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Theory of pseudo-differential operators of ultradifferentiable class

To the memory of Professor H. Kumano-go

By

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§1. Introduction.

Let $\{M_n\}_{n=0}^{\infty}$ be a sequence of positive numbers. We set $\mathcal{E}\{M_n\}=\{f(x)\in C^\infty(\boldsymbol{R}^l); \forall K$: compact in $\boldsymbol{R}^l$, $\exists C, R>0, \forall\alpha\in\boldsymbol{Z}_+^l, \sup_{u\in K}|\left(\frac{\partial}{\partial x}\right)^\alpha f|\leqq CR^{|\alpha|}M_{|\alpha|}\}$. We call this the ultradifferentiable space of class $\{M_n\}$ (=the ul.d. space of class $\{M_n\}$). A typical ul.d. class is a Gevrey class, i.e., $M_n=n!^\nu$ ($\nu\geqq 1$). When $\nu=1$, it is the real analytic class.

In the study of the spaces of the admissible data in the Cauchy problems, we were led to introduce some ul.d. classes wider than any Gevrey class. (See W. Matsumoto [29], [30] and [32].) In a systematic treatment of the problems in a ul.d. class, a theory of pseudo-differential operators (=ps.d.op's) of ul.d. class is required. However, it is not yet well investigated except the case of the Gevrey classes.

The theory of ps.d.op's of C^∞ class has been well studied. (See J.J. Kohn and L. Nirenberg [20], L. Hörmander [14], H. Kumano-go [26] and [27], etc.) Since their formulations are slightly different each other, we mean, in this paper, Kumano-go's theory [27] by the theory of ps.s.op's of C^∞ class. Hereafter, we shall try to construct a theory of ps.d.op's of general ul.d. class corresponding to that of C^∞ class. We are mainly interested in the ul.d. classes wider than any Gevrey class because that of Gevrey classes has been well investigated. (SeeL. Boutet de Monvel and P. Krée [7], L. Boutet de Monvel [6], L.R. Volevič [41], L. Hörmander [15], F. Treves [40], S. Hashimoto, T. Matsuzawa and Y. Morimoto [13], M.D. Bronstein [8], K. Taniguchi [37], [38] and [39], G. Métivier [33], M.S. Baouendi and C. Goulaouic [4], M.S. Baouendi, C. Goulaouic and G. Métivier [5], C. Iwasaki [16], K. Kataoka [17] and [18], T. Aoki [1], [2] and [3], L. Zanghirati [42], L. Cattabrige and L. Zanghirati [43], L. Rodino and L. Zanghirati [44], etc.)

A typical difference between the Gevrey classes and the ul.d. classes wider than any Gevrey class is characterized by the separativity condition:

$$\text{(S)} \qquad \exists H>1\,,\quad \forall n\in\boldsymbol{Z}_+\,,\quad (M_{2n})^{1/2n}\leqq H(M_n)^{1/n}\,.$$

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Every Gevrey class satisfies Condition (S) but any ul.d. class wider than all Gevrey classes does not. (See W. Matsumoto [31] and Paragraph 2.4.) L.R. Volevič [41] tried to widen the Gevrey classes to the general class $\{M_n\}$ assuming Condition (S).

By saying that $\sum_{i=0}^{\infty} p_i(x, \xi)$ is a formal symbol, we mean that this is a formal sum of asymptotic expansion of symbol. We denote temporarily the space of ps. d.op's of class $[M_n]$ with the indices ρ and δ in Hörmander's sense by $\boldsymbol{S}[M_n]$ $(=\boldsymbol{S}_{\rho\delta}[M_n])$, the space of those symbols by $\mathrm{S}[M_n]$ $(=\mathrm{S}_{\rho\delta}[M_n])$, and the space of formal symbols of the same class by $\mathcal{S}[M_n]$ $(=\mathcal{S}_{\rho\delta}[M_n])$. The more precise notation and the definitions of them will be given in Paragraphs 3.2 and 4.1. Under Condition (S), we can construct a true symbol from a formal symbol in the sense of class $[M_n]$. By virtue of this, in case of the ul.d. class with (S), the relation of asymptotic expansion gives a onto-homomorphism of star algebra from $\mathcal{S}_{\rho\delta}[M_n]$ to $\mathrm{S}_{\rho\delta}[M_n]$ modulo symbols of strong regularizers of class $[M_{n/(\rho-\delta)}]$. (Here, we say that an operator A is a strong regularizer of class $[M_n]$ when A is continuous from $\mathcal{E}'\{M_n\}$ to $\mathcal{E}\{M_n\}$.) Therefore, the investigation of the operations in $\boldsymbol{S}[M_n]$ is reduced to that in $\mathcal{S}[M_n]$. In $\mathcal{S}[M_n]$, the elementary operations, for example the operator product and the formal adjoint, they consist of the arithmetical operations and the derivation, which have the local property and then which are rather easily handled.

On the other hand, without Condition (S), it seems impossible to construct a true symbol from a formal symbol in the sense of class $[M_n]$. Namely, we cannot reduce the investigation of the operations in $\boldsymbol{S}[M_n]$ to that in $\mathcal{S}[M_n]$. We must consider them directly in $\boldsymbol{S}[M_n]$ (namely in $\mathrm{S}[M_n]$). As in the case of C^∞ class, we shall start from the following formula. (See, for example, H. Kumano-go [27] Theorem 1.7.)

$$(*)\qquad \sigma(P\circ Q) = \mathrm{Os}-\int e^{-\sqrt{-1}y\cdot\eta} p(x, \xi+\eta) q(x+y, \xi)\, dy\, đ\eta\,.$$

($P\circ Q$ is the operator product of P and Q, $\sigma(A)$ is the symbol of the ps.d.op. A and $đ_\eta$ is $(2\pi)^{-l}d\eta$.) Our consideration will become fairly complicated because the integration has not the local property. In case of the theory of ps.d.op's of ul.d. class, we cannot still expect that $\sigma(P\circ Q)$ itself belongs to $\mathrm{S}[M_n]$. Namely, we should consider some modulo class. (See Paragraph 5.2.) Therefore, we must divide the above integral into the main part which belongs to $\mathrm{S}[M_n]$ and the rest part which does to the modulo class. Here,this division depends on ξ. Thus, we need a cut-off function which depends on ξ. On the other hand, it is indispensable to assume the analyticity in ξ on the symbols, if we expect a theory which allows the asymptotic expansions of symbols of arbitrary length. (See the begining of Paragraph 4.1 and Paragraph 3.2.) After all, we might use a cut-off function of analytic class! Relaxing the analyticity in ξ to the pseudo-analyticity, we can barely construct a theory of ps.d.op's of general ul.d. class except the two points: Construction of parametrices of elliptic operators in $\boldsymbol{S}_{1\delta}[M_n]$ and Construction of true symbols in $\mathrm{S}[M_n]$ from formal symbols in $\mathcal{S}[M_n]$. This theory, on taking $[M_n]$ for the Gevrey classes, gives some better results than those known until now. (See Paragraphs 5.3 and 5.6 and K. Taniguchi [39].)

In this paper, we shall make much of the consideration on the reasonableness of the definitions and on not only affirmative results but also negative ones on the expected properties.

The auther deeply thanks Professor K. Taniguchi. Their discussions contributed to the framing of the ps.d. op's of ul.d. classes and to the reseach for the best possible results in §5.

§2. Notation, definitions and propositions of ultradifferentiable spaces.

2.1. *Notation and definitions.*

In the proofs in the following sections, we shall use the letters C and R for constants depending only on the symbols of ps.d.op's and the dimension l of x-space and they may not be the same at each line. We set $l_0=[l/2]+1$ and $l_1=[l/2(1-\delta)]+1$ $(0\leqq\delta<1)$.

"$K\subset\subset\Omega$" means that K is a compact subset of Ω and $\partial K\cap\partial\Omega=\phi$. "$K\to\Omega$" means that Ω is the union of the increasing sequence of the compact sets $\{K\}$.

Let $\boldsymbol{Z}_+$ be $\{0,1,2,3,\cdots\}$, and $\boldsymbol{R}_+=\{a\in\boldsymbol{R};\,a\geqq 0\}$. We denote the integral part of $k\in\boldsymbol{R}$ by $[k]$ and $\max\{k,0\}$ by k_+. We set $\log_+ x=\max\{\log x,1\}$ $(x>0)$. For $\alpha=(\alpha^1,\cdots,\alpha^l)$, $\alpha'=(\alpha^{1\prime},\cdots,\alpha^{l\prime})$ and β in $\boldsymbol{Z}_+^l$, we set $|\alpha|=\alpha^1+\cdots+\alpha^l$, $\alpha+\alpha'=(\alpha^1+\alpha^{1\prime},\cdots,\alpha^l+\alpha^{l\prime})$, $\alpha!=\alpha^1!\cdots\alpha^l!$, $\left(\dfrac{\partial}{\partial x}\right)^{\alpha}=\left(\dfrac{\partial}{\partial x_1}\right)^{\alpha^1}\cdots\left(\dfrac{\partial}{\partial x_l}\right)^{\alpha^l}$, $D_x^\alpha=(-\sqrt{-1})^{|\alpha|}\left(\dfrac{\partial}{\partial x}\right)^{\alpha}$, $p_{(\alpha)}^{(\beta)}(x,\xi)=D_x^\alpha\left(\dfrac{\partial}{\partial\xi}\right)^{\beta}p(x,\xi)$, $f_{(\alpha,\alpha')}(x,y)=D_x^\alpha D_y^{\alpha'}f(x,y)$ and $g^{(\beta,\beta')}(\xi,\eta)=\left(\dfrac{\partial}{\partial\xi}\right)^{\beta}\left(\dfrac{\partial}{\partial\eta}\right)^{\beta'}g(\xi,\eta)$. We always use α for the order of derivation on x and β for that on ξ in order to emphasize the regularity of derivatives on x, which is much more important than that on ξ in the theory of ps.d.op's of ul.d. class. (However, this use is reverse against other papers.) We denote $\alpha^j\leqq\alpha^{j\prime}$ $(1\leqq j\leqq l)$ by $\alpha\leqq\alpha'$. For $\alpha_j\in\boldsymbol{Z}_+^l$ $(1\leqq j\leqq k)$, we set $\begin{pmatrix}\alpha\\ \alpha_j\end{pmatrix}_k=\alpha!/\prod_{j=1}^k\alpha_j$ $(\sum\alpha_j=\alpha)$, and for $m\in\boldsymbol{Z}_+$ and $\alpha\in\boldsymbol{Z}_+^l$, $\begin{bmatrix}m\\ \alpha\end{bmatrix}_{(l)}=m!/\alpha!$ $(|\alpha|=m)$. Especially, we denote $\begin{pmatrix}\alpha\\ \alpha'\end{pmatrix}=\begin{pmatrix}\alpha\\ \alpha_j\end{pmatrix}_2$ $(\alpha_1=\alpha')$ and $\begin{bmatrix}m\\ k\end{bmatrix}=\begin{bmatrix}m\\ \alpha\end{bmatrix}_{(2)}$ $(m,k\in\boldsymbol{Z}_+,\ \alpha=(k,m-k))$.

We set $\hat{f}(\xi)\equiv\mathcal{F}[f](\xi)=\int e^{-\sqrt{-1}x\cdot\xi}f(x)dx$ (Fourier image of $f(x)$) and $\mathcal{F}^{-1}[f](x)=\int e^{\sqrt{-1}x\cdot\xi}\hat{f}(\xi)d\xi$ $[x\cdot\xi=x_1\xi_1+\cdots+x_l\xi_l$ and $d\xi=(2\pi)^{-l}d\xi]$, then it holds that $\mathcal{F}^{-1}\mathcal{F}=\mathcal{F}\mathcal{F}^{-1}=$Identity on $\mathcal{S}(\boldsymbol{R}^l)$ [or, more generally, on $\mathcal{D}_{L^2}^\infty(\boldsymbol{R}^l)$]. For f in $\mathcal{S}'(\boldsymbol{R}^l)$, $\hat{f}$ is defined by $\langle\hat{f},\varphi\rangle=\langle f,\hat{\varphi}\rangle$, $[\varphi\in\mathcal{S}(\boldsymbol{R}^l)]$. We denote $(\sum_{j=1}^l\xi_j^2)^{1/2}$ by $|\xi|$ and $(1+|\xi|^2)^{1/2}$ by $\langle\xi\rangle$.

Lemma 2.0.1. *For $\mu\in\boldsymbol{R}$, $\langle\xi\rangle^{\mu}$ ($|\xi|^{\mu}$, resp.) is extended analytically in a conic neighbourhood of the real axes $\{\zeta=\xi+\sqrt{-1}\eta;\ \xi,\ \eta\in\boldsymbol{R}^{l},\ |\eta|<\langle\xi\rangle/6l\ \ (<|\xi|/6l$, resp.)$\}$ and it satisfies*

$$|(\langle\xi\rangle^{\mu})^{(\beta)}|\leqq 2^{|\mu|/2}(6l)^{|\beta|}\langle\xi\rangle^{\mu-|\beta|}$$
$$|(|\xi|^{\mu})^{(\beta)}|\leqq 2^{|\mu|/2}(6l)^{|\beta|}|\xi|^{\mu-|\beta|},\quad resp.).$$

We shall often use this lemma without notice.

When $a(x,\xi)\in\mathcal{E}(\boldsymbol{R}^{2l})$ satisfies

$$\exists m\in\boldsymbol{R},\ \exists\tau\geqq 0,\ 0\leqq\exists\delta<1,\ \forall\alpha,\ \beta\in\boldsymbol{Z}_{+}^{l},\ \exists C(\alpha,\beta)>0\,,$$
$$|a_{(\alpha)}^{(\beta)}(x,\xi)|\leqq C\langle\xi\rangle^{m+\delta|\alpha|}\langle x\rangle^{\tau}\qquad\text{for}\quad(x,\xi)\in\boldsymbol{R}^{2l}\,,$$

we define the oscillatory integral of $a(x,\xi)$ as follows;

$$\text{(2.1)}\qquad \mathrm{Os}-\iint e^{-\sqrt{-1}x\cdot\xi}a(x,\xi)\mathrm{d}x\,d\!\!\!{}^{-}\!\xi = \lim_{\varepsilon\to 0}\iint e^{-\sqrt{-1}x\cdot\xi}\chi(\varepsilon x)\chi(\varepsilon\xi)a(x,\xi)\mathrm{d}x\,d\!\!\!{}^{-}\!\xi\,,$$

where χ belongs to $\mathcal{S}(\boldsymbol{R}^{l})$ and satisfies $\chi(0)=1$. This is well-defined. On the detailed properties of the oscillatory integrals, see H. Kumano-go [27], Chap. I.

For a sequence of positive numbers $\{M_n\}$, a positive constant R and a subset Ω of $\boldsymbol{R}^{l}$, we set $\mathcal{B}\{M_n\}_R(\Omega)=\{f(x)\in C^{\infty}(\Omega)$; there exists a positive constant C depending on $f(x)$ such that

$$|f_{(\alpha)}(x)|\leqq CR^{|\alpha|}M_{|\alpha|}\qquad\text{in }\Omega\text{ for }\forall\alpha\in\boldsymbol{Z}_{+}^{l}\}\,.$$

and

$$\mathcal{D}_{L^2}\{M_n\}_R=\{f(x)\in\mathcal{D}_{L^2}^{\infty}(\boldsymbol{R}^{l});$$
$$\|f\|_{\{M_n\},R}^2\equiv\sum_{\alpha\in\boldsymbol{Z}_{+}^{l}}\|f_{(\alpha)}\|_{L^2}^2/(R^{|\alpha|}M_{|\alpha|})^2<\infty\}\,.$$

$\mathcal{B}\{M_n\}_R(\Omega)$ is a Banach space with the norm of the infimum of C in the definition and $\mathcal{D}_{L^2}\{M_n\}_R$ is a Hilbert space with the natural inner product. We define the ultradifferentiable spaces (=the ul.d. spaces) of class $\{M_n\}$ as follows;

$$\mathcal{B}\{M_n\}(\Omega)\equiv\operatorname*{ind\,lim}_{R\to\infty}\mathcal{B}\{M_n\}_R(\Omega)\,,$$
$$\mathcal{E}\{M_n\}(\Omega)[\equiv C\{M_n\}(\Omega)]\equiv\operatorname*{proj\,lim}_{K\to\Omega}\operatorname*{ind\,lim}_{R\to\infty}\mathcal{B}\{M_n\}_R(K)\,,$$
$$\mathcal{D}\{M_n\}(\Omega)\equiv\operatorname*{ind\,lim}_{K\to\Omega}\operatorname*{ind\,lim}_{R\to\infty}\mathcal{B}\{M_n\}_R(K)\cap\mathcal{D}(K)\,,$$
$$\mathcal{D}_{L^2}\{M_n\}\equiv\operatorname*{ind\,lim}_{R\to\infty}\mathcal{D}_{L^2}\{M_n\}_R\,.$$

Obviously, it holds that $\mathcal{D}\{M_n\}(\Omega)\subsetneqq\mathcal{B}\{M_n\}(\Omega)\subsetneqq\mathcal{E}\{M_n\}(\Omega)$. Especially, for $M_n=n!^{\nu}$ ($\nu>0$), they are called the Gevrey class of order ν. When $\Omega=\boldsymbol{R}^{l}$, we write simply $\mathcal{B}\{M_n\}$, $\mathcal{E}\{M_n\}$ and $\mathcal{D}\{M_n\}$. We denote the strong dual space of a topological vector space X by X'. The dual space of the ul.d. space of class $\{M_n\}$ is called the space of ultradistributions of class $\{M_n\}$. On the topologies of

$\mathcal{E}\{M_n\}(\Omega)$, $\mathcal{D}\{M_n\}(\Omega)$ and those dual spaces, see H. Komatsu [22]. As $\mathcal{D}_{L^2}\{M_n\}_R$ is hilbertian, so is $\mathcal{D}_{L^2}'\{M_n\}_R$. Applying H. Komatsu [21], we have $\mathcal{D}'_{L^2}\{M_n\} = \operatorname{proj\,lim}_{R\to\infty} \mathcal{D}'_{L^2}\{M_n\}_R$, which is a Fréchet space. We shall characterize $\mathcal{D}_{L^2}\{M_n\}$ and $\mathcal{D}'_{L^2}\{M_n\}$ later on by the Fourier images.

Let Ω be a subset of $\boldsymbol{R}^{l_1}\times\boldsymbol{R}^{l_2}$. We shall also use

$$\mathcal{B}\{M_n, N_n\}_R(\Omega)=\{f(x)\in C^\infty(\Omega);\ \exists C>0,\ \forall\alpha_i\in\boldsymbol{Z}_+^{l_i} \quad (i=1,2)\,,\ |f_{(\alpha_1,\alpha_2)}|\leqq CR^{|\alpha_1|+|\alpha_2|}M_{|\alpha_1|}N_{|\alpha_2|} \text{ in } \Omega\}\,,$$

and

$$\mathcal{B}\{M_n, N_n\}(\Omega)=\operatorname*{ind\,lim}_{R\to\infty}\mathcal{B}\{M_n, N_n\}_R(\Omega)\,.$$

All propositions mentioned on $\mathcal{B}\{M_n\}(\Omega)$ in Paragraph 2.2 rest valid also on $\mathcal{B}\{M_n,N_n\}(\Omega)$ under corresponding assumptions on $\{M_n\}$ and $\{N_n\}$.

2.2. *Fundamental properties of ultradifferential spaces*[1] *and Assumption.*

By Kolmogoroff's theorem, we can rearrange $\{M_n\}$ to a logarithmicly convex one for $\mathcal{B}\{M_n\}$ and $\mathcal{D}\{M_n\}$. On $\mathcal{E}\{M_n\}$, we can also replace $\{M_n\}$ by a logarithmicly convex one when $\liminf_{n\to\infty}(M_n/n!)^{1/n}>0$, which is satisfied if $\mathcal{E}\{M_n\}$ is analytic or rather non-quasianalytic. (See S. Mandelbrojt [28] Chap. VI and W. Rudin [36].)

In case of $\mathcal{D}_{L^2}\{M_n\}$, owing to the Schwarz inequality we can replace $\{M_n\}$ by a logarithmicly convex one.

Throughout this paper, we always assume the following;

Assumption. $\{M_n\}$ satisfies

$$\text{(A)} \qquad \liminf_{n\to\infty}(M_n/n!)^{1/n}>0\,.$$

(By virtue of (A), we can assume that $\{M_n\}$ is logarithmicly convex. Moreover, as a replacement of finite elements of $\{M_n\}$ does not change the ul.d. class, we can also assume that $\{M_n\}$ is non-decreasing. Hereafter, under Assumption, we always assume that $\{M_n\}$ is logarithmicly convex and non-decreasing.)

Remark. Under Assumption, $\mathcal{E}\{M_n\}$ includes the real analytic functions.

Sometimes, we shall introduce supplementarily the following;

$$\text{(B)} \qquad \exists\nu>0\,,\quad \forall n\gg 1\,,\quad \log(M_{n+1}M_{n-1}/M_n^2)\geqq\nu/n\,.$$

Under (B), it holds that

$$\text{(B)}_1 \qquad \forall\bar{R}>1\,,\quad \exists d\in\boldsymbol{Z}_+\backslash\{0\}\,,\quad \forall n\geqq 1\,,\quad M_{dn+1}/M_{dn}>\bar{R}M_{n+1}/M_n\,.$$

On the other hand, (A) and (B) imply

1) S. Mandelbrojt [28] systematically investigated the fundamental properties of the ul.d. classes from the point of view of the theory of real functions.

$(\mathrm{B})_2$ $\quad \exists n_0>0, \ \forall n\geqq 0, \ (M_{n+n_0+1}/M_{n+n_0})-(M_{n+1}/M_n)\geqq 1.$

We have the following under Assumption.

Proposition 2.1. (Algebra, division, composition, etc.)

i) *$\mathcal{B}\{M_n\}$ and $\mathcal{E}\{M_n\}$ are algebras over $\boldsymbol{C}$ and products of the elements in $\mathcal{E}\{M_n\}$ by the ones in $\mathcal{D}\{M_n\}$ (of the elements in $\mathcal{B}\{M_n\}$ by the ones in $\mathcal{D}_{L^2}\{M_n\}$, resp.) belong to $\mathcal{D}\{M_n\}$ (to $\mathcal{D}_{L^2}\{M_n\}$, resp.).*

ii) *If $\{M_n\}$ satisfies the condition.*

$$(\mathrm{R}) \qquad H\geqq 1, \ n\gg 1, \ (M_m/m!)^{1/m}\leqq H(M_n/n!)^{1/n}, \ (1\leqq \forall m\leqq n),$$

$\mathcal{E}\{M_n\}$ [$\mathcal{B}\{M_n\}$, resp.] is closed under the derivation by non-vanishing elements (by uniformly non-vanishing elements, resp.].

iii) *If $\{M_n\}$ satisfies the condition*

$$(\mathrm{K}) \qquad \exists H\geqq 1, \ \forall n\gg 1, \ (M_m/m!)^{1/(m-1)}\leqq H(M_n/n!)^{1/(n-1)}, \ (2\leqq \forall m\leqq n),$$

$\mathcal{E}\{M_n\}$ and $\mathcal{B}\{M_n\}$ are closed under the composition, solving ordinary differential equations and the implicit function theorem.

i) is easily seen. ii) was shown in W. Rudin [36] and iii) was done in H. Komatsu [23], [24] and [25]. On the composition, we need a little more precise form, which is implied in the proof of the above iii) by H. Komatsu [23].

Lemma 2.0.2. *Let $\{M_n\}$ and $\{N_n\}$ are logarithmicly convex. We assume that $\{N_n\}$ satisfies Condition $(L)_1$ introduced below and $R_0=\limsup_{n\to\infty}(N_n/M_n)^{1/n}<\infty$ and that $g(y)\in C^N(\Omega_1)$ and $f_i(x)\in C^N(\Omega_2)$ satisfy*

$$|g_{(\gamma)}(y)|\leqq C_1R_1^{|\gamma|}M_{|\gamma|} \qquad (0\leqq|\gamma|\leqq N, \ \Omega_1\subset \boldsymbol{R}^m),$$
$$|f_{i(\alpha)}(x)|\leqq CR^{|\alpha|}N_{|\alpha|} \qquad (1\leqq|\alpha|\leqq N, \ 1\leqq i\leqq m, \ \Omega_2\subset \boldsymbol{R}^l).$$

If the range of $(f_i)_{1\leqq i\leqq m}$ is included in Ω_1, we have the following estimate.

$$(2.2) \qquad |(g\circ f)_{(\alpha)}|\leqq \begin{cases} C_1 & (\alpha=0), \\ C'(2R'R)^{|\alpha|}M_{|\alpha|} & (1\leqq|\alpha|\leqq N), \end{cases}$$

where $C'=mC_1R_1M_1CR_0/R'$ and $R'=HR_0+mCR_1$.

