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Global existence and convergence of solutions of Calabi flow on surfaces of genus $h \geq 2$

By

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Abstract

In this paper, based on a kind of Harnack estimate for the Calabi flow on surfaces, we show the long-time existence and convergence of solutions of 2-dimensional Calabi flow on surfaces (Σ, g_0) of genus $h \geq 2$ with any arbitrary background metric g_0.

1. Introduction

Let (Σ, g_0) be a Riemann surface with a given conformal class $[g_0]$ on Σ. We consider the following so-called Calabi flow on $(\Sigma, [g_0])$:

$$\frac{\partial g_{ij}}{\partial t} = (\Delta R) g_{ij}, \qquad g_{ij} \in [g_0]. \tag{1.1}$$

In fact, if $g = e^{2\lambda} g_0$, for a smooth function

$$\lambda : \Sigma \times [0, \infty) \to \mathbf{R},$$

Then equations (1.1) reduce to the following initial value problem of fourth order parabolic equation on $(\Sigma, [g_0])$:

$$\begin{cases} \dfrac{\partial \lambda}{\partial t} = \dfrac{1}{2} \triangle R \\ \lambda(p, 0) = \lambda_0(p) \\ g = e^{2\lambda} g_0 \\ \displaystyle\int_\Sigma e^{2\lambda_0}\, d\mu_0 = \int_\Sigma d\mu_0. \end{cases} \tag{1.2}$$

where $\Delta = \Delta_g$, $\Delta_0 = \Delta_{g_0}$, R is the scalar curvature with respect to the metric g, R_0 is the scalar curvature with respect to the metric g_0, $d\mu_0$ is the volume element of g_0, $d\mu$ is the volume element of g.

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For the background metric g_0 with constant Gaussian curvature, P. T. Chruściel proved that the following result ([Chru]):

Proposition 1.1. *Let (Σ, g_0) be a Riemann surface with the constant Gaussian curvature metric g_0. For any given smooth initial value λ_0, there exists a smooth solution $\lambda(t)$ of (1.2) on $\Sigma \times [0, \infty)$. Furthermore, the metric $g = e^{2\lambda(t)} g_0$ converges to one of the constant curvature metrics.*

In Chruściel's proof, the crucial step is the so-called Bondi mass loss formula, i.e.

$$\frac{d}{dt} \int_\Sigma e^{3\lambda} \, d\mu_0 \leq 0$$

if the background metric g_0 has constant Gaussian curvature. In general, it is not true for any arbitrary background metric g_0.

Here we generalize his results to the case of surfaces (Σ, g_0) of genus $h \geq 2$ with any arbitrary background metric g_0. The key step is, for any arbitrary background metric g_0, the Bondi mass may not decay, but we have a kind of Harnack estimate (Lemma 2.2) on the Bondi mass $\int_\Sigma e^{3\lambda} \, d\mu_0$ as following:

$$\frac{d}{dt} \int_\Sigma e^{3\lambda} \, d\mu_0 \leq C_1(g_0, R_0) + C_2(g_0, R_0) \int_\Sigma e^{-\lambda} \, d\mu_0. \tag{1.3}$$

That is

Theorem 1.2. *Let (Σ, g_0) be a closed Riemann surface with any arbitrary background metric g_0. For any given smooth initial value λ_0, if $\lambda(t)$ have a uniformly lower bound on $\Sigma \times [0, T)$, then there exists a smooth solution $\lambda(t)$ of (1.2) on $\Sigma \times [0, \infty)$. Furthermore, there exists a subsequence of solution, say $\lambda(t_j)$, such that $g = e^{2\lambda(t_j)} g_0$ converges to a constant curvature metric g_∞ as $t_j \to \infty$.*

Remark 1.1. The similar results hold for the 3-dimensional Calabi flow. We refer to [CW] for details.

Next, in case of closed Riemann surfaces with genus $h \geq 2$, we are able to control the uniformly lower bound on $\lambda(t)$ under the Calabi flow (1.2) (Lemma 4.2). Then, inspired by the papers of R. S. Hamilton ([H]) and M. Gursky ([G]), we are able to show the long time existence and convergence of solution of (1.2) on Σ with genus $h \geq 2$. Therefore we recapture the uniformization theorem on closed surfaces with genus $h \geq 2$. The significance of this approach is that the method may lead to a proof of the uniformization theorem on surfaces with finitely many conical singularities, which we will deal with in the forthcoming paper.

More precisely, we have

Theorem 1.3. *Let (Σ, g_0) be a closed surface of genus $h \geq 2$ with any arbitrary background metric g_0. For any given smooth initial value λ_0, there exists a smooth solution $\lambda(t)$ of (1.2) on $\Sigma \times [0, \infty)$. Furthermore, there exists a subsequence of* *solution, say $\lambda(t_j)$, such that $g = e^{2\lambda(t_j)}g_0$ converges to the constant negative curvature metric g_∞ as $t_j \to \infty$.*

One may think the problem here to be more difficult compared to the second order parabolic equations, due to a lack of the maximum principle for fourth order parabolic equations.

