t,s}\geq M\}$. Next apply the same lemma to $g_{t,s}$ and $-\sigma(C_j, R_{t,s})$ on $T_{t,s}(M)$, and we have

$$({}^*\sigma(C_j, R_{t,s}), dg_{t,s})_{T_{t,s}(M)}=0\,.$$

Since $u_{j,t,s}$ coinsides with $(-1/2\pi)\cdot\|\sigma(C_j, R_{t,s})\|^2_{R'_{t,s}}\cdot G_{j,t,s}$ on the component of $R_{t,s}-\bigcup_{k=1}^{N} C(j(k), t, s)-C(j, t, s)$ containing $f_{t,s}{}^{-1}(q)$ $(=h_{t,s}{}^{-1}(q))$, we have the assertion. q.e.d.

As in §4, there is a unique solution $B_{t,s}=(b_{j,;t,s})$ of the equation $Z_{t,s}=X_{t,s}\cdot B_{t,s}$, where $Z_{t,s}$ is the H-dimensional vector with the j-th component $z_{j;t,s}=\int_{C_j} {}^*dg_{t,s}$, namely,

$$\int_{C_j} {}^*dg_{t,s} = \sum_{k=1}^{H} b_{k;t,s} \cdot \int_{C_j} \phi(C_k, R_{t,s}) \quad (j = 1, \cdots, H)$$

for every (t, s) with a sufficiently small $|(t, s)|$. And similarly as in the proof of Lemma 4, we can show the following

Lemma 7. *For every j, it holds that*

1) $$b_{j;t,s} = O(||\sigma(C_j, R_{t,s})||_{R'_{t,s}}) \, , \quad \textit{and}$$

2) $$b_{j;t,s} = -||\sigma(C_j, R_{t,s})||^2_{R'_{t,s}} \cdot G_j(q) + o(||(t, s)||) = O(||(t, s)||)$$

as $|(t, s)|$ *tends to* 0.

Now for every (t, s) with a sufficiently small $|(t, s)|$, we set

$$\varphi_{t,s} = -i \cdot \phi(f_{t,s}{}^{-1}(q), R_{t,s}) - \sum_{j=1}^{H} b_{j;t,s} \cdot \phi(C_j, R_{t,s}) \, ,$$

then we can see from the definition that $\{\varphi_{t,s}\}$ satisfies the condition 2) in Theorem 1. Also we know that, under the assumption as in Theorem 3, $\phi(f_{t,s}{}^{-1}(q), R_{t,s})$ converges to $\phi(q, R_0)$ $(= \varphi_{0,0})$ strongly metrically with respect to $\{f_{t,s}\}$ ([7, Theorem 1]). Since $\sum_{j=1}^{H} b_{j;t,s} \cdot \phi(C_j, R_{t,s})$ converges to 0 by Lemma 7 and Theorem 6, $\varphi_{t,s}$ satisfies also the condition 1) in Theorem 1. And we know that $||\phi(f_{t,s}{}^{-1}(q), R_{t,s})||_{U_{j,t,s}}$ are uniformly bounded for every j ([7, Lemma 2]). Hence as in §4, we can see by Lemma 7-1) that $\{\varphi_{t,s}\}$ satisfies the condition 3) in Theorem 1.

Proof of Theorem 3. Set $\psi = \theta(d, R_0)$, then by the same argument as in §4, we can show that ψ satisfies the conditions A) and B) in Theorem 1. Applying Theorem 1 to the above $\{\varphi_{t,s}\}$ and ψ, we have

i) $$\iint_{R'_0} \omega_{t,s} = t \cdot \iint_{R'_0} -i \cdot \phi(q, R_0) \cdot \mu \wedge {}^*\theta(d, R_0) + o(|(t, s)|)$$

as $|(t, s)|$ tends to 0. Also we can show the equation

$$\int_d \operatorname{Re} \varphi_{t,s} - \int_d \operatorname{Re} \varphi_{0,0} = \operatorname{Re} \iint_{R'_0} \omega_{t,s}$$

from Lemma 8 below. Hence we conclude the desired formula similarly as in §4, by using Lemma 7-2), Corollary 2, and Theorem 5 q.e.d.