We introduce the following conditions for $k\in \boldsymbol{Z}_+$;

$(\mathrm{L})_k$ $\quad \exists C>0, \ \exists H\geqq 1, \ \forall n\gg 1,$
$$(M_{j+k}/j!)(M_{n-j+k}/(n-j)!)\leqq CH^nM_{n+k}/n!, \qquad (1\leqq \forall j\leqq n),$$

$(\mathrm{A.I})_k$ $\quad \exists H\geqq 1, \ \forall n\gg 1,$
$$(M_{m+k}/m!)^{1/m}\leqq H(M_{n+k}/n!)^{1/n}, \qquad (1\leqq \forall m\leqq n),$$

$(\mathrm{A.C})$ $\quad \exists H\geqq 1, \ \forall n\gg 1,$
$$M_m/(mM_{m-1})\leqq HM_n/(nM_{n-1}), \qquad (1\leqq \forall m\leqq n).$$

$(\mathrm{A.I})_k$ means that $A_n=(M_{n+k}/n!)^{1/n}$ is almost increasing. In fact, if $H=1$, $\{A_n\}$ is increasing. (A.C) means that $A_n=M_n/n!$ is almost logarithmicly convex. In fact, if $H=1$, $\{A_n\}$ is logarithmicly convex. (R) and (K) are equivalent to $(\mathrm{A.I})_0$ and $(\mathrm{A.I})_1$, respectively. $(\mathrm{A.I})_k$ implies $(\mathrm{L})_k$ and (A.C) with $H=1$ (i.e. $\{M_n/n!\}$ is logarithmicly convex) implies $(\mathrm{L})_k$ for every k in $\boldsymbol{Z}_+$. We shall assume $(\mathrm{L})_k$ in Theorem 5.3 and Corollary 5.4 for suitable k depending on the dimension l of x.

Throughout this paper, we use often cut-off functions. Therefore, we need introduce the non-quasianalyticity condition:

$$\text{(N.Q.A)} \qquad \sum_{n=0}^{\infty} M_n/M_{n+1}<\infty\,.$$

If and only if $\{M_n\}$ satisfies Condition (N.Q.A), $\mathcal{E}\{M_n\}$, $\mathcal{B}\{M_n\}$ and $\mathcal{D}\{M_n\}$ are not quasianalytic and $\mathcal{D}\{M_n\}$ contains non-zero elements.

Remark. Assumption follows from Condition (N.Q.A). (See W. Rudin [36].)

In the theory of ps.d.op's of ul.d. class, we wish a cut-off function of analytic class. Of course, it cannot exist. Therefore, we use a sequence of cut-off functions.

Lemma 2.0.3. *We take a logarithmicly convex sequence of positive numbers $\{L_n\}$ which satisfies Non-quasianalytic Condition (N.Q.A). There exist a sequence of cut-off functions $\{\psi_k(t)\}$ and two positive constants C and R independent of k and j, such that, for arbitrary k in $\boldsymbol{Z}_+$*

$$(2.3) \qquad \begin{cases} |D_x^j\psi_k(t)|\leqq \begin{cases} CR^jk^j & (0\leqq j\leqq k)\,, \\ CR^jL_j & (j\in \boldsymbol{Z}_+)\,, \end{cases} \\ \psi_k(t)=0\ \ (t\leqq 0)\,, \quad =1\ \ (t\geqq 1) \quad \text{and} \quad 0\leqq\psi_k(t)\leqq 1\,. \end{cases}$$

Proof. Let us take a non-negative function ϕ in $\mathcal{D}\{L_n\}$ which satisfies $\operatorname{supp}\phi\subset\{|t|\leqq 1\}$ and $\int\phi(t)\,dt=1$ and the characteristic function χ of $\{t\geqq 1/2\}$. We set $\psi_k(t)=\chi(t)*\phi(4t)*\underbrace{\phi(4kt)*\cdots*\phi(4kt)}_{k}$. Since $D_t^j\psi_k(t)$ is expressed by $(4k)^j\chi(t)*\phi(4t)*\underbrace{\phi'(4kt)*\cdots*\phi'(4kt)}_{j}*\underbrace{\phi(4kt)*\cdots*\phi(4kt)}_{k-j}$ for $j\leqq k$ and by $4^j\chi(t)*\phi_{(j)}(4t)$ $*\phi(4kt)*\cdots*\phi(4kt)$ for general j, the above properties are easily seen. Q.E.D.

Remark. The following inequality holds for k and j in $\boldsymbol{Z}_+$; $k^j\leqq e^kj!$.

2.3. *Associated function and Fourier images of elements in $\mathcal{D}_{L^2}\{M_n\}$ and in $\mathcal{D}_{L^2}'\{M_n\}$.*

Under Assumption, $\{M_n\}$ satisfies $\lim_{n\to\infty}(M_n)^{1/n}=\infty$ and it can be assumed that $\{M_n\}$ is logarithmicly convex. Therefore, setting $a_n=\log M_n$, $\{(n,a_n)\}$ forms a convex polygon with infinite sides. This is called the Newton polygon of $\{a_n\}$. Moreover, the following functions are well defined:

$$\left\{\begin{array}{ll} T(r) = \sup_{n\geqq 0} r^n/M_n & (r>0)\,, \\ H(t) = \sup_{n\geqq 0} \{nt - a_n\}\,. \end{array}\right. \tag{2.4}$$

The former is called the associated function of $\{M_n\}$ and the latter is done the trace function of $\{a_n\}$. $H(t)$ is increasing, convex and piecewise linear, so it has the right derivatives. We set

$$h(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_r H(t): \text{ the right derivative of } H(t)\,.$$

$h(t)$ is an increasing and diverging $\boldsymbol{Z}_+$-valued step function. The following relations hold;

$$T(r) = \exp H(\log r),\quad \{r\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)_r T(r)\}/T(r) = h(\log r)\,.$$

$T(r)$ and $H(t)$ are given as the maximums at $n=h(\log r)$ and at $n=h(t)$, respectively. Then, the following equality holds;

$$H(t) = t\ h(t) - a_{h(t)} \qquad (t \in \boldsymbol{R})\,. \tag{2.5}$$

These facts above rest also valid if we replace $h(t)$ by the left derivative of $H(t)$. We remark that $T(r)$ diverges more rapidly than any polynomial.

By virtue of Assumption, the following relations also hold;

$$M_n = \sup_{r>0} r^n/T(r)\,,\quad a_n = \sup_t \{nt - H(t)\}\,. \tag{2.6}$$

Two supremums are given as the maximums at r in $[M_n/M_{n-1}, M_{n+1}/M_n]$ and at t in $[a_n - a_{n-1}, a_{n+1} - a_n]$, respectively. Then, the following holds;

$$M_n = n(a_{n+1} - a_n) - H(a_{n+1} - a_n)\,. \tag{2.7}$$

If we set $a'_x = a_{n+1} - a_n$ for $n \leqq x \leqq n+1$ and allow multivalues at n in $\boldsymbol{Z}_+$ and if we allow multivalues at the points of discontinuity for $H(t)$, $n=h(t)$ is the inverse function of $t=a'_x$.

We say that $\{\tilde{M}_n\}$ ($\tilde{T}(r)$, resp.) is equivalent to $\{M_n\}$ ($T(r)$, resp) when there exists positive constants R_1, R_2, C_1 and C_2 such that $C_1 R_1^n M_n \leqq \tilde{M}_n \leqq C_2 R_2^n M_n$ ($C_1 T(R_1 r) \leqq \tilde{T}(r) \leqq C_2 T(R_2 r)$, resp.) If we change finite elements of $\{M_n\}$ ($T(r)$ on a bounded set, resp.), it is equivalent to the original one. Therefore, in the conditions on $\{M_n\}$ (on $T(r)$, resp.), we may always remove the restriction $n \gg 1$ ($r \gg 1$, resp.). We shall often use this fact without notice.

Let us set $\mathcal{F}[X] = \{\hat{f}(\xi); f \in X\}$ and $L^2[w(\xi)] = \{g(\xi):$ measurable and $g(\xi)w(\xi) \in L^2(\boldsymbol{R}^l)\}$. The following proposition is easily obtained.

Proposition 2.2.

i) $\mathcal{F}[\mathcal{D}_{L^2}\{M_n\}] = \operatorname{ind\,lim}_{R\to\infty} L^2[T(\langle\xi\rangle/R)]\,.$

ii) $\mathcal{F}[\mathcal{D}_{L^2}'\{M_n\}] = \operatorname{proj}\lim_{R\to\infty} L^2[T(\langle\xi\rangle/R)^{-1}]$.

Remark. As $T(r)$ diverges more rapidly than arbitrary polynomials, $\mathcal{D}_{L^2}\{M_n\}$ is strictly smaller than $\mathcal{D}_{L^2}^\infty(\boldsymbol{R}^l)$ and $\mathcal{D}_{L^2}'\{M_n\}$ is strictly larger than $\mathcal{D}_{L^2}'(\boldsymbol{R}^l)$.

2.4. *Differentiability and separativity.*

The ul.d. spaces are classified the differentiability and the separativity. We denote $a_n=O(g(n))$ or $a_n=o(g(n))$ according as $\limsup_{n\to\infty} a_n/g(n)<\infty$ or $\lim_{n\to\infty} a_n/g(n)=0$.

Differentiability Condition will play an essential role when we shall consider in Paragraph 4.3 whether $P(x,D)$ in $\boldsymbol{S}^{-\infty}[M_n]$ is a regularizer. It will also often appear in order to make the statements of theorems simple.

Proposition 2.3. (Differentiability.)
The following statements are equivalent.

(D.0) $\mathcal{B}\{M_n\}$ *is differentiable, that is,*
$f(x)\in\mathcal{B}\{M_n\}\Rightarrow\forall\alpha\in\boldsymbol{Z}_+^l, f_{(\alpha)}(x)\in\mathcal{B}\{M_n\}$.

(D.1) $\exists H>1,\quad \forall n\gg 1,\quad M_{n+1}\leqq H^n M_n$.

(D.2) $\exists H>1,\quad \forall n\gg 1,\quad (M_{n+1})^{1/(n+1)}\leqq H(M_n)^{1/n}$.

(D.3) $\log M_n = O(n^2)$.

(D.4) $\log(M_{n+1}/M_n) = O(n)$.

(D.5) $\exists\kappa>0,\quad \forall r\gg 1,\quad T(r)\geqq r^{\kappa\log r}$.

(D.5′) $\liminf_{t\to\infty} H(t)/t^2>0$.

(D.6) $\liminf_{t\to\infty} h(t)/t>0$.

(D.7) $\forall m\in\boldsymbol{Z}_+,\quad \exists H=H(m)>1,\quad \forall r\gg 1,\quad T(r)\geqq r^m T(r/H)$.

Remark. The following condition;

(D)$_s$ $\exists\bar{H}>1,\quad \forall n\gg 1,\quad M_{n+1}M_{n-1}/M_n^2\leqq\bar{H}$,

implies (D.1), but the converse is not always true.

We shall use the notation (D) on behalf of (D.j), $0\leqq j\leqq 7$. The above proposition was shown in S. Mandelbrojt [28], Chap. VI except the equivalence between (D.1) and (D.2). This equivalence and the assertion in the above remark are easily seen.

Proposition 2.4. (Weak separativity.)
The following statements are equivalent.

(W.S.0) $\mathcal{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})$ *is weakly separative. that is,*
$\exists\{N_n\}(N_n>0),\quad \mathcal{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})\subseteqq\mathcal{B}\{M_n, N_n\}(\boldsymbol{R}^{l_1}\times\boldsymbol{R}^{l_2})$.

(W.S.1) $\exists H>1,\quad \exists\{N_n\}(N_n>0),\quad \forall n, m\gg 1,\quad M_{n+m}\leqq H^{n+m}M_nN_m$.

(W.S.2) $\forall m>0, \quad \lim_{n\to\infty} (M_{n+m}/M_n)^{1/n} = 1\,.$

(W.S.3) $\log M_n = o(n^2)\,.$

(W.S.4) $\log (M_{n+1}/M_n) = o(n)\,.$

(W.S.5) $\forall \kappa>0, \quad \exists r_0>0, \quad \forall r\geqq r_0, \quad T(r)\geqq r^{\kappa \log r}.$

(W.S.5′) $\lim_{t\to\infty} H(t)/t^2 = \infty\,.$

(W.S.6) $\exists H>1, \quad \forall m>0, \quad \forall r\gg 1, \quad T(r)\geqq r^m T(r/H)\,.$

Remark. The following condition:

$(\mathrm{W.S})_s \quad \lim_{n\to\infty} M_{n+1}M_{n-1}/M_n^2 = 1\,,$

implies (W.S.1) but the converse is not always true. (This is easily seen.)

We shall denote (W.S.j), $0\leqq j\leqq 6$, by (W.S).

Separativity Condition introduced below play an essential role when we shall consider in Paragraph 4.3 whether $P(x, D)$ in $S^{-\infty}[M_n]$ is a strong regularizer. It will also have an essential role when we shall consider the construction of true symbols from formal symbols in sense of $[M_n]$ in §6.

Proposition 2.5. (Separativity.)
The following statements are equivalent.

(S.0) $\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})$ *is separative, that is,*

$$\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2}) = \mathscr{B}\{M_n, M_n\}(\boldsymbol{R}^{l_1+l_2})\,.$$

(S.1) $\exists H>1, \quad \forall n, m\gg 1, \quad M_{n+m}\leqq H^{n+m}M_nM_m\,.$

(S.2) $\exists H>1, \quad \forall n\gg 1, \quad (M_{2n})^{1/2n}\leqq H(M_n)^{1/n}\,.$

(S.3) $\exists H>1, \quad \forall n\gg 1, \quad M_{2n+1}/M_{2n}\leqq HM_{n+1}/M_n\,.$

(S.4) $\exists H>1, \quad \forall n\gg 1, \quad M_{n+1}/M_n\leqq H(M_n)^{1/n}\,.$

(S.5) $\exists H>1, \quad \forall n, m\gg 1, \quad \forall k\leqq n, \quad M_{k+m}\leqq H^{k+m}(M_{n+1}/M_n)^k M_m\,.$

(S.6) $\exists H>1, \quad \forall r\gg 1, \quad T(r)\geqq T(r/H)^2\,.$

We shall denote (S.j), $0\leqq j\leqq 6$, by (S).

Proposition 2.6. *Under Separativity Condition* (S), *the following equivalent conditions are satisfied. However, the converse is not always true.*

(S.7) $\exists \nu>0, \quad \forall n\gg 1, \quad M_n\leqq n!^{\nu}\,.$

(S.8) $\exists \nu>0, \quad \forall n\gg 1, \quad M_{n+1}/M_n\leqq n^{\nu}\,.$

(S.9) $\exists \kappa>0, \quad \forall r\gg 1, \quad T(r)\geqq \exp r^{\kappa}.$

Remark. By (S.7), each separative class is a subspace of a Gevrey class or a Gevrey class itself.

Condition (S.1) is called "stability under ultradifferential operators" in H. Komatsu [22], etc. On the other hand, L.R. Volevič said that $\{M_n\}$ is "admissible" in [41] when it satisfies (S.1). The proofs of Propositions 2.4, 2.5 and 2.6 were given in W. Matsumoto [31] except the equivalence between (S.2), (S.3), (S.4) and (S.5). The equivalence not yet proved will be shown in Appendix B. In the propositions in this paragraph, $\mathcal{B}\{M_n\}$ can be replaced by $\mathcal{E}\{M_n\}$ under Assumption. It can be also replaced by $\mathcal{D}\{M_n\}$ under (N.Q.A).

Now, we announce a proposition on inclusion.

Proposition 2.7.

i) $\mathcal{D}\{M_n\} \subset \mathcal{B}\{M_n\} \subset \mathcal{E}\{M_n\}$ *and* $\mathcal{D}\{M_n\} \subset \mathcal{D}_{L^2}\{M_n\}$.

The inclusion maps are all continuous. Under Condition (N.Q.A), $\mathcal{D}\{M_n\}$ is dense in $\mathcal{D}_{L^2}\{M_n\}$ and in $\mathcal{E}\{M_n\}$.

ii) *Under the differentiability condition (D) the following holds;*

$$\mathcal{D}\{M_n\} \subset \mathcal{D}_{L^2}\{M_n\} \subset \mathcal{B}\{M_n\} \subset \mathcal{E}\{M_n\}.$$

The inclusion maps are all continuous.

2.5. *Carleson's theorem.*

The following proposition will be applied in Paragraph 3.2. It was also applied· in L. Botet de Monvel and P. Krée [7] and L. Boutet de Monvel [6] when they constructed true symbols from formal symbols in sense of Gevrey classes.

Proposition 2.8. (L. Carleson [10])
Suppose Conditions (N.Q.A), (D) and the following

(C) $\exists R_0 \geqq 1, \quad \exists C_0 \geqq 0, \quad \forall r \gg 1,$

$$(2/\pi)\int_0^\infty \log T(rs)/(1+s^2)\,\mathrm{d}s \leqq \log T(R_0 r) + C_0\,.$$

Take an arbitrary sequence $(c_n)_{n=0}^\infty$ which satisfies

$$|c_n| \leqq C R^n M_n \qquad (n \geqq 0)\,,$$

for some $C>0$ and $R>0$. Then, we can find a function $g(t)$ in $\mathcal{B}\{M_n\}(\boldsymbol{R})$ such that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n g(0) = c_n\,, \qquad (n \geqq 0)\,.$$

Remark 1. Even if $\{M_n\}$ does not satisfy Condition (*D*) we can obtain the same result replacing Condition (C) by

$$(\mathrm{C}') \quad \begin{cases} \text{There exists a function } \tilde{T}(r) \text{ equivalent to } T(r) \text{ such that } \exists R_0 \geqq 1,\ \exists C_0 \geqq 0, \\ \exists \varepsilon > 0\,, \\ (2/\pi)\displaystyle\int_0^\infty \log \tilde{T}(rs)/(1+s^2)\,\mathrm{d}s + \{(1+\varepsilon)/2\}\log\log r \leqq \log T(R_0 r) + C_0\,, \\ \hfill (r \gg 1)\,, \\ \text{and} \quad \log \tilde{T}(e^t) + t/2 + \{(1+\varepsilon)/2\}\log t \quad \text{is convex for } t \gg 1\,. \end{cases}$$

Remark 2. The integral in Conditions (C) and (C′) converges if and only if $\{M_n\}$ satisfies Non-quasianalyticity condition (N.Q.A).

2.6. *Extension of* $\{M_n\}$ *on* $\boldsymbol{R}_+$.

As we consider $\boldsymbol{S}_{\rho\delta}$ on $\mathscr{B}\{M_n\}$, it is convenient to use non-integral numbers as the index n, for example, $n=k/(1-\delta)$ ($k\in\boldsymbol{Z}_+$, $0<\delta<1$). Although we can develop a theory of ps.d.op's using $[k/(1-\delta)]$: the integral part of $k/(1-\delta)$, $\{M_{[k/(1-\delta)]}\}$ is no longer logarithmicly convex, in general. In order to make clear the statements of theorems, we extend $\{M_n\}$ to a logarithmicly convex positive continuous function M_x ($x\in\boldsymbol{R}_+$). Of course, extension is not unique. Therefore, we chose an extension at the begining and we regard $\{M_n\}$ as the restriction of M_x on $\boldsymbol{Z}_+$.

A typical extension is the logarithmicly linear interpolation:

$$M_x = (M_n)^s(M_{n+1})^{1-s} \qquad (x=[x]+s \quad \text{and} \quad n<x<n+1)\,.$$

The graph of $(x, \log M_x)$ coincides with the Newton polygon of $\{\log M_n\}$. We set $a_x=\log M_x$. $\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)_r a_x$ coincides with a'_x introduced in Paragraph 2.3. Choosing this, the properties mentioned up to now on $\{M_n\}$ rest valide replacing n by x. Here, we must pay attension to the following; Let us take $\lambda>0$ and set $T_\lambda(r)=\sup_n r^n M_{n/\lambda}$ ($r>0$). Only the following relation holds good

$$T_\lambda(r)\leqq T(r^\lambda)\,.$$

However, as the Newton polygon of $\{\log M_{[n/\lambda]+1}\}$ stay in the upper side of that of $\{\log M_{n/\lambda}\}$, we have the relation:

$$T_\lambda(r)\geqq r^{-\lambda}T(r^\lambda)\,.$$

If $\{M_n\}$ satisfies Differentiability Condition (D), it holds that

$$T_\lambda(r)\geqq T(r^\lambda/H) \qquad (\exists H>1)\,.$$

On the other hand, in many cases, there is another natural extension. For example, $M_x=(x/\mathrm{e})^{\nu x}$ ($x\geqq 1$) gives a natural extension of $(n/\mathrm{e})^{\nu n}$, which is equivalent to $\{n!^\nu\}$ (the Gevrey class of order ν). If we adopt an extension which is not the logarithmicly linear interpolation,

$$\tilde{T}(r)=\sup_{x\geqq 0} r^x/M_x \ \ (r>0) \quad \text{and} \quad \tilde{H}(t)=\sup_{x\geqq 0}\{xt-a_x\}$$

do not coincide with $T(r)$ and $H(t)$, respectively ($a_x=\log M_x$). Nevertheless, under Differentiability Condition (D), $\tilde{T}(r)$ and $\tilde{H}(t)$ are equivalent to $T(r)$ and $H(t)$, respectively. Under a natural extension, sometimes it becomes very easy to obtain $\tilde{T}(r)$ and $\tilde{H}(t)$. After an extension, the relation between a_x and $H(t)$ is completely symmetric.

In this paper, we adopt always the logarithmicly linear interpolation. If Condition (D) is satisfied, we can replace it by another arbitrary extension.

We have the following lemma of the fractional derivatives.

Lemma 2.0.4. (1) *If* $\sup_x |f_{(\alpha)}(x)| \leqq CR^{|\alpha|}M_{|\alpha|}$ *for* $|\alpha|=n$ *and* $=n+1$, *the following holds for* $0\leqq\mu\leqq 1$;

$$\sup_{x,y} |f_{(\alpha)}(x+y)-f_{(\alpha)}(x)|/|y|^{\mu} \leqq 2^{1-\mu}CR^{|\alpha|+\mu}M_{|\alpha|+\mu}(|\alpha|=n)\,.$$

(2) *If* $\|f_{(\alpha)}(x)\|_{L^2} \leqq CR^{|\alpha|}M_{|\alpha|}$ *for* $|\alpha|=n$ *and* $=n+1$, *the following holds for* $0\leqq\mu\leqq 1$;

$$\|\langle\xi\rangle^{n+\mu}f(\xi)\|_{L^2} \leqq C\{(l+1)R\}^{n+\mu}M_{n+\mu}\,.$$

We shall often use this lemma without notice.

2.7. *Examples of* $\{M_n\}$.

In order to make clear the meaning of the conditions, we give some typical examples.

Example 1. $M_n^{(1)}(\nu, \mu)=\{n^{\nu}(\log n)^{\mu}\}^n$, $(n\gg 1,\ \nu>0,\ \mu\in\boldsymbol{R})$.
If $\mu=0$, this gives the Gevrey class of order ν.

Example 2. $M_n^{(2)}(\kappa, a, \nu)=n!^{\nu}\exp(an^{\kappa})$, $(n\gg 1,\ \kappa>1,\ a>0,\ \nu\in\boldsymbol{R})$.
$\mathcal{E}\{M_n^{(2)}(\kappa, a, 0)\}$ appeared in W. Matsumoto [29] and [30] and $\mathcal{E}\{M_n^{(2)}(2, a, 1)\}$ did in W. Matsumoto [32].

Example 3. $M_n^{(3)}(a, b, \nu)=n!^{\nu}\exp\{b\exp(an)\}$, $(n\gg 1,\ a,\ b>0,\ \nu\in\boldsymbol{R})$.

All of them satisfy Conditions (B), $(\text{A.I})_k$ ($k\in\boldsymbol{Z}_+$, then, of course, (R) and (K)) and (A.C). Here, we can take $H=1$. $\{M_n^{(1)}\}$ ($\nu\geqq 1$) and all of $\{M_n^{(2)}\}$ and $\{M_n^{(3)}\}$ satisfy Assumption. On the other conditions, we show when they are satisfied in the tables below.