In the paper of [Chru], the key C^0-estimate is the so called Bondi-Mass loss formula for the background metric g_0 with constant Gaussian curvature. In our present paper, for any arbitrary background metric g_0, the key C^0-estimate is the Bondi-mass type estimate (1.3).

We briefly describe the methods used in our proofs. In section 2, we will derive the key estimate of (1.2) from Bochner formula (Lemma 2.2).

In section 3, based on Lemma 2.2, we derive a kind of Harnack estimate as in Lemma 3.2, that is, if we assume the uniformly lower bound on λ, then $\lambda(t)$ has an uniformly upper bound and then the uniformly bounds on all $W_{k,2}$ norms which will imply the long time existence of (1.2). Moreover, inspired by Hamilton's work ([H]), we are able to prove convergence of a subsequence of the solution of (1.2).

Hence, in order to show the main Theorem 1.3, all we need is to find an uniformly lower bound on λ in case of Σ of genus $h \geq 2$ as in section 4.

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2. Bondi-Mass type estimate for the Calabi flow

For $g = e^{2\lambda}g_0$, $R_0 = R_{g_0}$, we have the following formulae for (1.2):

$$R = R_g = e^{-2\lambda}(R_0 - 2\Delta_0\lambda) \tag{2.1}$$

$$\Delta R = e^{-2\lambda}\Delta_0 R, \qquad \text{where } \Delta_0 = \Delta_{g_0},\ \Delta = \Delta_g \tag{2.2}$$

$$d\mu = e^{2\lambda}\, d\mu_0, \qquad \text{where } d\mu_0 = d\mu_{g_0},\ d\mu = d\mu_g \tag{2.3}$$

$$\frac{\partial}{\partial t} d\mu = \triangle R\, d\mu, \qquad \frac{\partial R}{\partial t} = -R\triangle R - \triangle^2 R \tag{2.4}$$

$$\int_\Sigma d\mu = \int_\Sigma e^{2\lambda}\, d\mu_0 = \int_\Sigma e^{2\lambda_0}\, d\mu_0 = \int_\Sigma d\mu_0. \tag{2.5}$$

Remark 2.1. (2.5) implies the volume will be fixed under the flow (1.2).

Then we have

Lemma 2.1. *Under the flow* (1.2), *we have*

$$\int_\Sigma R^2\, d\mu \leq C(R_0, \lambda_0),$$

for $0 \leq T \leq \infty$.

Proof. From the Bochner formula on surfaces and (2.4), we have

$$-\frac{1}{2}\frac{d}{dt}\int_\Sigma R^2\,d\mu = \int_\Sigma\left[-\frac{1}{2}R^2(\triangle R) + R(R\triangle R + \triangle^2 R)\right]d\mu$$

$$= \int_\Sigma[(\triangle R)^2 - R|\nabla R|^2]d\mu$$

$$= 2\int_\Sigma\left[\nabla_i\nabla_j R - \frac{1}{2}(\triangle R)g_{ij}\right]^2 d\mu.$$

This implies the Lemma.

Lemma 2.2. *Under the flow* (1.2), *for any arbitrary background metric* g_0, *we have*

$$\frac{d}{dt}\int_\Sigma e^{3\lambda}\,d\mu_0 \le C_1 + C_2\int_\Sigma e^{-\lambda}\,d\mu_0.$$

Proof. From (2.1) and (2.2), we have

$$\frac{d}{dt}\int e^{3\lambda}\,d\mu_0 = \frac{3}{2}\int e^{3\lambda}[e^{-2\lambda}\Delta_0(e^{-2\lambda}R_0) - 2e^{-2\lambda}\Delta_0(e^{-2\lambda}\Delta_0\lambda)]d\mu_0.$$

First we compute

$$-3\int e^{3\lambda}[e^{-2\lambda}\Delta_0(e^{-2\lambda}\Delta_0\lambda)]d\mu_0 = -3\int e^{-\lambda}(\Delta_0\lambda)^2 d\mu_0 - 3\int e^{-\lambda}\Delta_0\lambda|\overset{0}{\nabla}\lambda|^2 d\mu_0.$$

Now let $f = e^{-\lambda}$, then

$$-3\int e^{3\lambda}[e^{-2\lambda}\Delta_0(e^{-2\lambda}\Delta_0\lambda)]d\mu_0$$

$$= -3\int f^{-1}(\Delta_0 f)^2 d\mu_0 - 6\int f^{-3}|\overset{0}{\nabla}f|^4 d\mu_0 + 9\int f^{-2}\Delta_0 f|\overset{0}{\nabla}f|^2 d\mu_0.$$