Lemma 8. *For every (t, s), it holds that*

1) $$\int_d \operatorname{Re} \varphi_{t,s} - \int_d \operatorname{Re} \varphi_{0,0} = \iint_{R'_0} \operatorname{Re}(\varphi_{t,s} \circ f_{t,s}{}^{-1} - \varphi_{0,0}) \wedge {}^*\sigma(d, R_0) \, , \quad \textit{and}$$

2) $$\iint_{R'_0} \operatorname{Im}(\varphi_{t,s} \circ f_{t,s}{}^{-1} - \varphi_{0,0}) \wedge \sigma(d, R_0) = 0 \, .$$

Proof. Let $e_\delta(p)$ and F_δ be as in the proof of Lemma 5. Then from the definition, $\mathrm{Im}(F_\delta(\varphi_{t,s})-\varphi_{0,0})\in\Gamma_{e0}(R_0)$. Hence 2′) in the proof of Lemma 5 (with $d'=d$) is valid.

Next recall that there is a (smooth) closed differential α on R_0' such that $*\sigma(d, R_0)-\alpha\in\Gamma_{e0}(R_0)$ and the support of α is compact in $\tilde{R}_0=R_0'-\bar{U}\cup\bar{U}_q$ (cf. the proof of [5, Lemma 4]). Then we can see that

$$*\sigma(d, \tilde{R}_{t,s})-\alpha\circ f_{t,s}\in\Gamma_{e0}(\tilde{R}_{t,s}),$$

where $\tilde{R}_{t,s}=f_{t,s}{}^{-1}(\tilde{R}_0)$. Since $\mathrm{Re}\,\varphi_{t,s}\in\Gamma_h(\tilde{R}_{t,s})$, we have

$$\begin{aligned}\int_d \mathrm{Re}\,\varphi_{t,s} &= -(\mathrm{Re}\,\varphi_{t,s}, *(\alpha\circ f_{t,s}))_{\tilde{R}_{t,s}}\\ &= -(\mathrm{Re}\,\varphi_{t,s}\circ f_{t,s}{}^{-1}, *\alpha)_{\tilde{R}_0} = -(F_\delta(\mathrm{Re}\,\varphi_{t,s}), *\alpha)_{R_0'}.\end{aligned}$$

Hence we have

$$\begin{aligned}&\int_d \mathrm{Re}\,\varphi_{t,s}-\int_d \mathrm{Re}\,\varphi_{0,0}\\ &\quad= -(\mathrm{Re}\,(F_\delta(\varphi_{t,s})-\varphi_{0,0}), *\alpha)_{R_0'}\\ &\quad= (\mathrm{Re}\,(F_\delta(\varphi_{t,s})-\varphi_{0,0}), \sigma(d, R_0))_{R_0'}.\end{aligned}$$

Thus the assertions follows by the same argument as in the proof of Lemma 5.

q.e.d.

Remark. By using α as in the proof of Lemma 8, we can show rather directly the fact that metrical convergence implies convergence of periods (cf. [4, Corollary 3]).

Proof of Theorem 4. Set $\psi=-i\cdot\phi(q', R_0)$, then similarly as before, we can see that ψ satisfies the conditions A) and B) in Theorem 1. Hence applying Theorem 1 to the above $\{\varphi_{t,s}\}$ and this ψ, we have

$$\iint_{R_0'}\omega_{t,s}=t\cdot\iint_{R_0'}-i\cdot\phi(q, R_0)\cdot\mu\wedge*(-i\cdot\phi(q', R_0))+o(|(t, s)|)$$

as $|(t, s)|$ tends to 0. Next since $G_{j,t,s}\circ h_{t,s}{}^{-1}(q')$ converges to $G_j(q')$ ([7, Corollary 1]), we conclude by Lemma 7-2) and Lemma 9 below that

$$\begin{aligned}\mathrm{Re}\iint_{R_0'}\omega_{t,s} &= 2\pi(g_{0,0}(q')-g_{t,s}(f_{t,s}{}^{-1}(q')))\\ &\quad-\sum_{j=1}^{H}||\sigma(C_j, R_{t,s})||^2_{R_{t,s}'}\cdot G_j(q)\cdot G_{j,t,s}\circ h_{t,s}{}^{-1}(q')+o(||(t, s)||)\end{aligned}$$

as $|(t, s)|$ tends to 0. Hence the desired formula follows by Theorem 5. q.e.d.