Table 1.

	(N.Q.A)	(D) $(D)_s$	$(W.S)$ $(W.S)_s$	(S)
$M_n^{(1)}(\nu, \mu)$	$\nu>1$ or $\nu=1$ and $\mu>1$	all	all	all
$M_n^{(2)}(\kappa, a, \nu)$	all	$\kappa\leqq 2$	$\kappa<2$	nothing
$M_n^{(3)}(a, b, \nu)$	all	nothing	nothing	nothing

Table 2.

	$\tilde{T}(r)$ $(r\gg 1)$	(C) or (C')	(C^*)
$M_n^{(1)}(\nu, \mu)$	$\exp\{r^{1/\nu}(\log r)^{-\mu/\nu}\}$	$\nu>1$	$\nu>3/2$
$M_n^{(2)}(\kappa, a, 0)$	$\exp\{a^*(\log r)^{\kappa^*}\}$	all	all
$M_n^{(3)}(a, b, 0)$	$r^{a^{-1}}\{\log\log r-\log(abe)\}$	all	all

$((1/\kappa+1/\kappa^*)=1$, $(a\kappa)^{1/\kappa}\times(a^*\kappa^*)^{1/\kappa^*}=1$, Condition (C^*) will be introduced in §6.)

Remark. $\tilde{T}(r)$ is an equivalent function to $T(r)$. On $\{M_n^{(2)}(\kappa, a, 0)\}$ for $\kappa>3$ and on all $\{M_n^{(3)}(a, b, 0)\}$, $\tilde{T}(r)$ satisfies only $T(r)\leqq\tilde{T}(r)$ and $\sup_r r^n/\tilde{T}(r)=M_n$.

We give an available lemma to show the non-equivalence of classes.

Lemma 2.0.5. *Let $\{A_n\}$ and $\{B_n\}$ are two sequences of positive numbers. Suppose that $\{A_n\}$ is logarithmicly convex and that $\lim_{n\to\infty}(A_n)^{1/n}=\infty$ and $\liminf_{n\to\infty}(B_n/A_n)^{1/n}=0$. Then, there exists a periodic function in $\mathcal{B}\{A_n\}(\boldsymbol{R})$ which does not belong to $\mathcal{B}\{B_n\}(\boldsymbol{R})$.*

This lemma was given, for example, in S. Mandelbrojt [28] Chap. VI.

2.8. *Definition of pseudo-differential operators of class C^∞.*

We can consider the theory of ps.d.op.'s on a manifold. However, in this paper, in order to make clear our assertion, we consider it in $\boldsymbol{R}^l$ and stand on that of class $\mathcal{B}$ developed by H. Kumano-go [27]. We denote the space of the symbols $p(x, \xi)$ of the ps.d.op's of class C^∞, of order $m\,(\in\boldsymbol{R})$ and with $0\leqq\delta\leqq\rho\leqq1$ and $\delta<1$ in Hörmander's sense by $\mathrm{S}^m_{\rho\delta}$ that is,

$$
(2.8)\qquad \left\{\begin{array}{l} p(x,\xi)\in\mathrm{S}^m_{\rho\delta} \overset{\text{def}}{\Leftrightarrow} \forall\alpha,\beta\in\boldsymbol{Z}^l_+,\ \exists C(\alpha,\beta)>0\,, \\ \qquad |p^{(\beta)}_{(\alpha)}(x,\xi)|\leqq C\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|} \quad \text{in } \boldsymbol{R}^{2l}\,. \end{array}\right.
$$

We set $\mathrm{S}_{\rho\delta}=\bigcup_{m\in\boldsymbol{R}}\mathrm{S}^m_{\rho\delta}$.

We shall use the semi-norms $|p|^{(m)}_{j,k}=\sup_{\substack{x,\xi\\|\alpha|\leqq j,|\beta|\leqq k}}|p^{(\beta)}_{(\alpha)}(x,\xi)|\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|}$.

Remark. On $\mathrm{S}^{-\infty}$, we need only the following estimate

$$
(2.9)\qquad \forall N>0,\quad \forall\alpha,\beta\in\boldsymbol{Z}^l_+,\quad \exists C(N,\alpha,\beta)>0,\quad |p^{(\beta)}_{(\alpha)}(x,\xi)|\leqq C\langle\xi\rangle^{-N}\,.
$$

The ps.d.op. $P(x, D)$ with the symbol $p(x, \xi)$ is defined by

$$
(2.10)\quad P(x,D)u=\mathrm{Os}-\iint e^{\sqrt{-1}(x-y)\cdot\xi}p(x,\xi)u(y)\mathrm{d}y\,d\xi\,,\qquad (u\in\mathcal{B}(\boldsymbol{R}^l))\,.
$$

For $u\in\mathcal{D}^\infty_{L^2}(\boldsymbol{R}^l)$, it is represented by

$$
(2.10')\qquad P(x,D)u=\int e^{\sqrt{-1}x\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi\,.
$$

When P is continuous on $\mathcal{D}^\infty_{L^2}$, we define Pu for u in D'_{L^2} by the fllowing;

$$
\langle Pu,\phi\rangle=\langle u,\bar{P}^*\phi\rangle\qquad(\phi\in\mathcal{D}^\infty_{L^2}(\boldsymbol{R}^l))\,,
$$

where the symbol of $\bar{P}$ is the complex conjugate of that of P. We denote the space of ps.d.op's of order m by $\boldsymbol{S}^m_{\rho\delta}$ and set $\boldsymbol{S}_{\rho\delta}=\bigcup_{m\in\boldsymbol{R}}\boldsymbol{S}^m_{\rho\delta}$. Sometimes, we express the symbol of $P(x, D)$ by $\sigma(P(x, D))$.

We also denote the set of the formal sums $\sum_{i=0}^\infty p_i(x,\xi)$ with the order descent

$\rho-\delta$ by $\mathcal{S}^m_{\rho\delta}$, that is,

$$(2.11)\quad \begin{cases} \sum_{i=0}^{\infty} p_i(x, \xi) \in \mathcal{S}^m_{\rho\delta} \overset{\text{def}}{\Leftrightarrow} \forall i \in \boldsymbol{Z}_+ , \\ \qquad \forall \alpha, \beta \in \boldsymbol{Z}^l_+, \quad \exists r_0(i, \alpha, \beta) > 0, \quad \exists C(i, \alpha, \beta) > 0 , \\ \qquad |p_{i(\alpha)}^{(\beta)}(x, \xi)| \leqq C\langle\xi\rangle^{m-(\rho-\delta)i-\rho|\beta|+\delta|\alpha|} \\ \qquad\qquad \text{in} \quad \boldsymbol{R}^l \times \{\langle\xi\rangle \geqq r_0\} . \end{cases}$$

We set $\mathcal{S}_{\rho\delta} = \bigcup_{m\in\boldsymbol{R}} \mathcal{S}^m_{\rho\delta}$.

In $\mathcal{S}_{\rho\delta}$, we define "the operator product" and "the formal adjoint" as follows.

$$(2.12)\quad (\sum p_i)\circ(\sum q_j) = \sum_{k=0}^{\infty} r_k ,$$
$$r_k(x, \xi) = \sum_{i+j+|\gamma|=k} (\gamma!)^{-1} p_i^{(\gamma)}(x, \xi) q_{j(\gamma)}(x, \xi) ,$$

$$(2.13)\quad (\sum p_i)^* = \sum_{k=0}^{\infty} p_k^* ,$$
$$p_k^*(x, \xi) = \sum_{i+|\gamma|=k} (\gamma!)^{-1} \bar{p}_{i(\gamma)}^{(\gamma)}(x, \xi) .$$

We give the operator product as the product and the formal adjoint as the star operation to the $\boldsymbol{C}$-module $\mathcal{S}_{\rho\delta}$.

We say that the symbol $p(x, \xi)$ of order m[2] has an asymptotic expansion $\sum_{i=0}^{\infty} p_i(x, \xi)$ when the following is satisfied;

$$\forall N \in \boldsymbol{Z}_+, \quad \forall \alpha, \beta \in \boldsymbol{Z}^l_+, \quad \exists r_0(N, \alpha, \beta) > 0, \quad \exists C(N, \alpha, \beta) > 0 ,$$
$$(2.14)\quad |(p(x, \xi) - \sum_{i=0}^{N-1} p_i(x, \xi))_{(\alpha)}^{(\beta)}| \leqq C\langle\xi\rangle^{m-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|} \quad \text{in} \quad \boldsymbol{R}^l \times \{\langle\xi\rangle \geqq r_0\} .$$

We write the above relation by $p(x, \xi) \sim \sum p_i(x, \xi)$, and call $p(x, \xi)$ the true symbol and $\sum p_i(x, \xi)$ a formal symbol.

§3. Expected properties on pseudo-differential operators and properties on formal symbols of ultradifferentiable class (Property VI).

3.1. *Expected properties.*

In order to make simple the consideration of the possibility of a theory of ps.d.op's of ul.d. class, we restrict ourselves to an easily handled case. On the ul.d. spaces, some operators of infinite order are admited. In spite of this, the class of ps.d.op.'s of finite order is a star algebra so far as so is a wider one. Then, we consider only the class of finite order.

We settle our expectation in the following eight slogans. Of course, in applications, we use several of them depending on the problems and sometimes we

2) On the symbols, it seems better to say "of degree m" than "of order m". However, we use "order" not only on the operators but also on the symbols identifying both of them for simplicity.

wish other properties. However, we can say that the following eight are basic. We use the temporary notation introduced in §1. We also temporarily denote the relation of asymptotic expansion of class $[M_n]$ by $p(x, \xi) \underset{[M_n]}{\sim} \sum p_i(x, \xi)$.

I. *$\boldsymbol{S}[M_n]$, $\mathrm{S}[M_n]$ and $\mathcal{S}[M_n]$ are subsets of $\boldsymbol{S}_{\rho\delta}$, $\mathrm{S}_{\rho\delta}$ and $\mathcal{S}_{\rho\delta}$, respectively. The relation $p \underset{[M_n]}{\sim} \sum p_i$ implies $p \sim \sum p_i$.*

II. *$\boldsymbol{S}[M_n]$ contains all of the differential operators with coefficients in $\mathcal{B}\{M_n\}$.*

III. *Each ps.d.op. in $\boldsymbol{S}[M_n]$ is bounded on $\mathcal{D}_{L^2}\{N_n\}$ and on $\mathcal{B}\{N_n\}$ for suitable $\{N_n\}$.*

IV. *Each ps.d.op. in $\boldsymbol{S}^{-\infty}[M_n]$ is continuous from $\mathcal{D}'_{L^2}\{N_n\}$ to $\mathcal{D}_{L^2}\{N_n\}$ and from $\mathcal{E}'\{N_n\}$ to $\mathcal{E}\{N_n\}$ for a suitable $\{N_n\}$. (An operator which has the above property is called "a strong regularizer of class $\{N_n\}$".)*

(*or* IV'. *Each ps.d.op. in $\boldsymbol{S}^{-\infty}[M_n]$ is continuous from $\mathcal{D}'_{L^2}$ to $\mathcal{D}_{L^2}\{N_n\}$ and from $\mathcal{E}'$ to $\mathcal{E}\{N_n\}$ for a suitable $\{N_n\}$. (An operator which has the above property is called "a regularizer of class $\{N_n\}$".)*)

V. *$\boldsymbol{S}[M_n]$ is a star algebra over $\boldsymbol{C}$ (with or without modulo class $\boldsymbol{S}^{-\infty}[N_n]$) for a suitable $\{N_n\}$.*

VI. *$\mathcal{S}[M_n]$ is a star algebra over $\boldsymbol{C}$ with respect to the operator product.*

VII. *Every elliptic operator in $\boldsymbol{S}[M_n]$ has a parametrix in the same class.*

VIII. *For each $\sum p_i$ in $\mathcal{S}[M_n]$, there exists a true symbol in $\mathrm{S}[M_n]$ which satisfies $p \underset{[M_n]}{\sim} \sum p_i$.*

The ps.d.op's of C^∞ class satisfies all of the above properties removing $[M_n]$, $\{M_n\}$, and $\{N_n\}$.

First, we seek for a reasonable definitions of $\boldsymbol{S}[M_n]$ (that is, of $\mathrm{S}[M_n]$) and $\mathcal{S}[M_n]$. There are some possibilities and we want to choose a simple one.

3.2. *Definition of formal symbols and necessity of analytic estimate in ξ.*

We want to construct a theory which allows asymptotic expansions of symbols of arbitrary length. Then, we consider first the formal symbols.

Let $\{d_n\}_{n=0}$ be a non-decreasing sequence of positive numbers and $\{L_n\}$ be a positive and logarithmicly convex sequence with Non-quasianalytic Condition (N.Q.A).

Definition 3.1. We take real numbers $0 \leqq \delta \leqq \rho \leqq 1$, $\delta < 1$ and m.

$\sum p_i(x, \xi) \in \mathcal{S}^m_{\rho\delta}[M_n] \equiv \mathcal{S}^m_{\rho\delta}[M_n, L_n]$

$$\overset{\text{def}}{\Leftrightarrow} \quad \exists C>0, \ \exists R>0, \ \exists r_0>0, \ \forall \alpha, \beta \in \boldsymbol{Z}^l_+, \tag{3.1}$$
$$|p_{i(\alpha)}^{(\beta)}(x, \xi)| \leqq CR^{i+|\alpha+\beta|} M_{i+|\alpha|} \beta! \langle\xi\rangle^{m-(\rho-\delta)i-\rho|\beta|+\delta|\alpha|},$$
$$\text{for } (x, \xi) \in \boldsymbol{R}^l \times \{\langle\xi\rangle \geqq r_0 d_{i+|\beta|}\}.$$

$$\exists C>0, \ \exists R>0, \ \exists d'>0, \ \forall \alpha, \beta \in \boldsymbol{Z}^l_+, \tag{3.2}$$
$$|p_{i(\alpha)}^{(\beta)}(x, \xi)| \leqq CR^{i+|\alpha+\beta|} M_{i+|\alpha|} L_{i+|\beta|} \langle\xi\rangle^{m-(\rho-\delta)i-\rho|\beta|+\delta|\alpha|},$$
$$\text{for } (x, \xi) \in \boldsymbol{R}^l \times \{\langle\xi\rangle \geqq d'\}.$$

Remark. $\sum p_i(x, \xi)$ is a formal sum.

As far as we consider only the formal symbols, we can take $\{d_n\}$ arbitrarily because we use only the operations with the local property. On the other hand, in case of true symbols, the choice of $\{d_n\}$ becomes very important. Linked with the true symbols, we shall take $d_n=D_n^{1/\theta}=(M_{n+1}/M_n)^{1/\theta}$ and denote such space by $\mathcal{S}_{\rho\delta\theta}[M_n]$ $(0<\theta\leqq 1$ or $\theta=\infty)$. When $\theta=\infty$, we mean that $d_n=d_0(\forall n)$. In this case, $p_{i(\alpha)}$ is holomolphically extended in a conic neighbourhood of the real axes: $\{\zeta\in \boldsymbol{C}^l;\ |\mathrm{Im}\,\zeta|\leqq\varepsilon|\mathrm{Re}\,\zeta|^{\rho}$ and $|\zeta|\geqq r_0 d_0\}$ ($\exists\varepsilon>0$ independent of i) and it satisfies there

$$|p_{i(\alpha)}(x, \zeta)|\leqq CR^{i+|\alpha|}M_{i+|\alpha|}\langle\xi\rangle^{m-(\rho-\delta)i+\delta|\alpha|}\,, \qquad (\mathrm{Re}\,\zeta=\xi)\,. \tag{3.3}$$

We set $\mathcal{S}_{\rho\delta}[M_n]=\bigcup_{m\in\boldsymbol{R}}\mathcal{S}^m_{\rho\delta}[M_n]$.

Now, we give a comment on the analytic estimate on ξ. If $l=1$ and $p_i(x, \xi)$ is positively homogeneous in ξ, the analytic estimate on ξ of $p_i(x, \xi)$ is evident. In case of $l>1$, the requirement of the analytic estimate of the derivatives in ξ seems too strong. However, when $\{M_n\}$ satisfies Condition (C) and when $\{L_n\}$ satisfies (N.Q.A) and $\liminf_{n\to\infty}(n!/L_n)^{1/n}=0$, it is in general impossible to replace $\beta!$ in (3.1) by $L_{|\beta|}$ as long as we expect that $\mathcal{S}_{\rho\delta}[M_n]$ is closed under the operator product or under the star operation. In fact, in case of $\rho=1$ and $\delta=0$, we have the following counter-example.

First, we assume the closedness under the operator product. By Condition (C), there is a function $q(x_2)$ in $\mathcal{B}\{M_n\}(\boldsymbol{R})$ such that $q_{(n)}(0)=M_n$. (See Proposition 2.8.) Setting $N_{2n}=n!L_n$ and $N_{2n+1}=(N_{2n}N_{2n+2})^{1/2}$ $(n\geqq 0)$, $\liminf_{n\to\infty}(N_n/L_n)^{1/n}=0$ holds. Then, applying Lemma 2.0.5, there exists a periodic function $p(t)$ in $\mathcal{B}\{L_n\}(\boldsymbol{R})$ which does not belong to $\mathcal{B}\{N_n\}(\boldsymbol{R})$. We may assume that the period of $p(t)$ is one. Let $\chi(t)$ in $\mathcal{B}\{L_n\}(\boldsymbol{R})$ be

$$\chi(t)=\begin{cases}1 & 2\leqq t\leqq 3\,,\\ 0 & t\leqq 1 \quad\text{or}\quad t\geqq 4\,.\end{cases}$$

We set

$$p_0(\xi)=p\Big(\big(\sum_{k=2}^{l}\xi_k\big)/\xi_1\Big)\chi\Big(\sum_{k=2}^{l}(\xi_k^2/\xi_1^2)^{1/2}\Big)\,,$$
$$q_0(x)=q(x_2),\quad p_i(x, \xi)=q_i(x, \xi)=0 \qquad (i\geqq 1)\,.$$

Obviously, $p_0(x, \xi)$ and $q_0(x, \xi)$ are homogeneous of order 0 and satisfy (3.1) with $d_n=1$ $(\forall n\in\boldsymbol{Z}_+)$ replacing $\beta!$ by $L_{|\beta|}$ by virtue of Lemmas 2.0.1 and 2.0.2, because $\{n!\}$ satifies Condition (K).

Let $(\sum p_i)\circ(\sum q_i)$ be $\sum r_i(x, \xi)$. We take $x=O$, $\xi=(\rho, \rho t, 0, \cdots, 0)$, $\alpha=(0, m, 0, \cdots, 0)$ and $\beta=(0, n, 0, \cdots, 0)$, $(\rho>1,\ 2\leqq t\leqq 3)$. $r_{i(\alpha)}^{(\beta)}$ is given by

$$\begin{aligned} r_{i(\alpha)}^{(\beta)} &= (i!)^{-1}p^{(i+n)}(t)q_{(i+m)}(0)\rho^{-i-n}\\ &= \rho^{-i-n}(i!)^{-1}M_{i+m}p^{(i+n)}(t)\,. \end{aligned} \tag{3.4}$$

As $\sum r_i(x, \xi)$ satisfies (3.1) for some $C>0$ and $R>0$ replacing $\beta!$ by $L_{|\beta|}$, we have the following;

$$|p^{(i+n)}(t)| \leqq C(R/\sqrt{5})^{i+n} R^m i! L_n .$$

Taking $m=0$ and $i=n$, this implies

$$\sup_t |p^{(2n)}(t)| \leqq C(R/\sqrt{5})^{2n} N_{2n} . \tag{3.5}$$

Then, $p(t)$ belongs to $\mathcal{B}\{N_n\}$ by virtue of Kolmogoroff's theorem. Thus, we arrive at a contradiction. When we assume the closedness under the star operation, $r_0(x, \xi)=q_0(x)p_0(\xi)$ and $r_i(x, \xi)=0$ $(i \geqq 1)$ brings the same contradiction.

Remark. If we assume only (3.2), $\mathcal{S}_{\rho\delta}[M_n]$ becomes a star algebra. However, such class of formal symbols does not seem available in applications.

3.3. *Algebra of formal symbols (Property VI).*

We notice that $\mathcal{S}_{\rho\delta}[M_n]$ furnishes the operator product as the product. We can show the following theorem by the same way as L. Boutet de Monvel and P. Krée [7].

Theorem 3.1.

1) *$\mathcal{S}_{\rho\delta}[M_n]$ is a star algebra over $\boldsymbol{C}$, that is, when $\sum p_i$ and $\sum q_j$ belong to $\mathcal{S}^{m_1}_{\rho\delta}[M_n]$ and to $\mathcal{S}^{m_2}_{\rho\delta}[M_n]$, respectively, $(\sum p_i)\circ(\sum q_j)$ and $(\sum p_i)^*$ do to $\mathcal{S}^{m_1+m_2}_{\rho\delta}[M_n]$ and $\mathcal{S}^{m_1}_{\rho\delta}[M_n]$, respectively. (See* (2.12) *and* (2.13).)

2) *If a square matrix of formal symbol $\sum p_i$ in $\mathcal{S}^{m}_{\rho\delta}[M_n]$ is elliptic, that is,*

$$|\det\{p_0(x, \xi)/\langle\xi\rangle^m\}| \geqq C_0 \quad \text{in} \quad \bar{\Omega}\times\{\xi\in\bar{\Gamma}\,;\, \langle\xi\rangle \geqq R_0\},$$

for some positive constants C_0 and R_0 and for an open set Ω and an open conic set Γ in $\boldsymbol{R}^l$, there exists the inverse in $\mathcal{S}^{-m}_{\rho\delta}[M_n]$ on $\bar{\Omega}\times\{\xi\in\bar{\Gamma}\,;\, \langle\xi\rangle \geqq R_0\}$.

The structure of $\mathcal{S}_{\rho\delta}[M_n]$ is rather simpler than that of $\boldsymbol{S}_{\rho\delta\theta}[M_n]$ because all operations in $\mathcal{S}_{\rho\delta}[M_n]$ have the local property. On the formal symbols of class $[M_n]$, more profound results have been obtained. A remarkable one is "the perfect decomposition of formal symbol". T. Nishitani [35] proved it in the Gevrey classes. His proof holds good in the general class $[M_n]$ under Condition $(L)_0$ with $H=1$.

§4. Definition, continuity (Properties III and IV) and pseudo-local property of pseudo-differential operators of ultradifferentiable class.

4.1. *Definitions of symbols of pseudo-differential operators and of asymptotic expansions of them in class $[M_n]$.*

We regard $p_0(\xi)$ and $q_0(x)$ in Paragraph 3.2 as true symbols and assume the asymptotic expansion (4.3) introduced below replacing $\beta!$ by $L_{|\beta|}$. If $\{M_n\}$ satisfies Condition (D), (4.3) for N and $N+1$ implies (3.1) for N. Thus, if we want a theory of ps.d.op's of ul.d. class which allows the asymptotic expansions of symbols of arbitrary length and the star algebraic structure, the analytic estimate in ξ is indispensable also on true symbols. (See also (5.20) and (5.21) in Theorem 5.2.). On the other hand, as explained in § 1, we cannot adopt the process through the formal symbols in order to see the structure of the space of ps.d.op's in a ul.d. class without Condition (S).

We start from the formula $(*)$ in § 1. However, the derivatives of the integral on η near $\xi+\eta\sim O$ have not the estimates of $S_{\rho\delta}$ sense. (See Paragraph 5.2.) In order to treat this part separately, we wish a cut-off function depending on ξ. To make this cut-off consistent with the analytic estimate in ξ, we introduce the "pseudo-analytic" estimate.

Let us take d_n in Definition 3.1 for $(D_n)^{1/\theta}$, where $0<\theta\leqq 1$ or $\theta=\infty$ and $D_n=M_{n+1}/M_n$. We introduce supplementarily a positive and logarithmicly convex sequence $\{L_n\}$ with Non-quasianalytic Condition (N.Q.A).

Definition 4.1. We take $0\leqq\delta\leqq\rho\leqq 1$, $\delta<1$, $0\leqq\theta\leqq 1$ or $\theta=\infty$ and $m\in\boldsymbol{R}$.