Integrating by parts and from the Bochner-Lichnerowicz formula

$$\frac{1}{2}\Delta_0|\overset{0}{\nabla}f|^2 = |\overset{0}{\nabla}{}^2 f|^2 + \langle\overset{0}{\nabla}f,\overset{0}{\nabla}\Delta_0 f\rangle + Rc(\overset{0}{\nabla}f,\overset{0}{\nabla}f),$$

we have

$$\int f^{-2}\Delta_0 f|\overset{0}{\nabla}f|^2 d\mu_0 = \frac{2}{3}\int f^{-1}(\Delta_0 f)^2 d\mu_0 - \frac{2}{3}\int f^{-1}|\overset{0}{\nabla}{}^2 f|^2 d\mu_0$$

$$-\frac{2}{3}\int f^{-1}Rc(\overset{0}{\nabla}f,\overset{0}{\nabla}f)d\mu_0 + \frac{2}{3}\int f^{-3}|\overset{0}{\nabla}f|^4 d\mu_0.$$

Then

$$-3\int e^{3\lambda}[e^{-2\lambda}\Delta_0(e^{-2\lambda}\Delta_0\lambda)]d\mu_0$$

$$= -3\int f^{-1}(\Delta_0 f)^2 d\mu_0 - 6\int f^{-3}|\overset{0}{\nabla} f|^4 d\mu_0 + 9\int f^{-2}\Delta_0 f|\overset{0}{\nabla} f|^2 d\mu_0$$

$$= 3\int f^{-1}[(\Delta_0 f)^2 - 2|\overset{0}{\nabla}{}^2 f|^2]d\mu_0 - 6\int f^{-1}Rc(\overset{0}{\nabla} f, \overset{0}{\nabla} f)d\mu_0.$$

For $n = 2$, we have $Rc(g_0) = \frac{1}{2}R_0$, that is

$$-3\int e^{3\lambda}[e^{-2\lambda}\Delta_0(e^{-2\lambda}\Delta_0\lambda)]d\mu_0 = 3\int e^{\lambda}[(\Delta_0 e^{-\lambda})^2 - 2|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2]d\mu_0 - 3\int R_0 e^{-\lambda}|\overset{0}{\nabla}\lambda|^2 d\mu_0.$$

All these imply

$$\frac{d}{dt}\int e^{3\lambda}\,d\mu_0 = \frac{3}{2}\int e^{3\lambda}[e^{-2\lambda}\Delta_0(e^{-2\lambda}R_0) - 2e^{-2\lambda}\Delta_0(e^{-2\lambda}\Delta_0\lambda)]d\mu_0 \tag{2.6}$$

$$= 3\int e^{-\lambda}|\overset{0}{\nabla}\lambda|^2 R_0\,d\mu_0 - \frac{3}{2}\int e^{-\lambda}\langle\overset{0}{\nabla}\lambda, \overset{0}{\nabla} R_0\rangle d\mu_0$$

$$+ 3\int e^{\lambda}[(\Delta_0 e^{-\lambda})^2 - 2|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2]d\mu_0 - 3\int R_0 e^{-\lambda}|\overset{0}{\nabla}\lambda|^2 d\mu_0$$

$$= 3\int e^{\lambda}[(\Delta_0 e^{-\lambda})^2 - 2|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2]d\mu_0 - \frac{3}{2}\int e^{-\lambda}\langle\overset{0}{\nabla}\lambda, \overset{0}{\nabla} R_0\rangle d\mu_0.$$

But for $n = 2$,

$$(\Delta_0 e^{\lambda})^2 - 2|\overset{0}{\nabla}{}^2 e^{\lambda}|^2 \le 0.$$

Then

$$\frac{d}{dt}\int e^{3\lambda}d\mu_0 \le -\frac{3}{2}\int e^{-\lambda}\langle\overset{0}{\nabla}\lambda, \overset{0}{\nabla} R_0\rangle d\mu_0$$

$$\le -\frac{3}{2}\int e^{-\lambda}|\overset{0}{\nabla}\lambda|^2 R_0\,d\mu_0 + \frac{3}{2}\int e^{-\lambda}(\Delta_0\lambda)R_0\,d\mu_0.$$

But

$$2\int e^{-\lambda}(\Delta_0\lambda)R_0\,d\mu_0 = \int e^{-\lambda}(R_0 - e^{2\lambda}R)R_0\,d\mu_0$$

$$= -\int e^{\lambda}RR_0\,d\mu_0 + \int e^{-\lambda}R_0^2\,d\mu_0$$

$$\le C\int d\mu_0 + C\int R^2\,d\mu + \int e^{-\lambda}R_0^2\,d\mu_0$$

$$\le C + \int e^{-\lambda}R_0^2\,d\mu_0.$$

On the other hand,

$$2\Delta_0 e^{-\lambda} = e^{-\lambda}(-2\Delta_0\lambda + 2|\overset{0}{\nabla}\lambda|^2) = e^{-\lambda}[(e^{2\lambda}R - R_0) + 2|\overset{0}{\nabla}\lambda|^2].$$