Lemma 9. *For every t, it holds that*

1) $$\iint_{R_0'} \mathrm{Re}(\varphi_{t,s}\circ f_{t,s}^{-1}-\varphi_{0,0})\wedge dg(\cdot,\, q')=0\,, \quad \textit{and}$$

2) $$\iint_{R_0'} \mathrm{Im}(\varphi_{t,s}\circ f_{t,s}^{-1}-\varphi_{0,0})\wedge {}^*dg(\cdot,\, q')$$
$$= 2\pi(g_{0,0}(q')-g_{t,s}(f_{t,s}^{-1}(q')))+\sum_{j=1}^{H} b_{j;t,s}\cdot G_{j,t,s}\circ h_{t,s}^{-1}(q')\,.$$

Proof. Let $e_\delta(p)$ and F_δ be as in §4. Then since $\eta_{t,s}=\mathrm{Re}(F_\delta(\varphi_{t,s})-\varphi_{0,0})$ belongs to $\Gamma_c(R_0)$ and is harmonic on $U_{q'}$, we can apply [7, Lemma 4] (which remains valid for any pair of h and ω satisfying all conditions in the lemma except that they need to be smooth not everywhere but only on a neighbourhood of ∂D) to $h(p)=g(p,\, q')$ and $\omega=\eta_{t,s}$ on $S(M)=S-\{g(p,\, q')\geq M\}$ with a sufficiently large M, and obtain that $\iint_{S(M)} \eta_{t,s}\wedge dg(\cdot,\, q')=-\iint_{\partial S(M)} M\cdot\eta_{t,s}=0$. Since $g(p,\, q')\equiv 0$ on R_0-S, letting M become $+\infty$, we have

1′) $$\iint_{R_0'} \mathrm{Re}(F_\delta(\varphi_{t,s})-\varphi_{0,0})\wedge dg(\cdot,\, q')=0\,.$$

Next note that for any fixed $\delta>0$, we can take such an admissible family $\{h_{t,s}\}$ that $f_{t,s}^{-1}\equiv h_{t,s}^{-1}$ on the support of $e_\delta(p)$ for every $(t,\, s)$ with a sufficiently small $|(t,\, s)|$ (, by choosing W so that $e_\delta(p)\equiv 0$ on W in the proof of Theorem 6). And set

$$v_{t,s}=g_0-e_\delta\cdot(g_{t,s}\circ f_{t,s}^{-1}-\frac{1}{2\pi}\sum_{j=1}^{H} b_{j;t,s}\cdot G_{j,t,s}\circ h_{t,s}^{-1})\,.$$

Then clearly $v_{t,s}$ is a continuous Dirichlet potential on R_0' and harmonic on $U_{q'}$. Since $dG_{j,t,s}\circ h_{t,s}^{-1}\equiv -2\pi\cdot\mathrm{Im}\,\phi(C_j,\, R_{t,s})\circ h_{t,s}^{-1}$ on the support of $e_\delta(p)$, we have

$$dv_{t,s}-\mathrm{Im}(F_\delta(\varphi_{t,s})-\varphi_{0,0})=E_{t,s}\cdot de_\delta$$

with a suitable constant $E_{t,s}$ for every $(t,\, s)$ as above. Hence $w_{t,s}(p)=E_{t,s}\cdot(1-e_\delta(p))$ is a continuous Dirichlet potential on R_0' such that $w_{t,s}\equiv 0$ on $U_{q'}$ and $\mathrm{Im}(F_\delta(\varphi_{t,s})-\varphi_{0,0})=dv_{t,s}+dw_{t,s}$. Apply [7, Lemma 4] (generalized as above) to $h=v_{t,s}+w_{t,s}$ and $\omega={}^*dg(\cdot,\, q')$ on $S(M)$ with a sufficiently large M, and we have

$$\iint_{S(M)} \mathrm{Im}(F_\delta(\varphi_{t,s})-\varphi_{0,0})\wedge {}^*dg(\cdot,\, q')=\iint_{\partial S(M)} h\cdot {}^*dg(\cdot,\, q')=2\pi\, v_{t,s}(q')\,.$$

Hence letting M become $+\infty$, we have

2′) $$\iint_{R_0'} \mathrm{Im}(F_\delta(\varphi_{t,s})-\varphi_{0,0})\wedge {}^*dg(\cdot,\, q')$$
$$= 2\pi(g_0(q')-g_{t,s}\circ f_{t,s}^{-1}(q'))+\sum_{j=1}^{H} b_{j;t,s}\cdot G_{j,t,s}\circ h_{t,s}(q')\,.$$

Thus the assertions 1) and 2) follows from 1′) and 2′) similarly as before. q.e.d.

Appendix. A characterization of the conformal topology.