(i) $p(x, \xi)\in S^m_{\rho\delta\theta}[M_n]\ (=S^m_{\rho\delta\theta}[M_n, L_n]) \overset{\text{def}}{\Leftrightarrow} {}^\exists C,\ R>0,\quad {}^\forall\alpha,\ \beta\in\boldsymbol{Z}^l_+$,

$$(4.1)\qquad |p^{(\beta)}_{(\alpha)}(x, \xi)|\leq CR^{|\alpha+\beta|}M_{|\alpha|}\beta!\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|},\quad \text{for } (x, \xi)\in\boldsymbol{R}^l\times\{\langle\xi\rangle^\theta\geqq RD_{|\beta|}\},$$

$$(4.2)\qquad |p^{(\beta)}_{(\alpha)}(x, \xi)|\leq CR^{|\alpha+\beta|}M_{|\alpha|}L_{|\beta|}\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|},\qquad \text{for } (x, \xi)\in\boldsymbol{R}^l\times\boldsymbol{R}^l.$$

(ii) $S_{\rho\delta\theta}[M_n]=\bigcup_{m\in\boldsymbol{R}} S^m_{\rho\delta\theta}[M_n]$.

(iii) $\boldsymbol{S}^m_{\rho\delta\theta}[M_n]=\{P(x, D)\in\boldsymbol{S}_{\rho\delta};\ \sigma(P)\in S^m_{\rho\delta\theta}[M_n]\}$ and $\boldsymbol{S}_{\rho\delta\theta}[M_n]=\bigcup_{m\in\boldsymbol{R}}\boldsymbol{S}^m_{\rho\delta\theta}[M_n]$.

Remark 1. In case of $\theta=\infty$, we mean that (4.1) holds for $\langle\xi\rangle\geqq{}^\exists R_0$, where R_0 is independent of β. Hence, $p_{(\alpha)}$ is holomorphicly extended in a conic neighbourhood of the real axes: $\{\zeta\in\boldsymbol{C}^l;\ |\mathrm{Im}\,\zeta|<\varepsilon|\mathrm{Re}\,\zeta|^\rho,\ |\zeta|\geqq R_0\}$ $({}^\exists\varepsilon>0)$ and satisfies there

$$(4.1')\qquad |p_{(\alpha)}(x, \zeta)|\leqq CR^{|\alpha|}M_{|\alpha|}\langle\xi\rangle^{m+\delta|\alpha|},\quad (\mathrm{Re}\,\zeta=\xi).$$

On the other hand, when $\theta=0$, we do not expect (4.1). For $0<\theta\leqq 1$, we say that $p(x, \xi)$ is pseudo-analytic in ξ.

Remark 2. In Definition 4.1, the estimate (4.2) on β is of use when we consider the kernel of $P(x, D)$ in $\boldsymbol{S}_{\rho\delta\theta}[M_n]$. However, the regularity of the estimate of the derivatives in ξ in (4.2) is not essential in the other properties and we can replace $CR^{|\beta|}L_{|\beta|}$ by C_β which depends on the index β and the symbol $p(x, \xi)$.

Remark 3. $\boldsymbol{S}^m_{\rho\delta\theta}[M_n]\subseteqq\boldsymbol{S}^{m'}_{\rho'\delta'\theta'}[M_n]$ if and only if $\rho\geqq\rho'$, $\delta\leqq\delta'$, $\theta\geqq\theta'$ and $m\leqq m'$.

Remark 4. In case of $M_n=n!$, G. Métivier [33] relaxed the regularity on x when $\delta>0$; He replaced $\alpha!\langle\xi\rangle^{\delta|\alpha|}$ by $(\alpha!^{1/(1-\delta)}+\alpha!\langle\xi\rangle^{\delta|\alpha|})$. It is immediately generalized for general $\{M_n\}$ replacing $M_{|\alpha|}\langle\xi\rangle^{\delta|\alpha|}$ by $(M_{|\alpha|/(1-\delta)}+M_{|\alpha|}\langle\xi\rangle^{\delta|\alpha|})$. This has already been adopted by C. Iwasaki [16] and K. Taniguchi [39] in case of

Gevrey classes. This generalization however can be understood as the replacement of $S_{\rho\delta\theta}[M_n]$ by $S_{\rho 0\theta}[M_n^\delta(\xi)]$, $M_n^\delta(\xi)=\max\{M_{|\alpha|/(1-\delta)}, M_{|\alpha|}\langle\xi\rangle^{\delta|\alpha|}\}$. Our consideration rests valid for $S_{\rho 0\theta}[M_n^\delta(\xi)]$. (In Métivier's sense, $S_{\rho 0\theta}[M_n^\delta(\xi)]$ should be denoted, for example, by $S_{\rho\delta\theta}\langle M_{n/(1-\delta)}\rangle$.)

Definition 4.2. Let $p(x, \xi)$ belong to $S^m_{\rho\delta\theta}[M_n]$ and $\sum p_i(x, \xi)$ do to $\mathcal{S}^m_{\rho\delta\theta}[M_n]$.

$$p(x, \xi) \underset{[M_n]}{\overset{\rho\delta\theta m}{\sim}} \sum p_i(x, \xi) \overset{\text{def}}{\Leftrightarrow} \exists C, R>0, \quad \forall N\in \boldsymbol{Z}_+, \quad \forall\alpha, \beta\in\boldsymbol{Z}_+^l,$$

$$(4.3) \qquad |(p-\sum_{i<N} p_i)^{(\beta)}_{(\alpha)}| \leqq CR^{N+|\alpha+\beta|}M_{N+|\alpha|}\beta!\langle\xi\rangle^{m-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|}$$
$$\text{for} \quad (x, \xi)\in\boldsymbol{R}^l\times\{\langle\xi\rangle^\theta\geqq RD_{N+|\beta|}\}.$$

Remark 1. Taking $\alpha=\beta=0$, the righthand side of (4.3) becomes the smallest when N satisfies the relation $\langle\xi\rangle^{\rho-\delta}=RD_N$. Therefore, if and only if $\theta\geqq\rho-\delta$,we can benefit by the best possible estimate.

Remark 2. L. Boutet de Monvel and P. Krée [7] adopted $\theta=\infty$ in case of $M_n=n!^\nu$ $(\nu\geqq 1)$, $\rho=1$ and $\delta=0$. F. Treves [40] did $\theta=1$ in case of $M_n=n!$, $\rho=1$ and $\delta=0$. S. Hashimoto, T. Matsuzawa and Y. Morimoto [13] did $\theta=\rho-\delta$ in case of $M_n=n!^\nu$ $(\nu\geqq 1)$ and $0\leqq\delta<\rho\leqq 1$.

Under Definitions 3.1, 4.1 and 4.2, Properties I and II are satisfied.

In the case of $\theta=0$, that is, the case where the estimate (4.1) is not required, we cannot use the assymptotic expansions of symbols of arbitrary length. In such case, a theory of ps.d.op's of Gevrey class was constructed by K. Taniguchi [37]. On the other hand, S. Hashimoto, T. Matsuzawa and Y. Morimoto [13] constructed another theory standing on the calculus of formal symbols taking $\theta=\rho-\delta$ in Gevrey class. As we are interested in a theory which allows the asymptotic expansions of arbitrary length, the case of $\theta\geqq\rho-\delta$ becomes important. On the other hand, we shall see in Paragraphs 5.1 and 5.3 that the theory becomes clear if $\theta\leqq 1-\delta$.

4.2. *Continuity (Property III), kernel and pseudo-local property.*

We note that $\boldsymbol{S}^m_{\rho\delta\theta}[M_n]\subseteq\boldsymbol{S}^m_{\rho\delta 0}[M_n]$. As introduced in Paragraph 2.6, we use the logarithmicly linear interpolation M_x of $\{M_n\}$. We set $M_x=M_0$ for $x<0$. Now, we give a theorem on Property III, which stands on Calderón-Vaillancourt's theorem [9] and others.

Theorem 4.1. *Let us take $\kappa\geqq 1$ and $0\leqq\delta\leqq\rho\leqq 1$ $(\delta<1)$. We set $l_0=[l/2]+1$, $l_1=[l/2(1-\delta)]+1$, and $M_n^{(\kappa)}=\max\{M_{\kappa n}, M_nM_{\kappa\delta n}\}$. (If $\kappa\geqq 1/(1-\delta)$, it holds that $M_n^{(\kappa)}=M_{\kappa n}$.)*

1) (a) *$P(x, D)$ in $\boldsymbol{S}^m_{\rho\delta 0}[M_{n-2l_1}]$ is continuous from $\mathcal{D}_{L^2}\{M_{\kappa(n-m)}\}$ to $\mathcal{D}_{L^2}\{M_n^{(\kappa)}\}$. If $\{M_n\}$ satisfies Differentiability Condition (D), $P(x, D)$ in $\boldsymbol{S}_{\rho\delta 0}[M_n]$ is continuous from $\mathcal{D}_{L^2}\{M_{\kappa n}\}$ to $\mathcal{D}_{L^2}\{M_n^{(\kappa)}\}$.*

(b) *We define $P(x, D)u$ for u in $\mathcal{D}'_{L^2}\{M_n^{(\kappa)}\}$ by $\langle Pu, \varphi\rangle=\langle u, \bar{P}^*\varphi\rangle$ $(\forall\varphi\in\mathcal{D}_{L^2}\{M_n^{(\kappa)}\})$, where $\sigma(\bar{P})$ is the complex conjugate of $p(x, \xi)$.* $P(x, D)$ in

$S^m_{\rho\delta 0}[M_{n-2l_1}]$ *is continuous from* $\mathcal{D}'_{L^2}\{M^{(\kappa)}_n\}$ *to* $\mathcal{D}'_{L^2}\{M_{\kappa(n-m)}\}$. *If* $\{M_n\}$ satisfies (*D*), $P(x, D)$ *in* $S_{\rho\delta 0}[M_n]$ *is continuous from* $\mathcal{D}'_{L^2}\{M^{(\kappa)}_n\}$ *to* $\mathcal{D}'_{L^2}\{M_{\kappa n}\}$.

2) (a) $P(x, D)$ *in* $S^m_{\rho\delta 0}[M_n]$ *is continuous from* $\mathcal{B}\{M_{\kappa(n-m-2l_0-1)}\}$ *to* $\mathcal{B}\{M^{(\kappa)}_n\}$. *If* $\{M_n\}$ *satisfies* (*D*), *it is continuous from* $\mathcal{B}\{M_{\kappa n}\}$ *to* $\mathcal{B}\{M^{(\kappa)}_n\}$.

(b) *Similarly defining* $P(x, D)$ *for* u *in* $\mathcal{B}'\{M^{(\kappa)}_n\}$ *as for* u *in* $\mathcal{D}'_{L^2}\{M^{(\kappa)}_n\}$, $P(x, D)$ *in* $S^m_{\rho\delta 0}[M_n]$ *is continuous from* $\mathcal{B}'\{M^{(\kappa)}_n\}$ *to* $\mathcal{B}'\{M_{\kappa(n-m-2l_0-1)}\}$. *If* $\{M_n\}$ *satisfies* (*D*), *it is continuous from* $\mathcal{B}'\{M^{(\kappa)}_n\}$ *to* $B'\{M_{\kappa n}\}$.

(c) $P(x, D)$ *in* $S^m_{\rho\delta 0}[M_{n-2l_1}]$ *is continuous from* $\mathcal{D}\{M_{\kappa(n-m)}\}$ *to* $\mathcal{B}\{M^{(\kappa)}_{n+l_0}\}$.

Remark 1. In 1), in case of $\rho=1$, the gap $2l_1$ in $S[M_{n-2l_1}]$ may be replaced by 1. (See T. Muramatsu and M. Nagase [34].) However, it might not be 0. This is suggested by C.H. Ching's example [11]. In general case, we may at least replace $2l_1$ by $2l_0+1$. (See H.O. Cordes [12] and T. Kato [19].)

Remark 2. If $\kappa\geqq 1/(1-\delta)$, $M^{(\kappa)}_n$ coincides with $M_{\kappa n}$ by the relation $M_nM_m\leqq M_{n+m}$. If $\{M_n\}$ satisfies Separativity Condition (*S*), $\{M_nM_{\kappa\delta n}\}$ is equivalent to $\{M_{n+\kappa\delta n}\}$ and $M^{(\kappa)}_n=M_{\kappa n}$ implies $\kappa\geqq 1/(1-\delta)$. However, when $\{M_n\}$ does not satisfy (*S*), $\kappa\geqq 1/(1-\delta)$ rests only a sufficient condition for $M^{(\kappa)}_n=M_{\kappa n}$. For example, in case of $M_n=\exp(an^\nu)$ $(\nu>1,\ a>0)$, $M^{(\kappa)}_n=M_{\kappa n}$ if and only if $\kappa\geqq\{1/(1-\delta^\nu)\}^{1/\nu}$.

Remark 3. In order to obtain Theorem 4.1, we need not (pseudo-) analytic estimate in ξ.

Next, we consider the regularity of the kernel $K(x, y)$ of $P(x, D)$ in $S_{\rho\delta\theta}[M_n]$ on $R^l_x\times R^l_y\backslash\Delta$, Δ being the diagonal set, i.e. $\Delta=\{(x, y); x=y\}$. In the case of finite θ, the results are not sufficiently clear. We apply them to the pseudo-local property. In order to make the announcement simple, we restrict ourselves to the case of $\{M_n\}$ with condition (N.Q.A) in the following theorem.

Theorem 4.2. *Let us take* $\kappa\geqq 1$ *and an arbitrary open set* Ω *in* R^l. *We set* $M^{(\kappa)}_n=\max\{M_{\kappa n}, M_nM_{\kappa\delta n}\}$.

1) [*Case of* $\theta=\infty$.]

We set $M'_n=\max\{n!^{\delta/\rho}M_n, n!^{1/\rho}\}$, $M^{(\kappa)\prime}_n=\max\{M^{(\kappa)}_n, n!^{1/\rho}\}$ *and* $M^{(\kappa)\prime\prime}_n=\max\{M^{(\kappa)}_n, M'_n\}$.

(i) *The kernel* $K(x, y)$ *of* $P(x, D)$ *in* $S_{\rho\delta\infty}[M_n]$ *belongs to* $\mathcal{B}\{M'_n, n!^{1/\rho}\}(R^l_x\times R^l_y\backslash\Delta)$.

(ii) *If* $P(x, D)$ *belongs to* $S^m_{\rho\delta\infty}[M_n]$ *and if* u *in* $\mathcal{B}'\{M^{(\kappa)\prime}_n\}$ *satisfies*

$$(\omega)\qquad u|_\Omega\in\mathcal{E}\{M_{\kappa(n-(m+2l_0+1))}\}(\Omega),$$

$P(x, D)u|_\Omega$ *belongs to* $\mathcal{E}\{M^{(\kappa)\prime\prime}_n\}(\Omega)$.

When $\{M_n\}$ *satisfies* (*D*), *we can remove* $(m+2l_0+1)$ *in the above.*

2) [*Case where* $\{L_n\}$ *satisfies* (*S*).]

We set $M'_n=\max\{M_nL_{(\delta/\rho)n}, L_{n/\rho}\}$, $M^{(\kappa)\prime}_n=\max\{M^{(\kappa)}_n, L_{n/\rho}\}$ *and* $M^{(\kappa)\prime\prime}_n=\max\{M^{(\kappa)}_n, M'_n\}$.

(i) *The kernel* $K(x, y)$ *of* $P(x, D)$ *in* $\boldsymbol{S}_{\rho\delta\theta}[M_n]$ *belongs to* $\mathscr{B}\{M'_n, L_{n/\rho}\}(\boldsymbol{R}_x^l\times\boldsymbol{R}_y^l\setminus\Delta)$.
(ii) *If* $P(x, D)$ *belongs to* $\boldsymbol{S}^m_{\rho\delta\theta}[M_n]$ *and if* u *in* $\mathscr{B}'\{M_n^{(\kappa)}\}$ *satisfies* (ω), $P(x, D)u|_\Omega$ *belongs to* $\mathcal{E}\{M_n^{(\kappa)\prime\prime}\}(\Omega)$.
When $\{M_n\}$ *satisfies* (*D*), *we can remove* $(m+2l_0+1)$ *in Condition* (ω).
3) [*Case where* θ *satisfies* $0<\theta\leqq\rho$ *and* $\{M_n\}$ *does* (*S*).]

We set $M'_n=\max\{M_{(1+(\delta/\theta))n}, M_{n/\theta}\}$, $M_n^{(\kappa)\prime}=\max\{M_n^{(\kappa)}, M_{n/\theta}\}$ *and* $M_n^{(\kappa)\prime\prime}=\max\{M_n^{(\kappa)}, M'_n\}$.
(i) *The kernel of* $P(x, D)$ *in* $\boldsymbol{S}_{\rho\delta\theta}[M_n]$ *belongs to* $\mathscr{B}\{M'_n, M_{n/\theta}\}(\boldsymbol{R}_x^l\times\boldsymbol{R}_y^l\setminus\Delta)$.
(ii) *If* $P(x, D)$ *belongs to* $\boldsymbol{S}_{\rho\delta\theta}[M_n]$ *and if* u *in* $\mathscr{B}'\{M_n^{(\kappa)\prime}\}$ *satisfies*

$$(\omega') \qquad u|_\Omega\in\mathcal{E}\{M_{\kappa n}\}(\Omega),$$

$P(x, D)u|_\Omega$ *belongs to* $\mathcal{E}\{M_n^{(\kappa)\prime\prime}\}(\Omega)$.

Remark 1. On 1), first we give a remark for $M_n=n!^\nu$ ($\nu\geqq 1$). If $\kappa\geqq 1/\rho\nu$, it follows that $M_n^{(\kappa)\prime}=M_n^{(\kappa)\prime\prime}=M_n^{(\kappa)}$.

Next, we consider the case where $\{M_n\}$ satisfies $\varliminf_{n\to\infty}(\log M_n/(n\log n))=\infty$. In this case, automatically $M'_n=n!^{\delta/\rho}M_n$ and $M_n^{(\kappa)\prime}=M_n^{(\kappa)\prime\prime}=M_n^{(\kappa)}$, changing finite elements of $\{M_n\}$ if necessary.

Remark 2. When we consider problems on differential operators, we can often take $\{L_n\}$ an arbitrary sequence with (N.Q.A). For example, if we take $L_n=n!(\log n)^{2n}$, we have the same in Remark 1 for 2) replacing $\kappa\geqq 1/\rho\nu$ by $\kappa>1/\rho\nu$.

Remark 3. In 3), if $\kappa\geqq 1/\theta$, it holds that $M_n^{(\kappa)\prime}=M_n^{(\kappa)\prime\prime}=M_n^{(\kappa)}$.

Remark 4. See Remark 2 of Theorem 4.1 on $\{M_n^{(\kappa)}\}$.

Proof. We assume that $|x-y|\geqq d>0$. To see the regularity of kernel $K(x, y)$, we use the following;

$$(4.4)\qquad \begin{aligned} D_x^\alpha D_y^\beta K(x, y) &= \sum\binom{\alpha}{\alpha'}\lim_{\varepsilon\downarrow 0}\int e^{\sqrt{-1}(x-y)\cdot\xi}\chi(\varepsilon\xi)\xi^{\alpha-\alpha'+\beta}p_{(\alpha')}(x, \xi)d\xi \\ &= \sum\binom{\alpha}{\alpha'}\{I_1^{\alpha'}+I_2^{\alpha'}\}, \end{aligned}$$

$$(4.5)\qquad \left\{\begin{aligned} I_1^{\alpha'} &= \int_{\langle\xi\rangle<D} e^{\sqrt{-1}(x-y)\cdot\xi}\xi^{\alpha-\alpha'+\beta}p_{(\alpha')}(x, \xi)d\xi, \\ I_2^{\alpha'} &= \lim_{\varepsilon\downarrow 0}\int_{\langle\xi\rangle\geqq D} e^{\sqrt{-1}(x-y)\cdot\xi}\chi(\varepsilon\xi)\xi^{\alpha-\alpha'+\beta}p_{(\alpha')}(x, \xi)d\xi, \end{aligned}\right.$$

$$(\chi\in\mathcal{S} \text{ and } \chi(O)=1)$$

Obviously, we obtain

$$(4.6)\qquad |I_1^{\alpha'}|\leqq CR^{|\alpha'|}M_{|\alpha'|}D^{(\delta|\alpha'|+|\alpha-\alpha'|+|\beta|+m)_++l},$$

where $a_+=\max\{a, 0\}$.

On the other hand, $I_2^{\alpha'}$ is expressed by

$$(4.7)\quad I_2^{\alpha'}=\sum_{j=0}^{N-1}|x-y|^{-2(j+1)}\int_{\langle\xi\rangle=D}\left\{\left(\frac{d}{dr}\right)e^{\sqrt{-1}(x-y)\cdot\xi}\right\}\{(-\Delta_\xi)^j(\xi^{\alpha-\alpha'+\beta}p_{(\alpha')}(x,\xi))\}\,dS$$
$$+|x-y|^{-2N}\int_{\langle\xi\rangle\geq D}e^{\sqrt{-1}(x-y)\cdot\xi}(-\Delta_\xi)^N\{\xi^{\alpha-\alpha'+\beta}p_{(\alpha')}(x,\xi)\}\,d\xi$$

(dS is the surface element of $\{\xi;\langle\xi\rangle=D\}$).

Hence, we have

$$(4.8)\quad |I_2^{\alpha'}|\leqq\sum_{j=0}^{N-1}|x-y|^{-2j-1}CR^{|\alpha'|+2j}M_{|\alpha'|}\tilde{L}_{2j}D^{\delta|\alpha'|+|\alpha-\alpha'|+|\beta|-2\rho j+m+l-1}$$
$$+|x-y|^{-2N}CR^{|\alpha'|+2N}M_{|\alpha'|}\tilde{L}_{2N}(\max\{D,1\})^{\delta|\alpha'|+|\alpha-\alpha'|+|\beta|-2\rho N+m+l}.$$

Let us take $N=[(\delta|\alpha'|+|\alpha-\alpha'|+|\beta|+m+l)/2\rho]+1$.

In case of 1) (i), we can take $\tilde{L}_n=n!$ and $D=R_0$. Then, we obtain

$$(4.9)\qquad |I_1^{\alpha'}|+|I_2^{\alpha'}|\leqq C(d)R(d)^{|\alpha+\beta|}M_{|\alpha'|}\alpha'!^{\delta/\rho}(\alpha-\alpha')!^{1/\rho}\beta!^{1/\rho}.$$

In case of 2) (i), we can take $\tilde{L}_n=L_n$ and $D=0$. We obtain

$$(4.9')\qquad |I_2^{\alpha'}|\leqq C(d)R(d)^{|\alpha+\beta|}M_{|\alpha'|}L_{(\delta/\rho)|\alpha'|}L_{|\alpha-\alpha'|/\rho}L_{|\beta|/\rho}.$$

In case of 3) (i), we can take $\tilde{L}_n=n!$ and $D=RH(M_N)^{1/\theta N}$. ($\geqq R(D_N)^{1/\theta}$ by (S.4)). Thus, we obtain

$$(4.9'')\qquad |I_1^{\alpha'}|+|I_2^{\alpha'}|\leqq C(d)R(d)^{|\alpha+\beta|}M_{|\alpha'|}M_{(\delta/\theta)|\alpha'|}M_{|\alpha-\alpha'|/\theta}M_{|\beta|/\theta}.$$

When $\{N'_n\}$ and $\{N''_n\}$ are logarithmicly convex, it holds that $\max_{0\leqq m\leqq n}\{N'_m N''_{n-(1-\delta)m}\}$ $=\max\{N'_n N''_{\delta n}, N''_n\}$. Hence, we obtain 1) (i), 2) (i) and 3) (i).

The pseudo-local property follows from the regularity of the kernel of $P(x,D)$ on $\boldsymbol{R}_x^l\times\boldsymbol{R}_y^l\backslash\Delta$ and Theorem 4.1. Q.E.D.

4.3. *Definition of pseudo-differential operators of $-\infty$ order and regularizing property (Properties IV and IV').*

We shall see in Paragraph 5.2 that some modulo class is in general indispensable in order that $S_{\rho\delta\theta}[M_n]$ is an algebra. Take account of this, we expect that the modulo class is included in the space of pseudo-differential operators of $-\infty$ order. Hence, we *do not set* $S^{-\infty}[M_n]$ as $\bigcap_{m\in\boldsymbol{R}}S^m_{\rho\delta\theta}[M_n]$.

Let us take a positive and logarithmicly convex sequence $\{L_n\}$ with (N.Q.A). Let M_x be the logarithmicly linear interpolation of $\{M_n\}$.