Then

$$\begin{aligned} -2\int e^{-\lambda}|\overset{0}{\nabla}\lambda|^2 R_0\, d\mu_0 &\leq C\int e^{-\lambda}|\overset{0}{\nabla}\lambda|^2 d\mu_0 \\ &\leq C\int e^{-\lambda}(-e^{2\lambda}R + R_0)d\mu_0 \\ &\leq -C\int e^{\lambda}R\, d\mu_0 + C\int e^{-\lambda}R_0\, d\mu_0 \\ &\leq C + C\int e^{-\lambda}R_0\, d\mu_0. \end{aligned}$$

These imply

$$\frac{d}{dt}\int_\Sigma e^{3\lambda}\, d\mu_0 \leq C_1 + C_2\int_\Sigma e^{-\lambda}\, d\mu_0.$$

Corollary 2.3. *Under the flow* (1.2), *if we have the uniformly bound on* $\int_\Sigma e^{-\lambda}\, d\mu_0$ *for all* $0 \leq t \leq T$.
Then

$$\int_\Sigma e^{3\lambda}\, d\mu_0 \leq C_3 + C_4 t.$$

3. Harnack-type estimate and asymptotic convergence

In this section, based on Lemma 2.2, we will show the Harnack-type estimate (Lemma 3.2) and the C^0-bound of solutions of (1.2) if we assume the uniformly lower bound on λ. Thus, from Lemma 4.1 of [Chru], we will have the uniformly bounds on all $W_{k,2}$ norms which shows the long-time existence of solutions of (1.2). Finally, inspired by Hamilton's work ([H]), we are able to prove convergence of a subsequence of the solution of (1.2).

From now on, the constant C will denote the universal constant which is independent of t, for $t \in [0, \infty]$ and may vary from line to line.

Lemma 3.1. *For a fixed conformal class* $(\Sigma, [g_0])$ *with bounds on Volume and* $\int_\Sigma R^2\, d\mu$, *if*

$$\lambda \geq -C,$$

then

$$\int_\Sigma e^{-2\lambda}(\Delta_0\lambda)^2 d\mu_0 \leq C.$$

Proof. Since $\int_\Sigma R^2\, d\mu \le C$, we have

$$\int_\Sigma e^{-2\lambda}(R_0 - 2\Delta_0\lambda)^2 d\mu_0 \le C.$$

But $\lambda \ge -C$, this implies the Lemma.

Now we will derive the Harnack-type estimate of (1.2).

Lemma 3.2. *Assumptions being the same as in Theorem* 1.2, *under the flow* (1.2), *we have*

$$\int e^{3\lambda}\, d\mu_0 \le C,$$

for all $0 \le t \le \infty$.

Remark 3.1. *From Corollary* 2.3, *one has*

$$\int e^{3\lambda}\, d\mu_0 \le C_3 + C_4 t.$$

Proof. From Lemma 2.2 and (2.6), we have

$$\begin{aligned} \text{(3.1)} \quad \frac{d}{dt}\int_\Sigma e^{3\lambda}\, d\mu_0 &\le C' + C''\int_\Sigma e^{-\lambda}\, d\mu_0 - C'''\int e^{\lambda}[2|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2 - (\Delta_0 e^{-\lambda})^2] d\mu_0 \\ &\le C'''' - C'''\int e^{\lambda}[2|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2 - (\Delta_0 e^{-\lambda})^2] d\mu_0. \end{aligned}$$

Suppose

$$\sup_{[0,\infty)} \int e^{3\lambda}\, d\mu_0 = \infty.$$

Then there exists a subsequence $\{s_j\}$ with $s_j \to \infty$ such that

$$\int e^{3\lambda}\, d\mu_0|_{t=s_j} \to \infty$$

and

$$\left(\frac{d}{dt}\int_\Sigma e^{3\lambda}\, d\mu_0\right)(s_j) > 0.$$

But from (3.1), one has

$$\text{(3.2)} \qquad \int e^{\lambda}[2|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2 - (\Delta_0 e^{-\lambda})^2] d\mu_0|_{t=s_j} \le C.$$

Furthermore, from (3.2) and $\lambda \ge -C$, one has

$$\int [2|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2 - (\Delta_0 e^{-\lambda})^2] d\mu_0|_{t=s_j} \le C.$$

But the Bochner formula says

$$\int [|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2 - (\Delta_0 e^{-\lambda})^2] d\mu_0 = -\frac{1}{2}\int R_0 |\overset{0}{\nabla} e^{-\lambda}|^2 d\mu_0.$$