Let a Riemann surface R^* (with no nodes) be given, and consider the finitely augmented Teichmüller space $\hat{T}(R^*)$ of R^* (cf. [5, § 1, 1°)]). Fix a point R_0 in $\hat{T}(R^*)-T(R^*)$ once for all, and denote by $D(R_0)$ *the deformation space of* R_0 *in* $\hat{T}(R^*)$, namely, the subset of $\hat{T}(R^*)$ consisting of all points R such that there is a marking-preserving deformation of R to R_0. Next fix a marking-preserving deformation $(f^*; R^*, R_0)$ of R^* (with the identical mapping as the marking) to R_0. And, letting $N(R_0)=\{p_j\}_{j=1}^n$, we set $C_j^*=(f^*)^{-1}(p_j)$ with suitable orientation for every j. Recall that $\{C_j^*\}_{j=1}^n$ is a homotopically independent system of simple closed curves on R^*.

Now we choose a finite set $\{q_k\}_{k=1}^m$ of auxiliary points on $R^*-\bigcup_{j=1}^n C_j^*$ so that each component of $R^*-\bigcup_{j=1}^n C_j^*$ is either a non-parabolic part or a parabolic part containing (exactly) one point of $\{q_k\}_{k=1}^m$. And we consider the finitely augmented Teichmüller space $\hat{T}(R^{**})$ of $R^{**}=R^*-\bigcup_{k=1}^m \{q_k\}$, and the deformation space $D(R_{0^*})$ of $R_{0^*}=R_0-\bigcup_{k=1}^m \{f^*(q_k)\}$. Then there is a natural projection, say π, called the forgetful mapping from $D(R_{0^*})$ onto $D(R_0)$, and $R_n\in\hat{T}(R^*)$ converges to R_0 (as n tends to $+\infty$) if and only if $R_n\in D(R_0)$ for every sufficiently large n and a suitable lift R_n^* of R_n (i.e. $\pi(R_n^*)=R_n$) converges to R_{0^*} in $\hat{T}(R^{**})$. More precisely, if R_n converges to R_0, then there is an admissible sequence $\{(f_n; R_n, R_0)\}_{n=1}^\infty$ of deformations of R_n to R_0, and $R_n^*=R_n-\bigcup_{k=1}^N \{f_n^{-1}(f^*(q_k))\}$ considered as a point in $\hat{T}(R^{**})$ converges to R_{0^*}; the converse clearly holds.

So we will give a characterization of sequences in $\hat{T}(R^{**})$ converging to R_{0^*}. For this purpose, fix $R\in D(R_{0^*})$ and C_j^* arbitrarily. Here we may assume that C_j^* corresponds to none of nodes of R. Then by the assumption on auxiliary points, we can define a holomorphic differential on R, which is again called *the associated differential for* C_j^* considered as a loop on R, as follows; when $\sigma(C_j^*, R)\not\equiv 0$, then we set

$$\phi(C_j^*, R)=||\sigma(C_j^*, R)||_R^{-2}\cdot\theta(C_j^*, R).$$

When $\sigma(C_j^*, R)\equiv 0$, then C_j^* is a dividing curve on a component of $R'=R-N(R)$. Let W_1 and W_2 be the components of $R'-\bigcup_{j'=1}^{N'} C_{j'}^*$ whose boundary contains C_j^* and $-C_j^*$, respectively. Here $\{C_{j'}^*\}_{j'=1}^{N'}$ is the set of all C_k^* (considered as loops on R) corresponding to none of nodes of R (hence contains C_j^*). If both of W_1 and W_2 contain auxiliary points, say q_1 and q_2, respectively, then we set

$$\phi(C_j^*, R)=\frac{1}{2\pi i}\cdot\phi(q_1, q_2; R).$$

If only one of W_1 and W_2 contains an auxiliary point, say $q\in W_1$, then we can see that $\phi(q, R)\not\equiv 0$, and we set

$$\phi(C_j^*, R) = \frac{1}{2\pi i}\cdot\phi(q; R).$$

Then we can consider, on the component of R' containing C_j^*, the characteristic ring domain $W(C_j^*, R)$ of $\phi(C_j^*, R)$ for C_j^*, and we denote by $m(C_j^*, R)$ and $C(C_j^*, R)$ the modulus and the center trajectory, respectively, of $W(C_j^*, R)$, where $m(C_j^*, R)=0$ in case that $W(C_j^*, R)=\phi$. Also setting $m(C_j^*, R)=+\infty$ for every C_j^* not contained in $\{C_{j'}^*\}_{j'=1}^{N'}$, we can define $m(C_j^*, R)$ for every j (and every $R\in D(R_{0^*})$). Here we note the following

Lemma A1. *Let $\{C_{j'}^*\}_{j'=1}^{N'}$ be as above. If $m(C_{j'}^*, R)>2$ for every j', then $\{C(C_{j'}^*, R)\}_{j'=1}^{N'}$ are mutually disjoint.*

Proof. Suppose that there are two curves c_1 and c_2 in $\{C(C_{j'}^*, R)\}$ such that $c_1\cap c_2\neq\phi$. Then there is a component D_1 of $W(c_1, R)-c_1$ such that, for every curve γ in D_1 freely homotopic to c_1 (in $W(c_1, R)$), $c_2\cap\gamma$ consists of at least two points. Also it is clear that γ does not contained in $W(c_2, R)$ for every γ such as above.