Definition 4.3. i) $p(x,\xi)\in S^{-\infty}[M_n]\ (=S^{-\infty}[M_n, L_n])$
$\overset{\text{def}}{\Leftrightarrow}$ $p(x,\xi)\in\mathcal{E}\{M_n, L_n\}(\boldsymbol{R}_x^l\times\boldsymbol{R}_\xi^l)$ and $\exists C, R>0$, $\forall N\in\boldsymbol{R}_+$, $\forall\alpha,\beta\in\boldsymbol{Z}_+^l$,

$$(4.10)\qquad |p_{(\alpha)}^{(\beta)}(x,\xi)|\leqq CR^{N+|\alpha+\beta|}M_{N+|\alpha|}L_{|\beta|}\langle\xi\rangle^{-N},$$

for $(x,\xi)\in\boldsymbol{R}^l\times\{\langle\xi\rangle\geqq RD_N\}$.

ii) We set $S^{-\infty}[M_n]=\{P(x,D)\in S^{-\infty};\ \sigma(P)\in S^{-\infty}[M_n]\}$.

Remark 1. For $\langle\xi\rangle\geqq RD_{|\alpha|}$, (4.10) is equivalent to

$$|p^{(\beta)}_{(\alpha)}(x,\xi)|\leqq CR^{|\beta|}L_{|\beta|}\langle\xi\rangle^{|\alpha|}\{T(\langle\xi\rangle/R)\}^{-1}, \tag{4.10'}$$

where $T(r)$ is the associated function of $\{M_n\}$.

If $\{M_n\}$ satisfies Separativity Condition (S), (4.10) is equivalent to

$$|p^{(\beta)}_{(\alpha)}(x,\xi)|\leqq CR^{|\alpha+\beta|}M_{|\alpha|}L_{|\beta|}\{T(\langle\xi\rangle/R)\}^{-1}. \tag{4.10''}$$

Remark 2. We can replace $CR^{|\beta|}L_{|\beta|}$ by C_β which depends on the index β and on the symbol $p(x,\xi)$.

Now, we give three theorems on the regularizing power on $P(x,D)$ in $S^{-\infty}[M_n]$.

Theorem 4.3. *$P(x,D)$ in $S^{-\infty}[M_n]$ is continuous from $\mathcal{D}^\infty_{L^2}$ to $\mathcal{D}_{L^2}\{M_n\}$ and from $\mathcal{B}$ to $\mathcal{B}\{M_n\}$.*

When we consider the propagation of singularities, we often limit solutions to the elements in $\mathcal{D}'$. Thus, it is important to consider the necessary and sufficient condition in order that $P(x,D)$ in $S^{-\infty}[M_n]$ is continuous from $\mathcal{E}'$ to a ul.d. class and from $\mathcal{D}'_{L^2}$ to a ul.d. class in sense of L^2.

Theorem 4.4. *If and only if $\{M_n\}$ satisfies Differentiability Condition (D), $P(x,D)$ in $S^{-\infty}[M_n]$ is continuous from $\mathcal{D}'_{L^2}$ to $\mathcal{D}_{L^2}\{M_n\}$ and from $\mathcal{E}'$ to $\mathcal{B}\{M_n\}$, that is, $P(x,D)$ is a regularizer of class $\{M_n\}$.*

Every element in $S^{-\infty}$ is continuous from $\mathcal{D}'_{L^2}$ to $\mathcal{D}^\infty_{L^2}$ and from $\mathcal{E}'$ to $\mathcal{E}$. In case of $S^{-\infty}[M_n]$, we also expect the continuity from a space of ultradistributions to a ul.d. space of the same class.

Theorem 4.5. *If and only if $\{M_n\}$ satisfies Separativity Condition (S), every $P(x,D)$ in $S^{-\infty}[M_n]$ is continuous from $\mathcal{D}'_{L^2}\{M_n\}$ to $\mathcal{D}_{L^2}\{M_n\}$ and from $\mathcal{E}'\{M_n\}$ to $\mathcal{B}\{M_n\}$, that is, $P(x,D)$ is a strong regularizer of class $\{M_n\}$.*

Remark. In order to obtain Theorems 4.3, 4.4 and 4.5, it is sufficient to assume (4.10) for $|\beta|\leqq 2l_0$.

We can show Theorems 4.3 and 4.4 by the similar way as the proof of Theorem 4.5. Since the proof of Theorem 4.5 is a little long, we shall give it in Appendix A.

§5. Structure of star algebra of $S_{\rho\delta\theta}[M_n]$ and of $S^{-\infty}[M_n]$ (Property V) and Parametrices of elliptic operators in $S_{\rho\delta\theta}[M_n]$ (Property VII).

5.1. *Star algebra of $S^{-\infty}[M_n]$.*

As the structure of $S^{-\infty}[M_n]$ is comparatively simple, we first consider it.

Theorem 5.1. *Let us take $\lambda \leqq 1-\delta$ and let M_x be the logarithmicly linear interpolation of $\{M_n\}$.*

1) *For $P(x, D)$ and $Q(x, D)$ in $\boldsymbol{S}^{-\infty}[M_n, L_n]$, $P\circ Q(x, D)$ and $P^*(x, D)$ belong to $\boldsymbol{S}^{-\infty}[M_{n+l+1}, L_{n+2l_0}]$.*

2) *For $P(x, D)$ in $\boldsymbol{S}^{-\infty}[M_{n/\lambda}, L_n]$ and $Q(x, D)$ in $\boldsymbol{S}^m_{\rho\delta 0}[M_n, L_n]$, $P\circ Q(x, D)$ and $Q\circ P(x, D)$ belong to $\boldsymbol{S}^{-\infty}[M_{(n+[m_+]+l+1)/\lambda}, L_{n+2l_0}]$.*

3) *If $\{M_n\}$ and $\{L_n\}$ satisfy Differentiability Condition (D), $\boldsymbol{S}^{-\infty}[M_n, L_n]$ is a star algebra over $\boldsymbol{C}$ and a bimodule with operator domain $\boldsymbol{S}_{\rho\delta 0}[M_{(1-\delta)n}, L_n]$.*

Proof. $\sigma(P\circ Q)$ and $\sigma(P^*)$ are the left simplified symbols of the double symbols $p(x, \xi)q(x', \xi')$ and $p(x', \xi)$. Essentially, we show that the left simplified symbol of a double symbol in sense of ρ, δ, θ and of class $[M_n]$ belongs to $S_{\rho\delta\theta}[M_n]$. (The double symbols in sense of ρ, δ, θ and class $[M_n]$ is defined by the same way as Definitions 4.1, 4.3 in this paper and Definition 2.1 in Chap. II §2 of H. Kumano-go [27].) Therefore, we consider only the product. Since 3) is obvious by 1) and 2) and since 1) is similarly provable as 2), we give a proof only on 2). Further, in case of $Q\circ P$, our proof is rather easy than that in case of $P\circ Q$. Hence, we consider only $\sigma(P\circ Q)$.

We start from the formula on the symbol of operator product;

$$(5.1)\quad r(x, \xi) = \sigma(P\circ Q)(x, \xi) = \mathrm{Os}-\iint e^{-\sqrt{-1}y\cdot\eta} p(x, \xi+\eta) q(x+y, \xi) dy\, d\eta .$$

$r^{(\beta)}_{(\alpha)}(x, \xi)$ is given as follows;

$$(5.1')\quad r^{(\beta)}_{(\alpha)}(x, \xi) = \sum_{\alpha', \beta'} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \mathrm{Os}-\iint e^{-\sqrt{-1}y\cdot\eta} p^{(\beta')}_{(\alpha')}(x, \xi+\eta) q^{(\beta-\beta')}_{(\alpha-\alpha')}(x+y, \xi) dy\, d\eta .$$

Let us take a function $\phi(t)$ in $\mathscr{B}\{L_n\}$, such that

$$(5.2)\qquad \phi(t) = 1 \quad \text{for} \quad t \leqq 1/4, \quad =0 \quad \text{for} \quad t \geqq 1/2 \quad \text{and} \quad 0 \leqq \phi(t) \leqq 1 .$$

We set

$$(5.3)\qquad \chi_0(\eta, \xi) = \phi(|\eta|/\langle\xi\rangle) \quad \text{and} \quad \chi_1(\eta, \xi) = 1-\chi_0(\eta, \xi) .$$

$\chi_0(\eta, \xi)$ and $\chi_1(\eta, \xi)$ satisfy the following properties.

$$(5.4)\quad \begin{cases} |\eta| \leqq (1/2)\langle\xi\rangle \quad \text{and} \quad (1/2)\langle\xi\rangle \leqq \langle\xi+\eta\rangle \leqq (3/2)\langle\xi\rangle \quad \text{on} \quad \operatorname{supp} \chi_0, \\ |\eta| \geqq (1/4)\langle\xi\rangle \quad \text{and} \quad \langle\xi+\eta\rangle \leqq 5|\eta| \quad \text{on} \quad \operatorname{supp} \chi_1, \\ \exists C, R>0, \quad |\chi_i^{(\beta_1, \beta_2)}(\eta, \xi)| \leqq CR^{|\beta_1+\beta_2|} L_{|\beta_1+\beta_2|} |\eta|^{-|\beta_1|} \langle\xi\rangle^{-|\beta_2|} \\ \qquad\qquad\qquad \text{on} \quad \operatorname{supp} \chi_i \quad (i=0, 1) . \end{cases}$$

We define Ω_j inductively;

$$\Omega_1 = \{\eta \in \operatorname{supp} \chi_1;\ |\eta_1| \geqq |\eta|/\sqrt{l}\},$$

$$\Omega_j = \{\eta \in \operatorname{supp} \chi_1 \backslash (\bigcup_{i=1}^{j-1} \Omega_i);\ |\eta_j| \geqq |\eta|/\sqrt{l}\} \qquad (2 \leqq j \leqq l) .$$

It holds that $\operatorname{supp}\chi_1=\bigcup_{j=1}^{l}\Omega_j$. We devide the integral in (5.1′) as follows;

$$(5.5)\quad \mathrm{Os}-\iint e^{-\sqrt{-1}y\cdot\eta}p_{(\alpha')}^{(\beta')}(x,\xi+\eta)q_{(\alpha-\alpha')}^{(\beta-\beta')}(x+y,\xi)dy\,d\eta$$
$$=\sum_{j=0}^{l}\mathrm{Os}-\iint_{R_y^l\times\Omega_j}e^{-\sqrt{-1}y\cdot\eta}\langle y\rangle^{-2l_0}(1-\Delta_\eta)^{l_0}[\eta_j^{-(N+l+1)}D_{y_j}^{N+l+1}$$
$$\times\{\chi_j(\eta,\xi)p_{(\alpha')}^{(\beta')}(x,\xi+\eta)q_{(\alpha-\alpha')}^{(\beta-\beta')}(x+y,\xi)\}]dy\,d\eta=\sum_{j=0}^{l}I_j\,,$$
$$(\Omega_0=R_\eta^l\ \text{ and }\ \chi_j(\eta,\xi)=\chi_1(\eta,\xi),\ j=1,\cdots,l).$$

We consider first I_0 and secondarily I_j $(1\leqq j\leqq l)$.

(1) Using (5.4), for arbitrary N in Z_+, we arrive at

$$(5.6)\qquad |I_0|\leqq CR^{N+|\alpha+\beta|}M_{(N+|\alpha'|)/\lambda+|\alpha-\alpha'|}L_{|\beta|+2l_0}\langle\xi\rangle^{-N+\delta|\alpha-\alpha'|+m+l}\,.$$

Replacing N by $N+\delta|\alpha-\alpha'|+m+l$, we obtain

$$(5.6')\qquad |I_0|\leqq CR^{N+|\alpha+\beta|}M_{(N+|\alpha|+m+l)/\lambda}L_{|\beta|+2l_0}\langle\xi\rangle^{-N}\,.$$

(2) Taking account of $|\eta_j|\geqq|\eta|/\sqrt{l}\geqq\langle\xi\rangle/4\sqrt{l}$, we have

$$(5.7)\qquad |I_j|\leqq CR^{N+|\alpha+\beta|}M_{|\alpha'|/\lambda}M_{N+|\alpha-\alpha'|+l+1}L_{|\beta|+2l_0}\langle\xi\rangle^{-(1-\delta)N+\delta|\alpha-\alpha'|+m+\delta(l+1)}\,.$$

Here, since $N+|\alpha-\alpha'|+l+1$ rests an integer, we can regard N as non-negative real number. Replacing N by $\{N+\delta|\alpha-\alpha'|+m+\delta(l+1)\}/\lambda$, we arrive at

$$(5.7').\qquad |I_j|\leqq CR^{N+|\alpha+\beta|}M_{(N+|\alpha|+m+l+1)/\lambda}L_{|\beta|+2l_0}\langle\xi\rangle^{-N}\,.$$

(1) and (2) imply that $P\circ Q$ belongs to $S^{-\infty}[M_{(n+m+l+1)/\lambda},L_{n+2l_0}]$. Q.E.D.

5.2. *Necessity of modulo class in order that $S_{\rho\delta\theta}[M_n]$ is a star algebra.*

The symbol of the operator product of $P(x,D)$ and $Q(x,D)$ in $S_{\rho\delta}$ is given by

$$(5.8)\qquad \sigma(P\circ Q)=\mathrm{Os}-\int e^{-\sqrt{-1}y\cdot\eta}p(x,\xi+\eta)q(x+y,\xi)dy\,d\eta\,,$$

and this also belongs to $S_{\rho\delta}$. However, for P and Q in $S_{\rho\delta\theta}[M_n]$, $\sigma(P\circ Q)$ itself no longer belongs to $S_{\rho\delta\theta}[M_n]$ if Condition (B) is satisfied. We note that $p(\xi)=\langle\xi\rangle^{-2}$ belongs to $S_{10\infty}^{-2}[M_n]$ for arbitrary $\{M_n\}$. We can find a function $q(x)$ in $\mathcal{B}\{M_n\}$ such that $\sigma(P\circ Q)$ does not belong to $S_{101}^{-2}[M_n]$. (This means also $\sigma(R^*)$ does not belong to $S_{101}^{-2}[M_n]$ for $r(x,\xi)=q(x)p(\xi)$.) If we take $M_n=M_n^{(2)}(\kappa,a,\nu)\equiv n!^\nu e^{an^\kappa}$ $(\kappa>1,a>0,\nu\in R)$ or $M_n=M_n^{(3)}(a,b,\nu)\equiv n!^\nu e^{be^{an}}$ $(a>0,b>0,\nu\in R)$, it does not belong to $\bigcup_{\theta>0}S_{10\theta}^{-2}[M_n]$. In the rest part of this paragraph, we show this. We remark that $p(\xi)$ has the homogeneous expansion $\sum_{n=0}^{\infty}(-1)^n|\xi|^{-(2n+2)}$, which converges for $|\xi|>1$.

First, we study the properties of its derivatives. Let $\langle\xi'\rangle$ be $(1+\sum_{j=2}^{l}\xi_j^2)^{1/2}$.

Lemma 5.0.1. *Let us set* $\alpha_n=(n, 0, \cdots, 0)$.

(5.9) $$(-1)^n p^{(\alpha_n)}(\xi)>0 \quad \textit{for} \quad |\xi_1|>(n/2)\langle\xi'\rangle \quad \textit{and} \quad |\xi_1|\leqq(1/n)\langle\xi'\rangle .$$

(5.10) $$\int_0^A p^{(\alpha_n)}(\xi)d\xi_1>(2A/3)n!\langle\xi'\rangle^{-(n+2)}, \quad (n\gg 1, n\in 4\boldsymbol{Z}_+ \quad \textit{and} \quad 0<A\leqq(1/n)\langle\xi'\rangle).$$

Proof. From the following equality, these follow imediately:

(5.11) $$p^{(\alpha_n)}(\xi) = (-1)^n n! \sum_{k=0}^{[n/2]}\binom{n+1}{2k+1}(-1)^k\langle\xi'\rangle^{2k}\xi_1^{n-2k}(\xi_1^2+\langle\xi'\rangle^2)^{-(n+1)},$$

(5.12) $$\int_0^A p^{(\alpha_n)}(\xi)d\xi_1 = (-1)^{n-1}(n-1)! \sum_{k=0}^{[(n-1)/2]}\begin{bmatrix} n \\ 2k+1\end{bmatrix}\cdot \cdot A^{n-2k-1}(-1)^k\langle\xi'\rangle^{2k}(A^2+\langle\xi'\rangle^2)^{-n} . \qquad \text{Q.E.D.}$$

Next, we define $\mathcal{F}_{x\to\xi}[q(x)]=\hat{q}(\xi)$. As $\{M_n\}$ and $\{R^n M_n\}$ (R: a positive constant) give the same space, by Assumption, we may assume

(5.13) $$M_n\geqq n^n \ (n\geqq 1) \quad \text{and} \quad M_0=1 .$$

Setting $a_n=\log M_n$, we have

$$a_{n+1}-a_n\geqq a_n/n\geqq \log n .$$

Thus, $r(n)$ which satisfies $M_n=r(n)^n/T(r(n))$, has the minoration:

(5.14) $$r(n)\equiv M_{n+1}/M_n\equiv\exp(a_{n+1}-a_n)\geqq n. \quad \text{(See Paragraph 2.3.)}$$

By virtue of (5.14), we can take a subsequence $\{n(k)\}$ such that

(5.15) $$\begin{cases} 1) \ n(0)\geqq 4 \ \text{ and } \ r(n(0))\geqq 2 , \\ 2) \ n(k)\in 4\boldsymbol{Z}_+ \ \text{ and } \ n(k)>n(k-1) \ \ (k\geqq 1) , \\ 3) \ r(n(k))-r(n(k-1))\geqq \nu n(k) \ \ (k\geqq 1), (3/4<\nu<1) . \end{cases}$$

Let us set $\xi_1^{(k)}=r(n(k))=M_{n(k)+1}/M_{n(k)}$ $(=D_{n(k)})$ and $I_k=[-\xi_1^{(k)}, -\xi_1^{(k)}+1/n(k)]$ $(k\geqq 1)$. The following is satisfied;

(5.16) $$\operatorname{dist}(I_k, I_{k-1})\geqq(3/4)\, n(k) \quad (k\geqq {}^\exists k_0) .$$

We take $I'=\{\xi'\in\boldsymbol{R}^{l-1};\ |\xi'|\leqq 1\}$ and set

(5.17) $$\hat{q}(\xi) = \begin{cases} T[r(n(k))]^{-1}\, r(n(k))^{-2} \quad (\equiv c_k) & \text{on} \ \ I_k\times I' \ \ (k\geqq k_0) , \\ 0 & \text{otherwise.} \end{cases}$$

Since $|\hat{q}(\xi)|\leqq T(|\xi|)^{-1}|\xi|^{-2}$, $q(x)=\mathcal{F}^{-1}[\hat{q}]$ belongs to $\mathcal{B}\{M_n\}$.

Now, we are in a position to estimate $\sigma(P(D)\circ Q(x))$ $(\equiv r(x, \xi))$. It is represented by

(5.8′) $$r(x, \xi) = \int e^{\sqrt{-1}x\cdot\eta}p(\xi+\eta)\hat{q}(\eta)d\eta .$$

Then, for $n(k)$ and s in $4\boldsymbol{Z}_+$ satisfying $s\gg 1$, we have

$$r^{(\alpha_s)}_{(\alpha_{n(k)-s})}(x, \xi) = \int e^{\sqrt{-1}x\cdot\eta}\eta_1^{n(k)-s}p^{(\alpha_s)}(\xi+\eta)\hat{q}(\eta)d\eta .$$

Especially at $(x, \xi)=(O, \xi^{(k)})$, $\xi^{(k)}=(\xi_1^{(k)}, 0, \cdots, 0)$, it becomes

$$r^{(\alpha_s)}_{(\alpha_{n(k)-s})} = \sum_j c_j \int_{I_j\times I'} \eta_1^{n(k)-s}p^{(\alpha_s)}(\xi^{(k)}+\eta)d\eta , \tag{5.18}$$

where $\{c_j\}_{j=1}^{\infty}$ is given in (5.17). Since (5.9) brings

$$\eta_1^{n(k)-s}p^{(\alpha_s)}(\xi^{(k)}+\eta)>0 \quad \text{on} \quad I_j\times I' ,$$

applying (5.10), we can minorize $r^{(\alpha_s)}_{(\alpha_{n(k)-s})}(O, \xi^{(k)})$ as follows:

$$\begin{aligned} r^{(\alpha_s)}_{(\alpha_{n(k)-s})}(O, \xi^{(k)}) &\geqq c_k \int_{I_k\times I'} (\xi_1^{(k)})^{n(k)-s}p^{(\alpha_s)}(\xi^{(k)}+\eta)d\eta \\ &\geqq c_0 2^{-s/2}\{3n(k)\}^{-1}s!\,r(n(k))^{-2-s}M_{n(k)} , \end{aligned}$$

($(2\pi)^l c_0$ is the volume of the unit ball of dimension $l-1$.).

Therefore, for $n(k)\gg 1$, we have

$$\langle\xi^{(k)}\rangle^{2+s}\, r^{(\alpha_s)}_{(\alpha_{n(k)-s})}(O, \xi^{(k)}) \geqq c\, 2^{-n(k)/2}\, s!\, M_{n(k)} . \tag{5.18$'$}$$

Let us take $s=[\, n(k)/\log_+ \log_+ n(k)\,]$. As $n(k)-s>n(k)/\sqrt{2}$ for $n(k)\gg 1$ and as (5.16) holds, we arrive at

$$\begin{aligned} \langle\xi^{(k)}\rangle^{2+s}\, r^{(\alpha_s)}_{(\alpha_{n(k)-s})}(O, \xi^{(k)}) &\geqq c\, 2^{-n(k)}M_{n(k)-s}\, s!\, n(k)^{\{n(k)/\log_+\log_+ n(k)\}-1} \\ &\geqq c\, 3^{-n(k)}M_{n(k)-s}\, s!\{\exp(\log n(k)/\log_+\log_+ n(k))\}^{n(k)} . \end{aligned} \tag{5.18$''$}$$

On the other hand, by virtue of Condition (B), we have

$$D_{n(k)}/D_s \geqq \exp\{\sum_{j=s+1}^{n(k)} \nu/j\} \geqq (\,(1/2)\log_+ \log_+ n(k)\,)^{\nu} .$$

It means that

$$D_{n(k)} \geqq RD_s , \tag{5.19}$$

for arbitrary positive R and sufficiently large $n(k)$.

(5.18$''$) and (5.19) imply that $r(x, \xi)$ does not belong to $S^{-2}_{101}[M_n]$.

If $M_n=M_n^{(2)}(\kappa, a, \nu)\equiv n!^{\nu}e^{an^{\kappa}}$ ($\kappa>1$, $a>0$, $\nu\in\boldsymbol{R}$), taking $s=n^{1/\kappa'}$ ($1<\kappa'<\kappa$), and if $M_n=M_n^{(3)}(a, b, \nu)\equiv n!^{\nu}e^{be^{an}}$ ($a>0, b>0, \nu\in\boldsymbol{R}$), taking $s=n/2$, we can see that $r(x, \xi)$ does not belong to $\bigcup_{\theta>0} S^{-2}_{10\theta}[M_n]$.

5.3. *Star algebraic structure of* $\boldsymbol{S}_{\rho\delta\theta}[M_n]$ *(Property V).*

As we have shown in Paragraph 5.2, the integral near $\xi+\eta\sim O$ in (5.8$'$): $\int_{I_k\times I'} e^{\sqrt{-1}x\cdot\eta}p(\xi+\eta)\hat{q}(\eta)d\eta$ has not the estimate (4.1). On the other hand, we can show that the integral on $|\eta| \geqq d\langle\xi\rangle$ $(d>0)$ has the estimate (4.10) replacing $\{M_n\}$ by $\{M_{n/(1-\delta)}\}$. Thus, we can expect that $S_{\rho\delta\theta}[M_n]$ is a star algebra modulo $S^{-\infty}[M_{n/(1-\delta)}]$ (assuming $\theta \leqq 1$). We note that $l_0=[l/2]+1$, $l_1=[l/2(1-\delta)]+1$ and $D_n=M_{n+1}/M_n$.

Theorem 5.2. *Let $P(x, D)$ and $Q(x, D)$ belong to $S^m_{\rho\delta\theta}[M_n, L_n]$ and to $S^{m'}_{\rho\delta\theta}[M_n, L_n]$, respectively. We set $\sigma(P\circ Q)=r(x, \xi)$ and $\sigma(P^*)=p^*(x, \xi)$.*

I] *We assume one of the following three conditions;*

a) $\theta=0$,

b) $0 \leqq \rho < 1$ *and* $0<\theta \leqq 1$,

c) $\rho=1$, $0<\theta\leqq 1$ *and*

$(\mathrm{N.D})_s$ $\exists\varepsilon_0>0$, $\forall n\gg 1$, $D_n \geqq (\log n)^{\varepsilon_0 n}$.