It follows that

$$\int \left[|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2 - \frac{1}{2} R_0 |\overset{0}{\nabla} e^{-\lambda}|^2 \right] d\mu_0 |_{t=s_j} \leq C$$

and then, since $2|\overset{0}{\nabla}{}^2 e^{-\lambda}|^2 \geq (\Delta_0 e^{-\lambda})^2$

$$\int [(\Delta_0 e^{-\lambda})^2 - R_0 |\overset{0}{\nabla} e^{-\lambda}|^2] d\mu_0 |_{t=s_j} \leq C.$$

Now, since $\Delta_0 e^{-\lambda} = e^{-\lambda}(-\Delta_0 \lambda + |\overset{0}{\nabla}\lambda|^2)$, we compute

$$\begin{aligned}
\int_\Sigma e^{-2\lambda} |\overset{0}{\nabla}\lambda|^4 d\mu_0 &\leq C + 2\int_\Sigma e^{-2\lambda} \Delta_0 \lambda |\overset{0}{\nabla}\lambda|^2 d\mu_0 + \int_\Sigma e^{-2\lambda} R_0 |\overset{0}{\nabla}\lambda|^2 d\mu_0 \\
&\leq C + \frac{4}{\varepsilon_1}\int_\Sigma e^{-2\lambda}(\Delta_0\lambda)^2 d\mu_0 + \varepsilon_1 \int_\Sigma e^{-2\lambda} |\overset{0}{\nabla}\lambda|^4 d\mu_0 \\
&\quad + \frac{2}{\varepsilon_2}\int_\Sigma e^{-2\lambda}(R_0)^2 d\mu_0 + \varepsilon_2 \int_\Sigma e^{-2\lambda} |\overset{0}{\nabla}\lambda|^4 d\mu_0 \\
&\leq C\left(\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_2}\right) + (\varepsilon_1 + \varepsilon_2)\int_\Sigma e^{-2\lambda} |\overset{0}{\nabla}\lambda|^4 d\mu_0.
\end{aligned}$$

Choose small $\varepsilon_1, \varepsilon_2$, then

$$\int_\Sigma e^{-2\lambda} |\overset{0}{\nabla}\lambda|^4 d\mu_0 \leq C.$$

Now

$$\begin{aligned}
\int_\Sigma |\overset{0}{\nabla}\lambda|^2 d\mu_0 &= \int_\Sigma e^{\lambda} e^{-\lambda} |\overset{0}{\nabla}\lambda|^2 d\mu_0 \\
&\leq \left(\int e^{2\lambda}\, d\mu_0\right)^{1/2} \left(\int_\Sigma e^{-2\lambda} |\overset{0}{\nabla}\lambda|^4 d\mu_0\right)^{1/2} \\
&\leq C.
\end{aligned}$$

On the other hand, since $\lambda \geq -C$ and $\int e^{2\lambda}\, d\mu_0 \leq C$, we have $\lambda \in L^2$. Then, from Moser's inequality

$$\int e^{3(\lambda - \bar{\lambda})}\, d\mu_0 \leq C \exp(\|(\lambda - \bar{\lambda})\|_{W_{1,2}}) \leq C$$

and

$$\int e^{3\lambda}\, d\mu_0|_{t=s_j} \le C,$$

for all $0 \le s_j \le \infty$. This leads to a contradiction.

Then

Lemma 3.3. *Assumptions being the same as in Theorem* 1.2, *under the flow* (1.2), *there exists a constant* $C = C(\|\lambda_0\|_{W_{2,2}}, g_0)$ *such that*

$$\|\lambda(t)\|_{W_{2,2}} \le C, \qquad 0 \le t \le \infty,$$

in particular,

$$\sup_{p \in M_t} |\lambda(p,t)| \le C, \qquad 0 \le t \le \infty.$$

Proof. Now from Lemma 2.1, one obtains

$$\int e^{-2\lambda}\left(\Delta_0\lambda - \frac{1}{2}R_0\right)^2 d\mu_0 \le C,$$

and then

$$\begin{aligned}\int \left|\Delta_0\lambda - \frac{1}{2}R_0\right|^{6/5} d\mu_0 &= \int e^{(6/5)\lambda}\left|e^{-\lambda}\left(\Delta_0\lambda - \frac{1}{2}R_0\right)\right|^{6/5} d\mu_0 \\ &\le \left(\int e^{3\lambda}\, d\mu_0\right)^{2/5}\left(\int e^{-2\lambda}\left(\Delta_0\lambda - \frac{1}{2}R_0\right)^2 d\mu_0\right)^{3/5} \\ &\le C.\end{aligned}$$