Now consider $\theta=\phi(c_2, R)$ on $W(c_2, R)$, then $||\theta||^2_{W(c_2,R)}=2\cdot m(c_2, R)$, and it holds that $\int_{\gamma\cap W(c_2,R)}|\theta|\geq m(c_2, R)$ for every γ as above. Hence, recalling the definition of the extremal length, we have

$$1>\frac{2}{m(c_1, R)}=\lambda(c_1, D_1)\geq\frac{m(c_2, R)^2}{||\theta||^2_{W(c_2,R)}}=\frac{m(c_2, R)}{2}>1,$$

which is a contradiction. q.e.d.

In particular, when $m(C_{j'}^*, R)>2$ for every j', then we can construct a marked Riemann surface $R^\sharp$ with nodes from R, by cutting R along $\bigcup_{j'=1}^{N'} C(C_{j'}^*, R)$, attaching a once punctured disk to each border of $R-\bigcup_{j'=1}^{N'} C(C_{j'}^*, R)$, and fill two punctures corresponding to the same $C_{j'}^*$ by a single point for every j'. Such an $R^\sharp$ does not determined uniquely, but will be fixed arbitrarily for every $R\in D(R_{0^*})$. Then we have the following

Theorem A2. *In $D(R_{0^*})$, R_n converges to R_{0^*} if and only if*

i) $\lim_{n\to+\infty} m(C_j, R_n)=+\infty$ *for every j, and*

ii) $(R_n)^\sharp$ *(which is defined for every sufficiently large n) converges to R_{0^*} in the sense of the Teichmüller topology.*

Proof. First suppose that i) and ii) holds. In particular, there is a sequence of quasiconformal mapping f_n from $(R_n)^\sharp-N((R_n)^\sharp)$ onto $R_{0^*}-N(R_{0^*})$ for every n such that the maximal dilatation of f_n converges to 0 as n tends to $+\infty$. Then we can show similarly as in the proof of [2, Lemma 1] that for every neighbourhood W of $N(R_0)$, there is an N_0 such that $f_n^{-1}(W)$ contains $(R_n)^\sharp-(R_n-\bigcup_{j'=1}^{N'} C(C_{j'}^*, R_n))$

for every $n \geq N_0$. Hence we can construct an admissible sequence of deformations of R_n to R_{0*} by reforming $\{f_n\}$.

Conversely, suppose that R_n converges to R_{0*} in $D(R_{0*})$. Then by the same argument as in the proof of the 'only if' part of [7, Theorem 3], we can show that i) and ii) holds. q.e.d.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] Y. Kusunoki and F. Maitani, Variations of abelian differentials under quasiconformal deformations, Math. Z., **181** (1982), 435–450.

[2] M. Taniguchi, Remarks on topologies associated with squeezing a nondividing loop on compact Riemann surfaces, J. Math. Kyoto Univ., **19** (1979) 203–214.

[3] M. Taniguchi, On variation of periods of holomorphic Γ_{h0}-reproducing differentials, Proc. Japan Acad. Ser. A, **56** (1980) 315–317.

[4] M. Taniguchi, On convergence of holomorphic abelain differentials on the Teichmüller spaces of arbitrary Riemann surfaces, J. Math. Kyoto Univ., **24** (1984), 305–321.

[5] M. Taniguchi, Square integrable harmonic differentials on arbitrary Riemann surfaces with a finite number of nodes, ibid., **25** (1985), 597–617.

[6] M. Taniguchi, On the modulus of the free homotopy class of a simple closed curve on an arbitrary Rimann surface, Japanese J. Math., **12** (1986) 53–68.

[7] M. Taniguchi, Continuity of certain differentials on finitely augmented Teichmüller spaces and variational formulas of Schiffer-Spencer's type, Tohoku Math. J., **38** (1986) 281–295.

[8] M. Taniguchi, Supplements to my previous papers; a refinement and applications, J. Math. Kyoto Univ., **28** (1988), to appear.

[9] M. Taniguchi, Abelian differentials with normal behavior and complex pinching deformation, to appear.