We set $\bar{m}=0$ in case of (a) *and* (b) *and $\bar{m}=\min\{\tilde{m}\in \boldsymbol{Z}_+;\ \tilde{m}>\theta/\varepsilon_0(1-\delta)\}$ in case of* (c).

1) *$r(x, \xi)$ is divided as $r(x, \xi)=r_0(x, \xi)+r_\infty(x, \xi)$, where r_0 and r_∞ belong to $S^{m+m'}_{\rho\delta\theta}[M_{n+2l_1+\bar{m}}, L_{n+2l_0}]$ and to $S^{-\infty}[M_{(n+[m_+]+m'+l+2)/(1-\delta)}, L_{n+2l_0}]$, respectively. Further, if $\theta>0$, r_0 satisfies*

$$(5.20)\quad |(r_0-\sum_{i<N} r_i)^{(\beta)}_{(\alpha)}| \leqq CR^{N+|\alpha+\beta|} M_{N+|\alpha|+2l_1+\bar{m}}\beta!\langle\xi\rangle^{m+m'-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|}$$

$$\text{for}\quad \langle\xi\rangle^\theta \geqq RD_{N+|\beta|+2l_0},$$

$$r_i(x, \xi)=\sum_{|\gamma|=1}\gamma!^{-1}p^{(\gamma)}(x, \xi)q_{(\gamma)}(x, \xi)\qquad (i\in \boldsymbol{Z}_+).$$

In case of b), *if $\{M_n\}$ satisfies $(D)_s$ and $\{L_n\}$ does (D), $M_{n+2l_1+\bar{m}}$, D_{n+2l_0}, $M_{(n+[m_+]+m'+l+2)/(1-\delta)}$ and L_{n+2l_0} can be replaced by M_n, D_n, $M_{n/(1-\delta)}$ and L_n, respectively.*

2) *$p^*(x, \xi)$ is divided as $p^*(x, \xi)=p^*_0(x, \xi)+p^*_\infty(x, \xi)$, where p^*_0 and p^*_∞ belong to $S^m_{\rho\delta\theta}[M_{n+2l_1+\bar{m}}, L_{n+2l_0}]$ and to $S^{-\infty}[M_{(n+[m_+]+l+2)/(1-\delta)}, L_{n+2l_0}]$, respectively. Further, if $\theta>0$, p^*_0 satisfies*

$$(5.21)\qquad |(p^*_0-\sum_{i<N} p^*_i)^{(\beta)}_{(\alpha)}| \leqq CR^{N+|\alpha+\beta|} M_{N+|\alpha|+2l_1+\bar{m}}\beta!\langle\xi\rangle^{m-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|}$$

$$\text{for}\quad \langle\xi\rangle^\theta \geqq RD_{N+|\beta|+2l_0},$$

$$p^*_i(x, \xi)=\sum_{|\gamma|=i}\gamma!^{-1}\bar{p}^{(\gamma)}_{(\gamma)}(x, \xi),$$

where $\bar{p}$ is the complex conjugate of p.

In case of b), *if $\{M_n\}$ satisfies $(D)_s$ and $\{L_n\}$ does (D), $M_{n+2l_1+\bar{m}}$, D_{n+2l_0}, $M_{(n+[m_+]+l+2)/(1-\delta)}$ and L_{n+2l_0} can be replaced by M_n, D_n, $M_{n/(1-\delta)}$ and L_n, respectively.*

II] *We assume that $\rho=1$, $0<\theta\leqq 1-\delta$ and the followings;*

(B) $\nu>0$, $n\gg 1$, $\log(M_{n+1}M_{n-1}/M_n^2) \geqq \nu/n$,

$(D)_s$ $\exists \bar{H}\geqq 1$, $\forall n\gg 1$, $\forall m\leqq n$, D_n/D_{n-1} $(\equiv M_{n+1}M_{n-1}/M_n^2) \leqq \bar{H}$.

1) *$r(x, \xi)$ is divided as $r(x, \xi)=r_0(x, \xi)+r_\infty(x, \xi)$, where r_0 and r_∞ belong to $S^{m+m'}_{1\delta\theta}[M_n, L_{n+2l_0}]$ and to $S^{-\infty}[M_{n/(1-\delta)}, L_{n+2l_0}]$, respectively. Further, r_0 satisfies*

(5.20) *with* $\rho=1$. *If* $\{L_n\}$ *satisfies* (D), L_{n+2l_0} *can be replaced by* L_n.
2) $p^*(x, \xi)$ *is divided as* $p^*(x, \xi)=p_0^*(x, \xi)+p_\infty^*(x, \xi)$, *where* p_0^* *and* p_∞^* *belong to* $S^m_{1\delta\theta}[M_n, L_{n+2l_0}]$ *and to* $S^{-\infty}[M_{n/(1-\delta)}]$, *respectively. Further,* p_0^* *satisfies* (5.21) *with* $\rho=1$.
If $\{L_n\}$ *satisfies* (D), L_{n+2l_0} *can be replaced by* L_n.

Remark 1. $(D)_s$ is slightly stronger than (D) and $(N.D)_s$ is a little stronger than the negation of (D). $M_n^{(2)}(\kappa, a, \nu)$ in Paragraph 2.7 satisfies $(D)_s$ if and only if $\kappa\leqq 2$ and does $(N.D)_s$ if and only if $\kappa>2$.

Remark 2. We set $|p|^{(m)}_{j,k,\theta}=\sup|p^{(\beta)}_{(\alpha)}(x, \xi)|/\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|}$, where (x, ξ) runs over $\boldsymbol{R}^l\times\{\xi\in\boldsymbol{R}^l; \langle\xi\rangle^\theta\geqq RD_{|\beta|}\}$ and (α, β) runs over $\{\alpha\in\boldsymbol{Z}^l_+; |\alpha|\leqq j\}\times\{\beta\in\boldsymbol{Z}^l_+; |\beta|\leqq k\}$. In Case I, the following holds;

$$
\begin{aligned}
(5.22)\qquad & \exists C_0,\ R_0>0, \qquad \forall j,\ k\in\boldsymbol{Z}_+ ,\\
& |r_0|^{(m+m')}_{j,k,\theta}\leqq C_0R_0^{j+k}\max_{\substack{j'+j''=j\\ k'+k''=k}}|p|^{(m)}_{j',k'+2l_0,\theta}|q|^{(m')}_{j''+2l_1+\bar{m},k'',\theta} ,\\
& |p_0^*|^{(m)}_{j,k,\theta}\leqq C_0R_0^{j+k}|p|^{(m)}_{j+2l_1+\bar{m},k+2l_0,\theta} .
\end{aligned}
$$

However, in Case II, it is difficult to show (5.22). We can only show

$$
\begin{aligned}
(5.23)\qquad & \exists C_0,\ R_0>0, \qquad \forall j,\ k\in\boldsymbol{Z}_+ ,\\
& \sup_s|r_0|^{(m+m')}_{s,k,\theta}/(R_0R)^sM_s\\
& \leqq C_0R_0^k\max_{k'+k''=k}(\sup_s|p|^{(m)}_{s,k'+2l_0,\theta}/R^sM_s)(\sup_s|q|^{(m')}_{s,k'',\theta}/R^sM_s) ,\\
& \sup_s|p_0^*|^{(m)}_{s,k,\theta}/(R_0R)^sM_s\leqq C_0R_0^k(\sup_s|p|^{(m)}_{s,k+2l_0,\theta}/R^sM_s) .
\end{aligned}
$$

5.4. *Proof of Theorem* 5.2.—*Cut-off function*—

In order to prove the above theorem, we stand on the formula (5.8) $(=(*)$ in § 1) and the following (5.24).

$$
(5.8)\qquad \sigma(P\circ Q)=\mathrm{Os}-\iint e^{-\sqrt{-1}y\cdot\eta}p(x, \xi+\eta)q(x+y, \xi)dy\,đ\eta ,
$$

$$
(5.24)\qquad \sigma(P^*)=\mathrm{Os}-\iint e^{-\sqrt{-1}y\cdot\eta}p(x+y, \xi+\eta)dy\,đ\eta .
$$

Both of them are the left simplified symbols of the double symbols $p(x, \xi)q(x', \xi')$ and $p(x', \xi)$. We essentially estimate the semi-norms of the left simplified symbol by those of the double symbol. (On the double symbols, see, for example, H. Kumano-go [27] Chap II § 2.) Therefore, we consider only $\sigma(P\circ Q)$.

We want to divide the integral domain on η in (5.8) to

$$
|\eta|\leqq d_1\langle\xi\rangle \quad\text{and}\quad |\eta|\geqq d_2\langle\xi\rangle \qquad (0<d_2<d_1\leqq 1/2) .
$$

Hence, we want a cut-off function $\chi(\eta, \xi)$ which satisfies

$$\text{(5.25)}\qquad \chi(\eta,\xi)=\begin{cases}1 & 0\leqq|\eta|\leqq d\langle\xi\rangle \quad (0<{}^{\exists}d<1/2)\,,\\ 0 & |\eta|\geqq(1/2)\langle\xi\rangle\,,\end{cases}$$

and $0\leqq\chi(\eta,\xi)\leqq 1$.

If $\theta=0$ (Condition a) in Case I), such function is obtained by setting $\chi(\eta,\xi)=\chi^0(|\eta|/\langle\xi\rangle)$, where $\chi^0(t)$ belongs to $\mathcal{D}\{L_n\}$ and satisfies

$$\text{(5.25}'\text{)}\qquad \chi^0(t)=1 \quad \text{for } t\leqq 1/4, \qquad =0 \quad \text{for } t\geqq 1/2 \text{ and } 0\leqq\chi^0\leqq 1\,.$$

This $\chi(\eta,\xi)$ is adopted for the estimate (4.2) on $\sigma(P\circ Q)$.

In case of $\rho<1$ (Condition b) in Case I), we take $\{N_n\}$ as follows;

$$\text{(5.26)}\qquad N_n=\min\{L_n\,,\, n!(\log_+ n)^{an}\} \qquad (a>1)\,,$$

which satisfies (N.Q.A). We take $\chi^0(t)$ in $\mathcal{D}\{N_n\}$ which satisfies (5.25′) and set $\chi(\eta,\xi)=\chi^0(|\eta|/\langle\xi\rangle)$. Because supp $\chi^{0(j)}\subseteq\{1/4\leqq t\leqq 1/2\}$ for $j>0$, we have

$$(17)^{-1/2}\langle\eta\rangle\leqq|\eta|\leqq\langle\eta\rangle \quad \text{on} \quad \operatorname{supp}\chi^{(\beta_1,\beta_2)} \qquad \text{if} \quad |\beta_1+\beta_2|>0\,.$$

Then, $\chi(\eta,\xi)$ satifies (5.25) and

$$\text{(5.27)}\qquad \begin{aligned}&{}^{\exists}C,\ R>0\,, \qquad {}^{\forall}\beta_1,\ \beta_2\in\boldsymbol{Z}_+^l\,,\\ &|\chi^{(\beta_1,\beta_2)}(\eta,\xi)|\leqq CR^{|\beta_1+\beta_2|}N_{|\beta_1+\beta_2|}\langle\eta\rangle^{-|\beta_1|}\langle\xi\rangle^{-|\beta_2|}\,.\end{aligned}$$

As $\{N_n\}$ satisfies (D), restricting $\beta_1$to $|\beta_1|\leqq 2l_0$, it holds that

$${}^{\exists}R'>0,\ N_{|\beta_1+\beta_2|}\leqq R'^{|\beta_2|}N_{|\beta_2|}\leqq(lR')^{|\beta_2|}\beta_2!\,(\log_+|\beta_2|)^{a|\beta_2|}\,.$$

Further, if $\langle\xi\rangle^{\theta}\geqq D_n$, it holds that

$$\begin{aligned}&{}^{\exists}R'',\ R'''>0\,.\\ &\langle\xi\rangle^{-(1-\rho)n}\leqq D_n^{-\{(1-\rho)/\theta\}n}\leqq R''^n n^{-\{(1-\rho)/\theta\}n}\leqq R'''^n(\log_+ n)^{-an} \qquad (n\geqq 1)\,.\end{aligned}$$

Here, the second inequality holds good because $\{N_n\}$ satisfies (N.Q.A). (See Remark at Condition (N.Q.A).) Therefore, $\chi(\eta,\xi)$ satisfies also

$$\text{(5.27}'\text{)}\qquad \begin{aligned}&{}^{\exists}C,\ R>0,\ |\chi^{(\beta_1,\beta_2)}(\eta,\xi)|\leqq CR^{|\beta_2|}\beta_2!\langle\eta\rangle^{-|\beta_1|}\langle\xi\rangle^{-\rho|\beta_2|}\\ &\qquad\qquad \text{for} \quad \langle\xi\rangle^{\theta}\geqq RD_{|\beta_2|} \text{ and } |\beta_1|\leqq 2l_0\,.\end{aligned}$$

Thus, this is adopted to obtain the estimates (4.1) and (4.2) on $\sigma(P\circ Q)$.

In case of $\rho=1$ and $\theta>0$, if $(N.D)_s$ is satisfied (Condition c) in Case I), we adopt $\chi(\eta,\xi)$ in case of $\rho<1$ taking $a=\varepsilon_0(1-\delta)\bar{m}/\theta>1$ in (5.26). In this case, we use the estimate (5.27) and the relation

$$\text{(5.28)}\qquad \langle\xi\rangle^{-(1-\delta)\bar{m}}N_n\leqq n! \qquad \text{for} \quad \langle\xi\rangle^{\theta}\geqq D_n\,.$$

(5.28) follows from

$$\langle\xi\rangle^{-(1-\delta)\bar{m}}\leqq D_n^{-(1-\delta)\bar{m}/\theta}\leqq(\log_+ n)^{-a} \qquad \text{for} \quad \langle\xi\rangle^{\theta}\geqq D_n\,.$$

In the last case: $\rho=1$, $0<\theta\leqq 1-\delta$ and $\{M_n\}$ satisfies $(D)_s$, it is difficult to find a cut-off function satisfying (5.25) and (5.27′) even if we replace $\langle\eta\rangle^{-|\beta_1|}$ by $\langle\eta\rangle^{-\delta|\beta_1|}$. Then, we construct a cut-off function which satisfies (5.25) and a weakened estimate from (5.27′).

Lemma 5.0.2. *We assume (B) and $(D)_s$. There exists a function $\chi(\eta, \xi)$ with a parameter R_1 $(\geqq 1)$ which satisfies* (5.25) *and the following*;

(5.29) $\exists C, R>0$; *independent of* R_1,

$$|\chi^{(\beta_1,\beta_2)}(\eta, \xi)| \leqq \begin{cases} e^n CR^{|\beta_2|}\beta_2!\langle\eta\rangle^{-(1-\theta)|\beta_1|}\langle\xi\rangle^{-|\beta_2|}, & (1\leqq|\beta_2|\leqq n+2l_0), \\ CR^{|\beta_2|}L_{|\beta_2|}\langle\eta\rangle^{-(1-\theta)|\beta_1|}\langle\xi\rangle^{-|\beta_2|}, & (\forall\beta_2\in \boldsymbol{Z}_+^l), \end{cases}$$

for $eR_1D_n\leqq\langle\xi\rangle^\theta\leqq eR_1D_{n+1}$, $\eta\in\boldsymbol{R}^l$ *and* $|\beta_1|\leqq 2l_0$.

Proof. We can suppose that

(5.30) $$D_0\geqq 2.$$

We set

(5.30′) $$M_n = M_0 \quad \text{for} \quad -n_0\leqq n<0. \quad (n_0 \text{ is that in } (B)_2.)$$

i) [*Partition of unity in η-space.*]

Let us take $\chi^0(t)$ in $C^\infty(\boldsymbol{R})$ which satisfies

(5.31) $$\chi^0(t) = 1 \text{ for } t\leqq -1, \quad =0 \text{ for } t\geqq 0 \text{ and } 0\leqq\chi^0(t)\leqq 1.$$

Let R_0 $(\geqq 1)$ be a constant decided later on. We set

(5.32) $$\begin{cases} \chi_n(\eta) = \chi^0(|\eta|^\theta-R_0D_n)-\chi^0(|\eta|^\theta-R_0D_{n-1}) & (n\geqq 1), \\ \chi_0(\eta) = \chi^0(|\eta|^\theta-R_0D_0). \end{cases}$$

Obviously, it holds that $\sum_{n=0}^\infty \chi_n(\eta)\equiv 1$ and

(5.33) $$\sum_{j=0}^n \chi_j(\eta)\equiv\chi^0(|\eta|^\theta-R_0D_n)\equiv 1 \text{ on } \{|\eta|^\theta\leqq R_0D_n-1\} \text{ and}$$
$$R_0D_{n-1}-1\leqq|\eta|^\theta\leqq R_0D_n \text{ on supp } \chi_n(\eta) \quad (n\geqq 1).$$

m which satisfies $R_0D_n>R_0D_m-1$ is smaller than $n+n_0$ because of $(B)_2$ and $R_0\geqq 1$. Thus, we see that

(5.33′) $$\#\{n;\ \chi_n(\eta)\neq 0\}\leqq n_0 \quad \text{for} \quad \forall\eta\in\boldsymbol{R}^l.$$

Since it holds that $|\eta|\geqq 1$ on supp $\chi_n^{(\beta)}$ for arbitrary n and $\beta\neq 0$, Lemmas 2.0.1 and 2.0.2 imply

(5.34) $$\exists C>0, \quad |\chi_n^{(\beta)}(\eta)|\leqq C\langle\eta\rangle^{-(1-\theta)|\beta|} \quad \text{for} \quad \eta\in\boldsymbol{R}^l \text{ and } |\beta|\leqq 2l_0.$$

ii) [*Cut-off function in ξ-space.*]

Let R_1 $(\geqq 1)$ be a parameter. Using $\{\psi_n(t)\}$ in Lemma 2.0.3, we set

$$\phi_n(\xi) = \psi_{n+2l_0}(\theta \log \langle\xi\rangle - \log (R_1 D_n)) . \tag{5.35}$$

By virtue of Lemmas 2.0.1 and 2.0.2, it satisfies

$$\phi_n^{(\alpha)}(\xi) \leqq \begin{cases} e^n CK^{|\alpha|}\alpha!\langle\xi\rangle^{-|\alpha|} & (1\leqq|\alpha|\leqq n+2l_0) \\ CK^{|\alpha|}L_{|\alpha|}\langle\xi\rangle^{-|\alpha|} & (\forall\alpha\in \boldsymbol{Z}_+^l) , \end{cases} \tag{5.36}$$

$$\phi_n(\xi) = 1 \text{ for } \langle\xi\rangle^\theta \geqq eR_1D_n \quad \text{and} \quad = 0 \text{ for } \langle\xi\rangle^\theta \leqq R_1 D_n .$$

We set

$$k(n) = \max\{j;\ 0<\phi_j(\xi)<1 \text{ for some } \xi \text{ such that } \langle\xi\rangle^\theta \in [eR_1D_n,\ eR_1D_{n+1}]\} . \tag{5.37}$$

It holds that

$$D_{k(n)} < eD_{n+1} < e\bar{H}D_n , \tag{5.38}$$

by virtue of $(D)_s$. (Under Condition $(B)_1$, it holds that $k(n) < \exists\kappa_0 n$.)
We want the following property:

$$\forall n\in \boldsymbol{Z}_+,\ \exists d\in \boldsymbol{Z}_+\backslash\{0, 1\},\ \operatorname{supp}\sum_{j=0}^{k(n)} \chi_j \subseteqq \{\eta;\ |\eta|\leqq\langle\xi\rangle/2\} \text{ and} \tag{5.39}$$
$$\operatorname{supp}(1-\sum_{j=0}^{n}\chi_j)\subseteqq\{\eta;\ |\eta|\geqq d\langle\xi\rangle\} \quad \text{for} \quad \langle\xi\rangle^\theta\in[eR_1D_n,\ eR_1D_{n+1}] .$$

Now, we decide R_0 depending on R_1 and d independent of R_1;

$$R_0 = (2^\theta \bar{H})^{-1}R_1 \quad \text{and} \quad d = (1/2)(e\bar{H}^{n_0+2})^{-1/\theta} , \tag{5.40}$$

where $\bar{H}$ is that in $(D)_s$. This choice brings (5.39) by virtue of (5.30), (5.30′), (5.38), $(D)_s$ and $(B)_2$.

Let us set

$$\chi(\eta, \xi) = \sum_{n=0}^{\infty} \chi_n(\eta)\phi_n(\xi) . \tag{5.41}$$

As n is arbitrary in (5.39), (5.25) holds good. Further, $\chi(\eta, \xi)$ satisfies (5.29) by (5.34), (5.36) and (5.38). Q.E.D.

We notice that $\chi(\eta,\xi)$ in Lemma 5.0.2 satisfies

$$\operatorname{supp}\chi(\eta, \xi)\subseteqq\{(\eta, \xi);\ |\eta|\leqq\langle\xi\rangle/2\} \text{ and} \tag{5.42}$$
$$(1/2)\langle\xi\rangle\leqq\langle\xi+\eta\rangle\leqq(3/2)\langle\xi\rangle \quad \text{on} \quad \operatorname{supp}\chi(\eta, \xi) .$$

$$\operatorname{supp}\chi^{(\beta_1,\beta_2)}\subseteqq\{(\eta, \xi);\ d\langle\xi\rangle\leqq|\eta|\leqq\langle\xi\rangle/2\} \text{ and} \tag{5.42′}$$
$$\langle\xi+\eta\rangle\leqq(1+d^{-1})|\eta| \text{ on } \operatorname{supp}\chi^{(\beta_1,\beta_2)} , \quad \text{if} \quad |\beta_1|+|\beta_2|>0 .$$

$$\operatorname{supp}(1-\chi)\subseteqq\{(\eta, \xi);\ |\eta|\geqq d\langle\xi\rangle\} \text{ and} \tag{5.42″}$$
$$\langle\xi+\eta\rangle\leqq(1+d^{-1})|\eta| \quad \text{on} \quad \operatorname{supp}(1-\chi) .$$

5.5. *Proof of Theorem* 5.2. *—Continued—*

We can prove the theorem both in Cases I and II by the same idea. Since the proof is rather simple in Case I, we consider only Case II. As the estimate (4.2) on $\sigma(P\circ Q)$ is easily obtained, we show only (5.20), which coincides the estimate (4.1) on $\sigma(P\circ Q)$ when $N=0$.

We devide (5.8) into the following manner;

$$(5.43)\qquad \begin{aligned}\sigma(P\circ Q)(x,\xi) &= r_0(x,\xi)+r_\infty(x,\xi)\\ &\equiv \mathrm{Os-}\iint e^{-\sqrt{-1}y\cdot\eta}\chi(\eta,\xi)p(x,\xi+\eta)q(x+y,\xi)\mathrm{d}y\,d\eta\\ &\quad+\mathrm{Os-}\iint e^{-\sqrt{-1}y\cdot\eta}\{1-\chi(\eta,\xi)\}p(x,\xi+\eta)q(x+y,\xi)\mathrm{d}y\,d\eta\,.\end{aligned}$$

It is seen that $r_\infty(x,\xi)$ belongs to $S^{-\infty}[M_{(n+[m_+]+m'+l+2)/(1-\delta)}, L_{n+2l_0}]$ by the same way as in Case (2) of the proof of Theorem 5.1.

On r_0, we apply Taylor's formula on $p(x,\xi+\eta)$ at ξ and devide it in the following manner;

$$(5.44)\qquad \begin{aligned} r_0(x,\xi) &= \sum_{|\gamma|<N}\gamma!^{-1}p^{(\gamma)}(x,\xi)\,\mathrm{Os-}\iint e^{-\sqrt{-1}y\cdot\eta}q_{(\gamma)}(x+y,\xi)\mathrm{d}y\,d\eta\\ &\quad-\sum_{|\gamma|<N}\gamma!^{-1}p^{(\gamma)}(x,\xi)\,\mathrm{Os-}\iint e^{-\sqrt{-1}y\cdot\eta}\{1-\chi(\eta,\xi)\}q_{(\gamma)}(x+y,\xi)\mathrm{d}y\,d\eta\\ &\quad+\sum_{|\gamma|=N}\gamma!^{-1}N\,\mathrm{Os-}\iint e^{-\sqrt{-1}y\cdot\eta}\chi(\eta,\xi)q_{(\gamma)}(x+y,\xi)\mathrm{d}y\,d\eta\times\\ &\qquad\qquad\times\int_0^1(1-\theta)^{N-1}p^{(\gamma)}(x,\xi+\theta\eta)\,\mathrm{d}\theta\\ &\equiv \sum_{|\gamma|<N}I_0^\gamma-\sum_{|\gamma|<N}I_1^\gamma+\sum_{|\gamma|=N}I_1^\gamma\,.\end{aligned}$$

As $\mathrm{Os-}\iint e^{-\sqrt{-1}y\cdot\eta}f(x+y)\mathrm{d}y\,d\eta=f(x)$ holds for $f(x)$ in $\mathscr{B}(\boldsymbol{R}^l)$, we see that

$$(5.45)\qquad I_0^\gamma=\gamma!^{-1}p^{(\gamma)}(x,\xi)q_{(\gamma)}(x,\xi)\,.$$

It is rather easy to see that $|\sum_{|\gamma|<N}I_{1(\alpha)}^{\gamma(\beta)}|$ is majorized by the right-hand side of (5.20) because both of the second arguments of $p^{(\gamma)}$ and $q_{(\gamma)}$ are ξ. Then, we only show that $|\sum_{|\gamma|=N}I_{1(\alpha)}^{\gamma(\beta)}|$ is majorized by the right-hand side of (5.20).