This leads to

$$\|\Delta_0\lambda\|_{L_{6/5}} \le C.$$

Moreover, for $\phi = \lambda - \bar{\lambda}$, where $\bar{\lambda} = \dfrac{\int_\Sigma \lambda\, d\mu_0}{\int_\Sigma d\mu_0}$, by applying Sobolev inequality

$$\|\phi\|_{W_{2,6/5}} \le C.$$

Now based on Moser inequality, one has

$$|\bar{\lambda}| \le C.$$

All these imply

$$\|\lambda\|_{W_{2,6/5}} \le C,$$

and

$$\sup_{p \in \Sigma_t} |\lambda(p,t)| \le C.$$

Finally, from $\int e^{-2\lambda}(\Delta_0\lambda - \frac{1}{2}R_0)^2 d\mu_0 \le C$, we have

$$\|\lambda(t)\|_{W_{2,2}} \le C.$$

Moreover, we have

Lemma 3.4. *Assumptions being the same as in Theorem* 1.2, *under the flow* (1.2), *there exists a constant* $C = C(\|\lambda_0\|_{W_{2,2}}, g_0)$, $l \ge 2$ *such that*

$$\|\overset{0}{\nabla}{}^l \lambda(p,t)\|_{L_2} \le C,$$

$\forall t \in [0, \infty]$.

Remark 3.2. As soon as we have the $W_{2,2}$-estimate as in Lemma 3.3, the higher-order derivatives estimates will follow on the line of section 4 of [Chru].

Proof. The exactly same method as in [Chru, section 4], we have the uniformly bounds on all $W_{k,2}$ norms. More precisely, the key estimate is the following:

From Lemma 4.1 of [Chru], for every $l \in \mathbf{N}$, $l \ge 1$, there exist a constant $C < \infty$ such that

$$\frac{d}{dt}\|e^{-4\lambda}\overset{0}{\nabla}{}^l\lambda\|_{L_2}^2 \le -\|\overset{0}{\nabla}{}^{l+2}\lambda\|_{L_2}^2 + C\|\phi\|_{W_{2,2}}^2. \tag{3.3}$$

This, Lemma 3.3 and [Chru, Lemma 4.2] will imply the Lemma. For details, we refer to [Chru, section 4].

Then the long-time existence of solution of (1.2) follows easily.

Corollary 3.5. *Assumptions being the same as in Theorem* 1.2, *under the flow* (1.2), *there exists a subsequence* $\{t_j\}$ *such that*

$$\left|\nabla_i\nabla_j R - \frac{1}{2}(\triangle R)g_{ij}\right| \to 0$$

as $t_j \to \infty$.

Proof. Since

$$-\frac{1}{2}\frac{d}{dt}\int_\Sigma R^2\, d\mu = 2\int_\Sigma \left[\nabla_i\nabla_j R - \frac{1}{2}(\triangle R)g_{ij}\right]^2 d\mu.$$

Then

$$\int_0^\infty \int_\Sigma \left[\nabla_i\nabla_j R - \frac{1}{2}(\triangle R)g_{ij}\right]^2 d\mu dt < \infty.$$

Hence there exists a subsequence $\{t_j\}$ such that

$$\int_\Sigma \left[\nabla_i\nabla_j R - \frac{1}{2}(\triangle R)g_{ij}\right]^2 d\mu|_{t=t_j} \to 0$$

as $t_j \to \infty$. But from the previous lemma, we have

$$\|\lambda\|_{W_{k,2}} \leq C$$

for all $0 \leq t \leq \infty$. Then the Lemma follows easily from the interpolation inequality ([A]).

Next, let

$$M_{ij} = \nabla_i \nabla_j R - \frac{1}{2}(\triangle R) g_{ij}.$$

Inspired by Hamilton's work ([H]), that is, from the previous corollary, there exists a subsequence $\{t_j\}$ such that

$$M_{ij} = 0.$$

as $t_j \to \infty$.

The same notation as in [H], a metric g_{ij} with $M_{ij} = 0$ is a soliton solution for the Calabi flow on surfaces. Then we have the same result as [H, Theorem 10.1.]:

Lemma 3.6. *On a compact surface, assumptions being the same as in Theorem* 1.2, *there are no soliton other than constant curvature.*

Proof. As pointed out by B. Chow, one may use the Kazdan-Warner identity ([F]). That is, for any conformal vector field V,

$$\int V \cdot \nabla R \, d\mu = \int (div V) R \, d\mu = 0.$$

Now if g is a soliton solution for the Calabi flow, then ∇R is a conformal vector field. Thus

$$0 = \int (\Delta R) R \, d\mu = -\int |\nabla R|^2 d\mu.$$

This implies

$$R \equiv r$$

where $r = \dfrac{\int_\Sigma R \, d\mu}{\int_\Sigma d\mu}$.