In order to make clear the dependence of every constant, we specify the meaning of the constants C and R in this paragraph. We denote the maximum of R's in (4.1) and (4.2) for $\sigma(P)$ and $\sigma(Q)$ and in (5.29) by R again. We may assume $R\geqq 1$. We also denote the maximum of C's in (4.1) and (4.2) for $\sigma(P)$ (for $\sigma(Q)$, resp.) by C_p (by C_q, resp.). Further, we use C only for that in (5.29). We have the following representation:

$$(5.46)\qquad I_{1(\alpha)}^{\gamma(\beta)} = \gamma!^{-1}N\sum\binom{\alpha}{\alpha_i}_2\binom{\beta}{\beta_j}_3 \mathrm{Os}-\iint e^{-\sqrt{-1}y\cdot\eta}\chi^{(0,\beta_3)}(\eta,\xi)\times$$
$$\times q_{(\gamma+\alpha_2)}^{(\beta_2)}(x+y,\xi)\mathrm{d}y\,d\eta\int_0^1(1-\theta)^{N-1}p_{(\alpha_1)}^{(\gamma+\beta_1)}(x,\xi+\theta\eta)\mathrm{d}\theta$$
$$\equiv \gamma!^{-1}N\sum\binom{\alpha}{\alpha_i}_2\binom{\beta}{\beta_j}_3 I_1(\alpha_i,\beta_j)\,.$$

Let us set $R_2=\mathrm{e}R_1$ and restrict $\langle\xi\rangle$ to $R_2D_n\leqq\langle\xi\rangle^\theta\leqq R_2D_{n+1}$ and $N+|\beta|$ to $N+|\beta|\leqq n$, where R_1 is that in Lemma 5.0.2.

In (4.1), we may replace $D_{|\beta|}$ by $D_{|\beta|-2l_0}$ due to Condition $(D)_s$.

In case of $\beta_3=0$, following the proof of Lemma 2.4 in H. Kumano-go [27], we obtain

$$(5.47)\qquad |I_1^\gamma(\alpha_i,\beta_j)|\leqq N^{-1}\gamma!C_1C_pC_q(2^\rho 3R^2)^{N+|\alpha+\beta|}M_{N+|\alpha|+2l_1}\beta!\times$$
$$\times\langle\xi\rangle^{m+m'-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|}\,,$$

where C_1 is independent of N, α and β.

We estimate $|I_1^\gamma(\alpha_i,\beta_j)|$ in case of $\beta_3\neq 0$.

i) (*Case of* $N+|\alpha_2|+2l_1\geqq n$)

$I_1^\gamma(\alpha_i,\beta_j)$ is expressed as follows;

$$(5.48)\qquad I_1^\gamma(\alpha_i,\beta_j)=\iint e^{-\sqrt{-1}y\cdot\eta}\mathrm{d}y\,d\eta\langle y\rangle^{-2l_0}(1-\Delta_\eta)^{l_0}[|\eta|^{-2l_1}(-\Delta_y)^{l_1}\times$$
$$\times\{\chi^{(0,\beta_3)}(\eta,\xi)q_{(\gamma+\alpha_2)}^{(\beta_2)}(x+y,\xi)\int_0^1(1-\theta)^{N-1}p_{(\alpha_1)}^{(\gamma+\beta_1)}(x,\xi+\theta\eta)\mathrm{d}\theta\}]\,.$$

Then, by virtue of (5.42), (5.42′), $N+|\alpha_2|+2l_1\geqq n$ and Lemma 2.0.1, we have

$$(5.48')\qquad |I_1^\gamma(\alpha_i,\beta_j)|\leqq N^{-1}\gamma!C_1'C_pC_q(2^\rho 3\mathrm{e}R^2)^{N+|\alpha+\beta|}\times$$
$$\times M_{N+|\alpha|+2l_1}\beta!\langle\xi\rangle^{m+m'-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|}\,,$$

where C_1' is independent of N, α and β.

ii) (*Case of* $N+|\alpha_2|+2l_1<n$)

In (5.48), we replace $|\eta|^{-2l_1}(-\Delta_y)^{l_1}$ by $|\eta|^{-2[(n-N-|\alpha_2|)/2]}(-\Delta_y)^{[(n-N-|\alpha_2|)/2]}$. Then, we obtain the following instead of (5.48′);

$$(5.49)\quad |I_1^\gamma(\alpha_i,\beta_j)|\leqq N^{-1}\gamma!C_0C_pC_qe^n\{2^\delta d^{-1}R\}^{2([(n-N-|\alpha_2|)/2]-l_1)}\times$$
$$\times(2^\rho 3R^2)^{N+|\alpha+\beta|}M_nM_{|\alpha_1|}\beta!\langle\xi\rangle^{m+m'-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|-2(1-\delta)([(n-N-|\alpha_2|)/2]-l_1)}\,,$$

where we used (5.42), (5.42′) and Lemma 2.0.1.

As $\langle\xi\rangle^\theta$ is restricted to $[R_2D_n, R_2D_{n+1}]$ and θ is not greater than $1-\delta$, we have

$$(5.50)\qquad M_n\langle\xi\rangle^{-2(1-\delta)([(n-N-|\alpha_2|)/2]-l_1)}$$
$$\leqq M_n(R_2D_n)^{-2\{(1-\delta)/\theta\}([(n-N-|\alpha_2|)/2]-l_1)}$$
$$\leqq(\mathrm{e}R_1)^{-2[(n-N-|\alpha_2|)/2]+2l_1}M_{N+|\alpha_2|+2l_1+1}\,.$$

Let us choose $R_1=\max\{2^\delta d^{-1}R,\ 2^\rho 3\mathrm{e}R^2H^{2l_1+1}\}$, where H is that in Condition (D.1). We arrive at

$$(5.49') \qquad |I_1^{\gamma}(\alpha_i, \beta_j)| \leqq N^{-1}\gamma! C_1'' C_p C_q (2^{\rho}3eR^2)^{N+|\alpha+\beta|} M_{N+|\alpha|+2l_1+1}\beta! \times \times \langle\xi\rangle^{m+m'-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|},$$

where C_1'' is independent of N, α and β.

Applying (D.1) $2l_1+1$ times, by virtue of (5.47), (5.48′) and (5.49′), we obtain the following;

$$(5.51) \qquad |I_{1(\alpha)}^{\gamma(\beta)}| \leqq C_2 C_p C_q R_2^{N+|\alpha+\beta|} M_{N+|\alpha|}\beta! \langle\xi\rangle^{m+m'-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|}, \quad \text{for} \quad \langle\xi\rangle^{\theta} \geqq R_2 D_{N+|\beta|},$$

where C_2 is independent of N, α and β. Q.E.D.

We give a remark in Case I c). When $\beta_3 \neq 0$, we obtain in the first place the following estimate by the same way as in i) in the above proof replacing $2l_1$ by $2l_1+\bar{m}$;

$$(5.52) \qquad |I_1^{\gamma}(\alpha_i, \beta_j)| \leqq N^{-1}\bar{C} C_p C_q R_1^{N+|\alpha+\beta|} M_{N+|\alpha|+2l_1+\bar{m}} N_{N+|\beta|+2l_0} \times \times \langle\xi\rangle^{m+m'-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|-(1-\delta)\bar{m}}.$$

By virtue of (5.28), we arrive at

$$(5.52') \qquad |I_1^{\gamma}(\alpha_i, \beta_j)| \leqq N^{-1}\gamma! \bar{C}' C_p C_q R_1'^{N+|\alpha+\beta|} M_{N+|\alpha|+2l_1+\bar{m}}\beta! \times \times \langle\xi\rangle^{m+m'-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|} \quad \text{for} \quad \langle\xi\rangle^{\theta} \geqq R_1 D_{N+|\beta|+2l_1+\bar{m}}.$$

5.6. *Multiproducts of pseudo-differential operators and parametrices of elliptic operators in $\boldsymbol{S}_{\rho\delta\theta}[M_n]$ (Property VII).*

If we can construct a true symbol in $S_{\rho\delta\theta}[M_n]$ from every formal symbol in $\mathcal{S}_{\rho\delta\theta}[M_n]$, Theorem 3.1 2) brings immediately the existence of a parametrix of elliptic operator in $\boldsymbol{S}_{\rho\delta\theta}[M_n]$. However, unfortunately, the construction of true symbols seems impossible if $\{M_n\}$ does not satisfy Separativity Condition (S). In this paragraph, we shall try to construct a symbol of a parametrix of elliptic operator through a Neumann series. In order to show the convergence of this series, we need a sharp estimate on the symbol of the multiproduct of ps.d.op's in $\boldsymbol{S}_{\rho\delta\theta}[M_n]$. Such estimate was obtained in the case of Gevrey classes by K. Taniguchi [39] and by C. Iwasaki [16]. Their proofs are available in case of general $\{M_n\}$.

Theorem 5.3. *We suppose $(L)_{2l_1}$ on $\{M_n\}$, $(L)_{2l_0}$ on $\{L_n\}$ and one of* a), b) *and* c) *in Theorem* 5.2, *Case I. We assume the following estimates*;

$$\exists C_j>0,\ \exists R>0,\ \forall\alpha,\ \beta\in\boldsymbol{Z}_+^l,$$

$$(5.53) \qquad |p_{j(\alpha)}^{(\beta)}(x, \xi)| \leqq C_j R^{|\alpha+\beta|} M_{|\alpha|}\beta! \langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|}, \quad \textit{for} \quad (x, \xi)\in\boldsymbol{R}^l\times\{\xi;\ \langle\xi\rangle^{\theta}\geqq RD_{|\beta|}\},$$

$$(5.54) \qquad |p_{j(\alpha)}^{(\beta)}(x, \xi)| \leqq C_j R^{|\alpha+\beta|} M_{|\alpha|} L_{|\beta|} \langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|}, \quad \textit{for} \quad (x, \xi)\in\boldsymbol{R}^l\times\boldsymbol{R}^l,\ (1\leqq j\leqq m).$$

Then, we can devide $\sigma(P_1\circ\cdots\circ P_m)(x, \xi)$ to $r_0^m(x, \xi)+r_\infty^m(x, \xi)$ which satisfy the *followings*

$$\exists C_0,\ R_0>0,\ \forall\alpha,\ \beta\in \boldsymbol{Z}^l_+,$$

$$|r^{m(\beta)}_{0\,(\alpha)}(x,\ \xi)|\leqq C^m_0(\prod_{j=1}^m C_j)(R_0R)^{|\alpha+\beta|}M_{|\alpha|+2l_1+\bar m}\beta!\langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|} \quad \text{for} \quad (x,\ \xi)\in \boldsymbol{R}^l\times\{\langle\xi\rangle^\theta\geqq R_0RD_{|\beta|+2l_0}\}, \tag{5.55}$$

$$|r^{m(\beta)}_{0\,(\alpha)}(x,\ \xi)|\leqq C^m_0(\prod_{j=1}^m C_j)(R_0R)^{|\alpha+\beta|}M_{|\alpha|+2l_1+\bar m}L_{|\beta|+2l_0}\langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|} \quad \text{for} \quad (x,\ \xi)\in \boldsymbol{R}^l\times \boldsymbol{R}^l, \tag{5.56}$$

and

$$|r^{m(\beta)}_{\infty(\alpha)}(x,\ \xi)|\leqq C^m_0(\prod_{j=1}^m C_j)(R_0R)^{|\alpha+\beta|}M_{(N+|\alpha|+2l_1)/(1-\delta)}L_{|\beta|+2l_0}\langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|} \quad \text{for} \quad (x,\ \xi)\in \boldsymbol{R}^l\times \boldsymbol{R}^l, \tag{5.57}$$

where C_0 *and* R_0 *are independent of* m, α, β *and the symbols* $\{p_j\}$. *In case of* b), *if* $\{M_n\}$ *satisfies* $(D)_s$ *and* $\{L_n\}$ *does* (D), $M_{|\alpha|+2l_1+\bar m}$, $D_{|\beta|+2l_0}$, $M_{(N+|\alpha|+2l_1)/(1-\delta)}$ *and* $L_{|\beta|+2l_0}$ *are replaced by* $M_{|\alpha|}$, $D_{|\beta|}$, $M_{(N+|\alpha|)/(1-\delta)}$ *and* $L_{|\beta|}$, *respectively.*

Theorem 5.3 implies the following.

Corollary 5.4. *We assume* $\rho>\delta$, $(L)_{4l_1+1}$ *on* $\{M_n\}$, $(L)_{4l_0+1}$ *on* $\{L_n\}$ *and one of* a), b) *and* c) *in Theorem* 5.2 *Case I. Let a square matrix* $P(x, D)$ *with elements in* $\boldsymbol{S}^m_{\rho\delta\theta}[M_n]$ *be elliptic, that is,*

$$\exists C_0,\ R_0>0\,, \quad |\det\{p(x,\ \xi)/\langle\xi\rangle^m\}|\geqq C_0 \qquad \text{for} \quad \langle\xi\rangle\geqq R_0\,. \tag{5.58}$$

Then, there exists $Q(x, D)$ *in* $\boldsymbol{S}^{-m}_{\rho\delta\theta}[M_{n+6l_1+\bar m+1}, L_{n+6l_0+1}]$, *such that* $P\circ Q-I$ *and* $Q\circ P-I$ *belong to* $\boldsymbol{S}^{-\infty}[M_{(n+6l_1+|m|+1)/(1-\delta)}, L_{n+6l_0+1}]$. *In case of* b), *if* $\{M_n\}$ *satisfies* $(D)_s$ *and* $\{L_n\}$ *does* (D), $M_{n+6l_1+\bar m+1}$, $M_{(n+6l_1+|m|+1)/(1-\delta)}$ *and* L_{n+6l_0+1} *are replaced by* M_n, $M_{n/(1-\delta)}$ *and* L_n, *respectively.*

Remark 1. The assumption of the ellipticity of $P(x,D)$ can be weakened to (H_2) and (H_3) in S. Hashimoto, T. Matsuzawa and Y. Morimoto [13]. (See K. taniguchi [39].)

Remark 2. If $\{M_n\}$ satisfies Condition (S), we can obtain the same result in case of $0\leqq\delta<\underline{\rho\leqq 1}$ replacing modulo class by $\underline{\boldsymbol{S}^{-\infty}[M_{n/(\rho-\delta)}]}$. (See Corollary 6.4.)

§ 6. Construction of true symbol in $S_{\rho\delta\theta}[M_n]$ (Property VIII) and parametrices of elliptic operators in $S_{\rho\delta\theta}[M_n]$ (Property VII).

In the previous paragraph, we constructed a parametrix of elliptic operator in $\boldsymbol{S}_{\rho\delta\theta}[M_n]$ through a Neumann series except the case of $\rho=1$ and $\theta>0$. In order to clear this exceptional case, we return to the idea developed by L. Boutet de Monvel and P. Krée [7] and F. Treves [40]: construction of true symbol from formal symbol. However, in case without Condition (S), the results are not satisfactory.

Theorem 6.1. *We suppose that* $0<\theta\leqq\rho-\delta$.
1) *We take arbitrary k in* $\boldsymbol{Z}_+$. *For each formal symbol* $\sum p_i(x,\xi)$ *in* $\mathcal{S}^m_{\rho\delta\theta}[M_{n-k-1}]$, *there exists a true symbol* $p(x,\xi;k)$ *which satisfies* (2.8) *limitting* $|\alpha|\leqq k$ *and* $|\beta|\leqq k$ *and also does*

$$\exists R>0,\ \forall N\in\boldsymbol{Z}_+,\ \alpha,\ \beta\in\boldsymbol{Z}^l_+\ (|\alpha|,\ |\beta|\leqq k),\ \exists C_\alpha>0\,,$$

$$|\{p(x,\xi)-\sum_{i<N}p_i(x,\xi)\}^{(\beta)}_{(\alpha)}|\leqq C_\alpha R^{N+|\beta|}M_N\beta!\langle\xi\rangle^{m-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|}\,, \tag{6.1}$$

$$\text{for}\quad (x,\xi)\in\boldsymbol{R}^l\times\{\langle\xi\rangle^\theta\geqq RD_N\}.$$

If $\{M_n\}$ *satisfies* $(D)_s$, *we can replace* M_{n-k-1} *by* M_n.
2) *We assume Condition* $(W.S)_s$ *on* $\{M_n\}$. *For each formal symbol* $\sum p_i(x,\xi)$ *in* $\mathcal{S}^m_{\rho\delta\theta}[M_n]$, *there exists a true symbol* $p(x,\xi)$ *in* $\mathrm{S}^m_{\rho\delta}$ *which satisfies* (6.1) *for arbitrary* α *and* β.

On the other hand, there exist a positive sequence $\{L'_n\}$ *and another true symbol* $\tilde{p}(x,\xi)$ *in* $\mathrm{S}^m_{\rho\delta\theta}[M_n, L'_n]$ *which satisfies.*

$$\tilde{p}(x,\xi)\sim\sum p_i(x,\xi)\,.$$

3) *We assume Condition* (S) *on* $\{M_n\}$. *For each formal symbol* $\sum p_i(x,\xi)$ *in* $\mathcal{S}^m_{\rho\delta\theta}[M_n, L_n]$, *there exists a sequence of positive numbers* $\{L'_n\}$ $(L'_n\geqq L_n)$ *and a true symbol* $p(x,\xi)$ *in* $\mathrm{S}^m_{\rho\delta\theta}[M_n, L'_n]$ *which satisfies*

$$p(x,\xi)\underset{[M_n]}{\overset{\rho\delta\theta m}{\sim}}\sum p_i(x,\xi)\,.$$

Remark. In 2) and 3), if $\{L_n\}$ satisfies Condition (B), $\{L'_n\}$ is taken as $\{L_{(d+1)n}\}$, where d is that in $(B)_1$ for $\bar{R}=eH$, H is that in (S.3).

Theorem 6.2. *We assume* (D), (C) *and the following*:
(C^*) $\exists R_1,\ R_2$ *such that* $R_0^2<R_1<R_2$ *and*

$$T(R_1r)T(r/R_2)/T(r)^2\quad\textit{is bounded,}$$

where $T(r)$ *is the associated function of* $\{M_n\}$ *and* R_0 *is that in Condition* (C). *Then, the results* 2) *and* 3) *in Theorem* 6.1 *hold replacing* θ *by* ∞ *and relaxing* $(D)_s$ *by* (D) *and* $(W.S)_s$ *by* $(W.S)$.

Remark. In case of $\theta=\infty$, we also obtain the result corresponding to 1) in Theorem 6.1. As it is complicated, we omit it.

The following is brought by Theorems 6.1 and 6.2 and the proof of Theorem 4.5.

Corollary 6.3. *We assume Condition* (S) *and take* $\theta=\rho-\delta$ *or* $\theta=\infty$. *We also assume the conditions in Theorem* 6.2 *when* $\theta=\infty$. *Then, the relation of asymptotic expansion* $p(x,\xi)\underset{[M_n]}{\overset{\rho\delta\theta m}{\sim}}\sum p_i(x,\xi)$ *gives a onto-homomorphism of star algebra from*

$\mathcal{S}_{\rho\delta\theta}[M_n]$ to $\mathrm{S}_{\rho\delta\theta}[M_n]$ modulo symbols of $\{M_{n/(\rho-\delta)}\}$ strong regularizers.

We can show Theorem 6.1 following F. Treves [40] and S. Hashimoto, T. Matsuzawa and Y. Morimoto [13]. On the other hand, we can show Theorem 6.2 following L. Boutet de Monvel and P. Krée [7] and L. Boutet de Monvel [6]. In order to make clear the role of Condition (S), we present the proof of Theorem 6.1 Case 3).

Proof of Theorem 6.1 *Case* 3). Under Assumption, we can assume $D_n \geqq n$. Using $\{\psi_k(t)\}$ in Lemma 2.0.3, we set

$$\chi_k(\xi) = \psi_k(\theta \log \langle\xi\rangle - \log (R_1 D_k)), \tag{6.2}$$

where $R_1 = H^2 R$ and H is that in (S.3). $\chi_k(\xi)$ satisfies

$$|\chi_k^{(\beta)}(\xi)| \leqq \begin{cases} C e^k R^{|\beta|} \beta! \langle\xi\rangle^{-|\beta|} & (0 \leqq |\beta| \leqq k), \\ C R^{|\beta|} L_{|\beta|} \langle\xi\rangle^{-|\beta|} & (\beta \in \boldsymbol{Z}_+^l). \end{cases} \tag{6.3}$$

Let us set

$$p(x, \xi) = \sum_{i=0}^{\infty} p_i(x, \xi) \chi_i(\xi). \tag{6.4}$$

On each compact set in $\boldsymbol{R}_\xi^l$, the right-hand side of (6.4) is a finite sum and then it converges and belongs to $\mathcal{E}(\boldsymbol{R}^l \times \boldsymbol{R}^l)$.

For arbitrary n in $\boldsymbol{Z}_+ \backslash \{0\}$, we limit ξ to $G_n = \{ eR_1 D_n \leqq \langle\xi\rangle^\theta \leqq eR_1 D_{n+1} \}$. Since supp χ_i is included in $\{ \langle\xi\rangle^\theta \geqq R_1 D_i \}$, if supp $\chi_i \cap G_n \neq \phi$, it holds that

$$D_i \leqq eD_{n+1} \leqq eHD_n. \tag{6.5}$$

We set $d(n) = \max\{i;\ \text{supp}\, \chi_i \cap G_n \neq \phi\}$. Since, by virtue of (S.3) and (6.5), we have

$$D_{|\beta|+i} \leqq HD_{\max(n,i)} \leqq eH^2 D_n, \qquad (|\beta| \leqq n,\ i \leqq d(n)), \tag{6.6}$$

we can apply the estimate (3.1) to $p_{i(\alpha)}^{(\beta)}(x, \xi)$ on G_n for $|\beta| \leqq n$ and $i \leqq d(n)$.

Paying attention to $\chi_i(\xi) \equiv 1$ on G_n if $i \leqq n$, we see that $p(x, \xi) - \sum_{i<N} p_i(x, \xi) = \sum_{N \leqq i \leqq d(n)} p_i(x, \xi) \chi_i(\xi)$ on G_n for $N \leqq n$. Applying (S.1), we obtain

$$\begin{aligned} |\{p - \sum_{i<N} p_i\}_{(\alpha)}^{(\beta)}| \leqq\, & C(HR)^{N+|\alpha+\beta|} M_{|\alpha|} \beta! \langle\xi\rangle^{m-(\rho-\delta)N - \rho|\beta| + \delta|\alpha|} \times \\ & \times \sum_{N \leqq i \leqq d(n)} (HR)^{i-N} M_i / (eR_1 D_n)^{i-N} \sum_{\beta' \leqq \beta} 1. \end{aligned} \tag{6.7}$$

As it holds that $M_i/(eHD_n)^{i-N} \leqq M_N$ $(i \leqq d(n))$ by virtue of (6.5) and $\#\{\beta' \leqq \beta\} \leqq 2^{|\beta|+l-1}$, we arrive at

$$(6.7') \qquad |\{p-\sum_{i<N} p_i\}_{(\alpha)}^{(\beta)}| \leqq CR^{N+|\alpha+\beta|} M_N M_{|\alpha|} \beta! \langle\xi\rangle^{m-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|}$$
$$\text{on } G_n \text{ for } |\beta| \leqq n\,.$$

On the other hand, on G_n and for $|\beta|>n$, we have

$$(6.8) \qquad |p_{(\alpha)}^{(\beta)}| \leqq C(HR)^{|\alpha+\beta|} M_{|\alpha|} L_{d(n)+|\beta|} \langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|} \times$$
$$\times \sum_{0\leqq i\leqq d(n)} (HR)^i M_i/(eR_1 D_n)^i \sum_{\beta'\leqq\beta} 1$$
$$\leqq C' R'^{|\alpha+\beta|} M_{|\alpha|} L_{d(|\beta|)+|\beta|} \langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|}\,. \qquad \text{Q.E.D.}$$

We remark that (6.6) holds if and only if $\{M_n\}$ satisfies Condition (S). (See (S.3).) Further, in order to obtain (6.7), we need

$$\exists R>0, \quad N\leqq \forall i \leqq d(N), \quad \forall k\leqq N, \quad M_{i+k} \leqq R^{i+k} M_{N+k}(D_N)^{i-N}\,.$$

This is (S.5) itself.

On the other hand, in the proof of Theorem 6.2, we use the following;

For some positive R independent of k, the norm of $\{M_{n+k}\}_{n=0}^{\infty}$ in $l^2\{R^n M_n\}$ is bounded by $R^k M_k$, where $l^2\{N_n\} = \{\{c_n\}_{n=0}^{\infty};\ \|\{c_n\}\|^2 \equiv \sum_{n=0}^{\infty} (c_n/N_n)^2 < \infty\}$. This is equivalent to Condition (S.1).

Thus, as long as we adopt the method by F. Treves or L. Boutet de Monvel and P. Krée, the construction of true symbol in $S_{\rho\delta\theta}[M_n]$ from formal symbol in $\mathcal{S}_{\rho\delta\theta}[M_n]$ goes well if and only if $\{M_n\}$ satisfies (S).

By Theorems 6.1 and 6.2, we immediately obtain the following;

Corollary 6.4. *We restrict θ to $0<\theta\leqq\rho-\delta$ or $\theta=\infty$. We assume the same conditions in Theorem 6.2 when $\theta=\infty$. We assume also that a matrix $P(x, D)$ with elements in $\boldsymbol{S}_{\rho\delta\theta}^m[M_n]$ is elliptic, that is,*

$$(6.9) \qquad \exists C_0>0,\ \exists r_0>0,\ \det|\{p(x, \xi)/\langle\xi\rangle^m\}| \geqq C_0 \quad \textit{for} \quad (x, \xi)\in \boldsymbol{R}^l\times\{\langle\xi\rangle\geqq r_0\}\,.$$

1) *We take arbitrary k in $\boldsymbol{Z}_+$. There exists a ps.d.op $Q(X, D; k)$ which is continuous from $\mathcal{D}_{L^2}{}^{-m+j}$ to $\mathcal{D}_{L^2}{}^{j}$ $(0\leqq j\leqq k-2l_1)$. Further, it satisfies*

$$\exists R>0,\ \forall N\in\boldsymbol{Z}_+,\ \alpha,\ \beta\in\boldsymbol{Z}_+^l\ (|\alpha|,\ |\beta|\leqq k),\ \exists C_\alpha>0\,,$$
$$(6.10) \qquad |\sigma(P\circ Q-I)_{(\alpha)}^{(\beta)}|,\ |\sigma(Q\circ P-I)_{(\alpha)}^{(\beta)}| \leqq C_\alpha R^{N+|\beta|} M_{N+k+1}\beta! \langle\xi\rangle^{m-(\rho-\delta)N-\rho|\beta|+\delta|\alpha|}$$
$$\textit{for} \quad (x, \xi)\in\boldsymbol{R}^l\times\{\langle\xi\rangle^\theta\geqq RD_N\}\,.$$

If $\{M_n\}$ satisfies $(D)_s$, we can replace M_{N+k+1} by M_N.

2) *We assume Condition $(W.S)_s$ when $\theta\leqq\rho-\delta$ and $(W.S)$ when $\theta=\infty$. There exists a ps.d.op $Q(x, D)$ in $\boldsymbol{S}^{-m}$ which satisfies* (6.10) *for arbitrary α and β. On the other hand, there exist a positive sequence $\{L_n'\}$ and another ps.d.op $\tilde{Q}(x, D)$ in $\boldsymbol{S}_{\rho\delta\theta}^{-m}[M_n, L_n']$, where $P\circ\tilde{Q}-I$ and $\tilde{Q}\circ P-I$ belong to $\boldsymbol{S}^{-\infty}$.*

3) *We assume Condition (S). There exists a sequence of positive numbers $\{L_n'\}$ $(L_n'\geqq L_n)$ and a ps.d.op $Q(x, D)$ in $\boldsymbol{S}_{\rho\delta\theta}^{-m}[M_n, L_n']$ such that $P\circ Q-I$ and $Q\circ P-I$ belong to $\boldsymbol{S}^{-\infty}[M_{n/\theta'}]$, $\theta'=\min\{\theta, \rho-\delta\}$.*

Appendix A. Proof of Theorem 4.5.

A.1. *Proof of Theorem* 4.5, *sufficiency.*

Under Separativity Condition, we show only the continuity of $P(x, D)$ in $S^{-\infty}[M_n]$ from $\mathscr{D}'_{L^2}\{M_n\}$ to $\mathscr{D}_{L^2}\{M_n\}$ because the continuity from $\mathscr{E}'\{M_n\}$ to $\mathscr{B}\{M_n\}$ follows immediately from this by virtue of Sobolev's lemma. We use a "regularized associated function" $\tilde{T}(r)$. Using a non-negative function $\rho(r)$ in $\mathscr{D}(\boldsymbol{R})$ with its support in $[-1, 1]$ and $\|\rho\|_{L^1}=1$, we set $\tilde{T}(r)=\int\rho(r-s)T(s)ds$ $(r\in\boldsymbol{R})$ where $T(s)$ $(s<0)$ is defined by $T(-s)$. Then, we easily see that $\tilde{T}(r)$ belongs to $\mathscr{E}(\boldsymbol{R})$ and satisfies

$$T(r/2)\leqq\tilde{T}(r)\leqq T(2r) \qquad (r\geqq 2)\,, \tag{A.1}$$

$$|D_r^j\tilde{T}(r)|\leqq C_jT(2r) \qquad (r\in\boldsymbol{R},\ j\in\boldsymbol{Z}_+)\,. \tag{A.2}$$

Owing to (A.1), we can use $\tilde{T}(r)$ instead of the associated function $T(r)$.

By virtue of (S.6) in Proposition 2.5 and (4.10′), we have

$$\begin{gathered}\exists r_0>0,\ \exists C,\ R>0,\ \forall\alpha\in\boldsymbol{Z}_+^l,\ \beta\in\boldsymbol{Z}_+^l\ (|\beta|\leqq 2l_0)\,,\\ |\xi^\gamma p_{(\alpha)}^{(\beta)}(x,\xi)|\leqq CR^{|\alpha|}M_{|\alpha|}\langle\xi\rangle^{|\gamma|}\tilde{T}(\langle\xi\rangle/r_0)^{-3}\,.\end{gathered} \tag{A.3}$$

On the other hand, let B be a bounded set in $\mathscr{D}'_{L^2}\{M_n\}$. Applying Proposition 2.2, we have the representation of the Fourier image of u in B:

$$\forall r_0>0,\ \exists v(x)=v(x, r_0)\in L^2(\boldsymbol{R}^l),\ \hat{u}(\xi)=\tilde{T}(\langle\xi\rangle/r_0)\hat{v}(\xi)\,.$$

Then, $D_x^\alpha(P(x, D)u)$ is expressed as follows:

$$\begin{aligned}(Pu)_{(\alpha)}&=\sum_{\alpha'}\binom{\alpha}{\alpha'}\int e^{\sqrt{-1}x\cdot\xi}\xi^{\alpha-\alpha'}p_{(\alpha')}(x,\xi)\hat{u}(\xi)d\xi\\ &=\sum_{\alpha'}\binom{\alpha}{\alpha'}\int\langle x-y\rangle^{-2l_0}e^{\sqrt{-1}(x-y)\cdot\xi}v(y)dy\times\\ &\quad\times\int(1-\Delta_\xi)^{l_0}\{\xi^{\alpha-\alpha'}p_{(\alpha')}(x,\xi)\tilde{T}(\langle\xi\rangle/r_0)\}\,d\xi\,.\end{aligned} \tag{A.4}$$

The above $(1-\Delta_\xi)^{l_0}\{\cdots\}$ is majorized by $CR'^{|\alpha|}M_{|\alpha|}\tilde{T}(\langle\xi\rangle/r_0)^{-1}$ in virtue of (2.6) and (S.1). Applying Schwarz' inequality, it is majorized as follows;

$$|(Pu)_{(\alpha)}|^2\leqq(C'R'^{|\alpha|}M_{|\alpha|})^2\int\langle x-y\rangle^{-2l_0}|v(y)|^2dy\,.$$

Thus, we obtain $\|Pu\|_{\{M_n\},2R'}<\infty$ and this show that the image of B by $P(x, D)$ is bounded in $\mathscr{D}_{L^2}\{M_n\}$. Q.E.D.

A.2. *Proof of Theorem* 4.5, *necessity.* —*Preliminary*—

Assuming that Separativity Condition (S) is not satisfied, we show that there exists a symbol $p(\xi)$ in $S^{-\infty}[M_n]$ and an ultradistribution u in $\mathscr{E}'\{M_n\}$ such that

$P(D)u$ does not belong to $\mathcal{E}\{M_n\}$.

Having regard to Theorem 4.4, we may assume Differentiability Condition (D) on $\{M_n\}$. Preliminarily, we consider the trace function of $\{\log M_n\}$. We use the notation in Paragraph 2.3. By virtue of Proposition 2.5, if $\{M_n\}$ does not satisfy Separativity Condition (S), there exists a sequence $\{r_j\}$ such that

$$\text{(A.5)} \qquad \lim_{j\to\infty} r_j = \infty \quad \text{and} \quad \forall R^0>1,\ \exists j_0\geqq 1,\ \forall j\geqq j_0,\ T(r_j/R_0)^2/T(r_j)>1\,.$$

Setting $t_j=\log r_j$ and $3\gamma=\log R_0$, (5.5) means

$$\text{(A.5}'\text{)} \qquad \lim_{j\to\infty} t_j = \infty \quad \text{and} \quad \forall \gamma>0,\ \exists j_0\geqq 1,\ \forall j\geqq j_0,\ 2H(t_j-3\gamma)-H(t_j)>0\,.$$

We set $\tau_j=t_j-2\gamma$. (A.5′) brings

$$\text{(A.5}''\text{)} \qquad 2H(\tau_j-\gamma)-H(\tau_j)>2\gamma h(\tau_j)\,.$$

Thus, it follows that

$$\text{(A.6)} \qquad 2H(t-\gamma)-H(t)>\gamma h(t) \qquad \text{on} \quad [\tau_j-\gamma,\ \tau_j]\,.$$

In fact, as $h(t)$ is non-decreasing, we have

$$\begin{aligned}\text{(A.7)} \qquad & [2H(\tau_j-\gamma)-H(\tau_j)]-[2H(t-\gamma)-H(t)]\\ & \leqq \{2h(\tau_j-\gamma)-h(t)\}(\tau_j-t)\\ & \leqq h(\tau_j-\gamma)(\tau_j-t)\leqq \gamma h(\tau_j) \qquad \text{for} \quad \tau_j-\gamma\leqq t\leqq \tau_j\,.\end{aligned}$$

Combining (A.5″) and (A.7), we obtain (A.6).

Standing on the inequality (A.6), we have a lemma.

Lemma A.1. *There is a function $G(t)$ which satisfies*

$$\text{(A.8)} \qquad \forall k\in \boldsymbol{Z}_+,\ \exists s(k)>0,\ G(t)\leqq H(t-k)-k \qquad \textit{for} \quad t\geqq s(k)\,,$$

$$\begin{aligned}\text{(A.9)} \qquad & \forall k\in \boldsymbol{Z}_+,\ \exists \tau(k)>0,\ 2G(t)-H(t)\geqq 0 \qquad \textit{on} \quad [\tau(k)-k,\ \tau(k)]\,.\\ & \textit{and } \tau(k)>s(k)+k>\tau(k-1)+k\,.\end{aligned}$$

Proof. Let us set

$$H_k(t) = H(t-k)-k\equiv (t-k)h(t-k)-a_{h(t-k)}-k\,.$$

We define inductively $s(k)$ and $\tau(k)$ and construct $G(t)$ which satisfies

$$G(t) = H_k(t) \qquad \text{for} \quad s(k)\leqq t\leqq \tau(k)\,.$$

Step 1. We set $s(0)=0$, $\tau(0)=1$ and $G(t)=H_0(t)$ on $[0, 1]$.
Step 2. Let k be a natural number. Suppose that $s(k-1)$, $\tau(k-1)$ ($s(k-1)+k-1<\tau(k-1)$) and $G(t)$ for $t\leqq\tau(k-1)$ are already defined and that $G(t)$ satisfies

$$\text{(A.10)} \qquad G(t) = H_{k-1}(t) \qquad \text{for} \quad s(k-1)\leqq t\leqq \tau(k-1)\,.$$

Let us set

(A.11) $F_{k-1}(t)=\{t-(k-1)\}h(\tau(k-1)-(k-1))-a_{h(\tau(k-1)-(k-1))}-(k-1)$

and

(A.12) $$s(k)=\min\{t;\ t>\tau(k-1) \text{ and } F_{k-1}(t)=H_k(t)\}.$$

Since $H_k(t)>F_{k-1}(t)$ for sufficiently large t and

$$H_k(t)|_{t=\tau(k-1)+1}=F_{k-1}(\tau(k-1))-1,$$

$s(k)$ is well-defined and larger than $\tau(k-1)+1$. We define

(A.13) $$G(t)=F_{k-1}(t) \qquad \text{on} \quad [\tau(k-1), s(k)].$$

Then, it follows that

(A.14) $$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_r (H_k(t))|_{t=s(k)}=h(s(k)-k)$$
$$\geqq h(\tau(k-1)-k+1)\geqq\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\text{left}} G(t)|_{t=s(k)}.$$

Let us take $\gamma=k$ in (A.6) and take one of the element of $\{\tau_j\}$ $(j\geqq j_0)$ such that

(A.15) $$\tau_j-k>s(k) \quad \text{and} \quad h(t)\geqq 2 \qquad \text{for} \quad t\geqq\tau_j-k.$$

We set

(A.16) $$\tau(k)=\tau_j \quad \text{and} \quad G(t)=H_k(t) \qquad \text{on} \quad [s(k), \tau(k)].$$

Thus, $s(k)$, $\tau(k)$ $(k\in \boldsymbol{Z}_+)$ and $G(t)$ $(t\geqq 0)$ are defined. They also satisfy (A.9) by virtue of (A.6) and (A.15). Q.E.D.

Remark. A function is a trace function of some $\{a_n\}$ when it is increasing, convex and piecewise linear with integer valued slopes. $G(t)$ defined in the above lemma is a trace function.

A.3. *Proof of Theorem* 4.5, *necessity.* —*Continued*—

Let us set

$$S(r)=\exp G(\log r) \quad \text{and} \quad N_n=\sup_r r^n/S(r).$$

Obviously, $S(r)$ is the associated function of $\{N_n\}$. (A.8) implies

(A.17) $$\forall R>0,\ N_n\geqq R^{n+1}M_n \qquad (n\gg 1).$$

Moreover, (A.8) and (A.9) mean

(A.18) $$\forall k\in \boldsymbol{Z}_+,\ S(r)\leqq \mathrm{e}^{-k}T(r/\mathrm{e}^k),$$

and

(A.19) $$\forall k\in \boldsymbol{Z}_+, \quad S(r)^2/T(r)\geqq 1 \qquad \text{on} \quad [r_k^{(1)}, r_k^{(2)}],$$

where $r_k^{(1)}=e^{-k+\tau(k)}$ and $r_k^{(2)}=e^{\tau(k)}$.

$\tilde{T}(z)=\sum_{j=0}^{\infty} z^j/(2^j M_j)$ is an entire function and satisfies

$$\text{(A.20)}\qquad \begin{aligned} &\forall d>1,\quad \exists C(d)>0,\quad T(r/2)\leqq\tilde{T}(r)\leqq C(d)T(dr/2) \qquad (r\geqq 0)\,,\\ &\forall n\in \boldsymbol{Z}_+,\quad |D_z^n\tilde{T}(z)|\leqq 2^{-n}\tilde{T}(|z|) \qquad (z\in\boldsymbol{C})\,, \end{aligned}$$

because of $M_n/M_{n-1}\geqq n$ $(n\geqq 1)$. [See (5.14)] We set

$$\text{(A.21)}\qquad p(\xi)=\tilde{T}(\langle\xi\rangle)^{-1}\,.$$

By virtue of Proposition 2.1 ii), this is real analytic in $\boldsymbol{R}^l$ and satisfies

$$\text{(A.22)}\qquad \exists C,\ R>0,\quad |p^{(\beta)}(\xi)|\leqq CR^{|\beta|}\beta!\tilde{T}(\langle\xi\rangle)^{-1}\,,$$

because $n!$ satisfies the condition (R) with $H=1$. (See also the proof of Theorem A in W. Rudin [36].) This means that $p(\xi)$ belongs to $S^{-\infty}[M_n]$.

Now, we show that the image of $\mathcal{E}'\{M_n\}$ by $P(D)$ is not included in $\mathcal{E}\{M_n\}$. First, we assume Weak Separativity Condition $(W.S)$ on $\{M_n\}$. Under this restriction, we can take

$$\text{(A.23)}\qquad G(t)\geqq t^2\,.$$

In fact, as $F_{k-1}\ll H_k(t)$ and $t^2\ll H_k(t)$ $(t\gg 1)$, $\tilde{s}(k)=\min\{t;\ t>\tau(k-1)$ and $\tilde{F}_{k-1}(t)=H_k(t)\}$ is well defined. Here, $\tilde{F}_k(t)$ is defined by $\max\{F_k(t),\ t^2\}$. If we use $\tilde{F}(t)$ and $\tilde{s}(k)$ instead of $F_k(t)$ and $s(k)$, $G(t)$ satisfies (A.23).
(A.23) means $\{N_k\}$ satisfies Differentiability Condition (D). We set

$$\text{(A.24)}\qquad u=\sum_{j=0}^{\infty}(-1)^j H^{2j}\delta^{(2j,0,\cdots,0)}/N_{2j}\,,$$

where H is that in Condition (D.1) on $\{N_n\}$ and $\delta^{(\alpha)}=\left(\dfrac{\partial}{\partial x}\right)^{\alpha}\delta$. u belongs to $\mathcal{E}'\{M_n\}$ by virtue of (A.17) and $P(D)u$ belongs to $\mathcal{E}$ due to Theorem 4.4. If $P(D)u$ further belongs to $\mathcal{E}\{M_n\}$, $|\langle u,\ P(D)u\rangle|$ must be bounded. However, we have

$$\text{(A.25)}\qquad \exists R>0,\ \forall d>1,\ \exists C(d)>0,\quad S(|\xi_1|)\leqq\hat{u}(\xi)\leqq C(d)S(dH|\xi_1|)(|\xi_1|>1)\,,$$

because of Differentiability Condition on $\{N_n\}$. Then, it is seen that

$$\text{(A.26)}\qquad \begin{aligned} &\langle\sum_{j=0}^{n}(-1)^j H^{2j}\delta^{(2j,0,\cdots,0)}/N_{2j},\ P(D)u\rangle\\ &=\int\{\sum_{j=0}^{n}(H\xi_1)^{2j}/N_{2j}\}\tilde{T}(\langle\xi\rangle)^{-1}\{\sum_{k=0}^{\infty}(H\xi_1)^{2k}/N_{2k}\}\,d\xi\\ &\geqq C\int_{\substack{\xi_1\geqq\sqrt{2}\\ h(\log\xi_1)\leqq 2n}} S(|\xi_1|)^2 T(|\xi_1|)^{-1}d\xi_1\int_{|\xi'|\leqq 1}d\xi'\to\infty\qquad(n\to\infty)\\ &\qquad\qquad [\,\xi'=(\xi_2,\cdots,\xi_l)\,]\,, \end{aligned}$$

by (A.19), (A.20) with $d=\sqrt{2}$ and (A.25). This implies that $P(D)u$ does not belong to $\mathcal{E}\{M_n\}$.

In case that $\{M_n\}$ satisfies only Differentiability Condition, modifying $p(\xi)$ to

$$p(\xi)=\tilde{T}(\langle\xi\rangle)^{-1}\phi(\xi_1)$$

and u to

$$u=\sum_{j=0}^{\infty}(-\sqrt{-1})^j\delta^{(j,0,\cdots,0)}/N_j\,,$$

we arrive at the same conclusion, where

$$\text{(A.27)}\qquad \begin{cases} 0\leqq\phi(\xi_1)\leqq 1, & \phi(\xi_1)\in\mathcal{B}\{L_n\}(\boldsymbol{R}),\\ \phi(\xi_1)=\begin{cases}1 & \xi_1\geqq\sqrt{2}\,,\\ 0 & \xi_1\leqq 1\,.\end{cases} \end{cases}$$

Q.E.D.

Appendix B. Equivalence between (S.2), (S.3), (S.4) and (S.5).

We rewrite (S.j) $(1\leqq j\leqq 5)$ using $a_n=\log M_n$ and $a'_n=a_{n+1}-a_n$.

(1) $\exists\gamma>0,\ \forall n,\ m\gg 1,\qquad a_{n+m}\leqq a_n+a_m+(n+m)\gamma,$

(2) $\exists\gamma>0,\ \forall n\gg 1,\qquad a_{2n}/(2n)\leqq(a_n/n)+\gamma,$

(3) $\exists\gamma>0,\ \forall n\gg 1,\qquad a'_{2n}\leqq a'_n+\gamma,$

(4) $\exists\gamma>0,\ \forall n\gg 1,\qquad a'_n\leqq(a_n/n)+\gamma,$

(5) $\exists\gamma>0,\ \forall n,\ m\gg 1,\ k\leqq n,\qquad a_{k+m}\leqq ka'_n+a_m+(k+m)\gamma.$

Proof. i) The equivalence between (2) and (4) follows from

$$\text{(B.1)}\qquad na'_n\leqq\sum_{j=n}^{2n-1}a'_j=a_{2n}-a_n\leqq na'_{2n}\,.$$

ii) Note that (1) and (2) are equivalent. (5) is derived from (1) and the relations $a_k/k\leqq a'_k$ and $k\leqq n$.

iii) (5)$\Rightarrow$(3). In (5), we take $m=sn$ $(s\in\boldsymbol{Z}_+\backslash\{0\})$ and $k=n$. Then, it holds that $a_{(s+1)n}\leqq na'_n+a_{sn}+(s+1)n\gamma$. Therefore, we have

$$na'_{sn}\leqq a_{(s+1)n}-a_{sn}\leqq na'_n+(s+1)n\gamma\,.$$

If we set $s=2$, the above inequality implies (3).

iv) (3)$\Rightarrow$(2). For $n\geqq 3$, by virtue of (3), it holds that

$$\text{(B.2)}\qquad a'_{2n}\leqq a'_n+\gamma\leqq a'_{[n/2]+1}+2\gamma\leqq a'_{n-1}+2\gamma\,.$$

Then, we have

$$\begin{aligned} a_{4n}-a_{2n}&=\sum_{j=2n}^{4n-1}a'_j\leqq 2\sum_{j=n+1}^{2n}a'_{2j}\leqq 2\sum_{j=n+1}^{2n}(a'_{j-1}+2\gamma)\\ &\leqq 2(a_{2n}-a_n)+4n\gamma\,,\end{aligned}$$

that is,

(B.3) $$(a_{4n}/4n)-(3/2)(a_{2n}/2n)+(1/2)(a_n/n)\leqq\gamma\ .$$

We set $b_n=n\{(a_{2n}/2n)-(a_n/n)\}$. Then, (B.3) means that

(B.3′) $$b_{2n}-b_n\leqq 2\gamma n\ .$$

This implies

$$a_{2^{n+1}}-2a_{2^n}\leqq\gamma 2^{n+2}+2(b_4-8\gamma)\leqq\gamma' 2^n \qquad (\exists\gamma'>0)\ .$$

Thus, for $2^{n-1}\leqq k\leqq 2^n$, we obtain

$$a_{2k}-2a_k\leqq a_{2^{n+1}}-2a_{2^n}\leqq\gamma' 2^n\leqq(2\gamma')k\ ,$$

because $a_{2n}-2a_n$ is increasing on n. Q.E.D.

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