Then

Theorem 3.7. *On a compact surface, the same assumptions as in Theorem* 1.2, *there exists a subsequence of solutions $\lambda(t_j)$ of the Calabi flow*

$$\frac{\partial \lambda}{\partial t} = \frac{1}{2} \triangle R$$

such that $e^{2\lambda(t_j)} g_0$ converges to a constant curvature metric as $t_j \to \infty$.

Then Theorem 1.2 follows easily.

Remark 3.3. In view of this section, we reduce the proof of our main Theorem 1.3 to finding a uniformly lower bound on λ which we will deal with in the final section.

4. Find a bound on λ

In view of the section 3, we reduce the proof of our main Theorem 1.3 to finding a uniformly lower bound on λ in case of Σ with genus $h \geq 2$. In the following, we will follow the notion as in [G].

Definition 4.1. We say that $\lambda(t)$ satisfies the property $(*)$ if there is a point $x \in \Sigma$, positive constants ρ, ε, C such that, for $g = e^{2\lambda} g_0$

$$\int_{B(x,\rho)} e^{-\varepsilon\lambda(t)}\, d\mu_0 \leq C, \qquad \text{for all } t. \tag{$*$}$$

Lemma 4.1. *For a fixed conformal class $(\Sigma, e^{2\lambda} g_0)$ with bounds on Volume and $\int_\Sigma R^2\, d\mu$ and λ satisfies the property $(*)$, then there are positive constants C_0, δ_0 such that*

$$\int_\Sigma e^{-\delta_0\lambda}\, d\mu_0 \leq C_0'. \tag{4.1}$$

As a consequence, there is a constant C_0 such that

$$\lambda \geq -C_0. \tag{4.2}$$

Proof. First we observe, for $0 \leq \delta \leq \frac{1}{2}$,

$$\Delta_0 e^{-2\delta\lambda} = e^{-2\delta\lambda}(-2\delta\Delta_0\lambda + 4\delta^2 |\overset{0}{\nabla}\lambda|^2) = e^{-2\delta\lambda}[\delta(e^{2\lambda}R - R_0) + 4\delta^2 |\overset{0}{\nabla}\lambda|^2].$$

Then

$$\begin{aligned}
\int_\Sigma |\overset{0}{\nabla} e^{-\delta\lambda}|^2 d\mu_0 &= \delta^2 \int_\Sigma e^{-2\delta\lambda} |\overset{0}{\nabla}\lambda|^2 d\mu_0 \\
&= -\frac{\delta}{4}\int_\Sigma e^{2(1-\delta)\lambda} R\, d\mu_0 + \frac{\delta}{4}\int_\Sigma e^{-2\delta\lambda} R_0\, d\mu_0 \\
&= -\frac{\delta}{4}\int_\Sigma e^{-2\delta\lambda} R\, d\mu + \frac{\delta}{4}\int_\Sigma e^{-2\delta\lambda} R_0\, d\mu_0 \\
&\leq C\int_\Sigma e^{-4\delta\lambda}\, d\mu + C\int_\Sigma R^2\, d\mu + \frac{\delta}{4}\int_\Sigma e^{-2\delta\lambda} R_0\, d\mu_0 \\
&= C\int_\Sigma e^{(2-4\delta)\lambda}\, d\mu_0 + C\int_\Sigma R^2\, d\mu + \frac{\delta}{4}\int_\Sigma e^{-2\delta\lambda} R_0\, d\mu_0 \\
&\leq C + \frac{\delta}{4}\|R_0\|_{L_r}\int_\Sigma e^{-2\delta\lambda}\, d\mu_0.
\end{aligned} \tag{4.3}$$

Now let λ_1 denote the first non-zero eigenvalue of Δ_0, by Rayleigh inequality, we have

$$\int_\Sigma e^{-2\delta\lambda}\,d\mu_0 \le \frac{(\int_\Sigma e^{-\delta\lambda}\,d\mu_0)^2}{(\int_\Sigma d\mu_0)} + \frac{1}{\lambda_1}\int_\Sigma |\overset{0}{\nabla} e^{-\delta\lambda}|^2 d\mu_0 \tag{4.4}$$

$$\le \frac{(\int_\Sigma e^{-\delta\lambda}\,d\mu_0)^2}{(\int_\Sigma d\mu_0)} + C + \frac{\delta}{4\lambda_1}\|R_0\|_{L_r}\int_\Sigma e^{-2\delta\lambda}\,d\mu_0.$$

But for $\delta < \varepsilon$, one has

$$\int_\Sigma e^{-\delta\lambda}\,d\mu_0 = \int_{B_p} e^{-\delta\lambda}\,d\mu_0 + \int_{B_p^c} e^{-\delta\lambda}\,d\mu_0 \le C + \left(\int_{B_p^c} e^{-2\delta\lambda}\,d\mu_0\right)^{1/2}\left(\int_{B_p^c} d\mu_0\right)^{1/2},$$

and then, for any $\eta > 0$

$$\frac{(\int_\Sigma e^{-\delta\lambda}\,d\mu_0)^2}{(\int_\Sigma d\mu_0)} \le C(\eta) + (1+\eta)\frac{(\int_{B_p^c} d\mu_0)}{(\int_\Sigma d\mu_0)}\int_\Sigma e^{-2\delta\lambda}\,d\mu_0.$$

This implies

$$\int_\Sigma e^{-2\delta\lambda}\,d\mu_0 \le C(\eta) + (1+\eta)\frac{(\int_{B_p^c} d\mu_0)}{(\int_\Sigma d\mu_0)}\int_\Sigma e^{-2\delta\lambda}\,d\mu_0 + \frac{\delta}{4\lambda_1}\|R_0\|_{L_r}\int_\Sigma e^{-2\delta\lambda}\,d\mu_0.$$

Then choose η, δ small enough and take $\delta_0 = 2\delta$, which giving us (4.1).

To see that (4.2) follows from (4.1). Let $G(x,.)$ denote the Green's function for Δ_{g_0} with singularity at $x \in \Sigma$. Then $G(x,.) \ge 0$ and $\|G(x,.)\|_{L_p} \le C$, for any $p < \infty$.

Now

$$e^{-(\delta_0/2)\lambda}(x) = \frac{\int e^{-(\delta_0/2)\lambda}\,d\mu_0}{\int d\mu_0} - \int G(x,.)\Delta_0(e^{-(\delta_0/2)\lambda})d\mu_0 \tag{4.5}$$

$$\le C - \int G(x,.)e^{-(\delta_0/2)\lambda}\left\{-\frac{\delta_0}{2}\Delta_0\lambda + \frac{\delta_0^2}{4}|\overset{0}{\nabla}\lambda|^2\right\}d\mu_0$$

$$\le C - \int G(x,.)\left[\frac{\delta_0}{4}e^{(2-(\delta_0/2))\lambda}R - \frac{\delta_0}{4}e^{-(\delta_0/2)\lambda}R_0\right]d\mu_0$$

$$\le C + C\left(\int G^p e^{-(\delta_0/2)p\lambda}\,d\mu\right)^{1/p}\left(\int R^q\,d\mu\right)^{1/q}$$

$$+ C\left(\int G^2\,d\mu_0\right)^{1/2}\left(\int e^{-\delta_0\lambda}\,d\mu_0\right)^{1/2}$$

$$\le C + C\left(\int G^p e^{-(\delta_0/2)p\lambda}\,d\mu\right)^{1/p}\left(\int R^q\,d\mu\right)^{1/q},$$

for $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

For $1 < q < 2$ and $-\frac{\delta_0}{2}p = -2$, we have

$$e^{-(\delta_0/2)\lambda}(x) \leq C + C\left(\int G^p\, d\mu_0\right)^{1/p} \leq C.$$

This completes the proof of Lemma.

Lemma 4.2. *For a fixed conformal class with the bounds of the volume and $\int_\Sigma R^2\, d\mu$ on Σ with genus $h \geq 2$, then $\lambda(t)$ satisfies $(*)$.*

Remark 4.1. In [G], Gursky proved the following: Let (M, g_0) be a closed n-manifold with the Yamabe constant $Q < 0$, $n \geq 3$. For $g \in [g_0]$, say $g = e^{2\lambda}g_0$. If $\int d\mu \leq V$ and $\int_M R^p\, d\mu \leq \beta^2$, $p > \frac{n}{2}$, for some positive constants V, β, then there exists $H = H(V, \beta)$ such that $\lambda \geq -H$. We will follow their proof. That is, we may prove the Lemma 4.2 by assuming there is a metric with negative scalar curvature. On the other hand, since the genus $h \geq 2$, and (background) metric is different from this metric with negative scalar curvature by a fixed conformal factor, then Lemma 4.2 holds in general as soon as we have the bounds of the volume and $\int_\Sigma R^2\, d\mu$.

Proof. From (4.3), we may take $\delta = \frac{1}{2}$, then

$$\int_\Sigma |\overset{0}{\nabla} e^{-(1/2)\lambda}|^2 d\mu_0 \leq C + \frac{1}{8}\int_\Sigma e^{-\lambda} R_0\, d\mu_0.$$

But for Σ with genus greater than one, as in [G, section 5] or [CY], we may assume that $-R_0 \geq k_0 > 0$, then

$$k_0 \int_\Sigma e^{-\lambda}\, d\mu_0 \leq -\int_\Sigma e^{-\lambda} R_0\, d\mu_0 \leq 8C$$

and then

$$\int_\Sigma e^{-\lambda}\, d\mu_0 \leq C_0.$$

From the previous section, Lemma 4.1 and Lemma 4.2, Theorem 1.3 follows easily.

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