J. Math. Kyoto Univ. (JMKYAZ)
44-1 (2004), 173–180

Some properties of subharmonic functions on complete Riemannian manifolds and their geometric applications

By

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Abstract

This paper investigates the global behavior of subharmonic functions on a complete noncompact simply-connected Riemannian manifold. The authors obtain some Liouville-type theorems, a comparison theorem for the strong parabolicity of a manifold and their applications to geometry.

1. Introduction

Let M be a Riemannian manifold with dimension $n \geq 2$. For $u \in C^2(M)$, let Δu denote the Laplacian of u. A function $u \in C^2(M)$ is said to be subharmonic (resp. harmonic) if $\Delta u \geq 0$ (resp. $\Delta u = 0$).

The classical Liouville theorem says that a subharmonic function defined over $\mathbf{R^2}$ (or a harmonic function defined over $\mathbf{R^n}$) and bounded above is constant. Huber [1] proved that a complete two-dimensional Riemannian manifold with non-negative curvature does not admit a nonconstant negative subharmonic function. Karp [2] found that a complete noncompact Riemannian manifold admits no nonconstant negative subharmonic function if it has moderate volume growth (see next section of this paper). Some further results about the properties of subharmonic and harmonic functions on complete Riemannian manifolds were obtained by many authors, such as Greene and Wu [3], Huber [1], Karp [2] and Yau [4] (also see Hildebrandt [5] and Karp [6]).

In the present paper, we continue to study subharmonic functions on complete noncompact Riemannian manifolds. Our attention is on the connections between the geometry of a noncompact manifold and the global behavior of its subharmonic functions. Our main results are Theorems 3.1, 3.2 (a Liouville-type theorem), 3.3 (a comparison theorem for strong parabolicity) and their applications such as Corollary 3.2.

This paper is organized as follows. In Section 2 we fix some notations and definitions, and we recall some known results. In Section 3 we give our main

2000 *Mathematics Subject Classification(s).* 53C21, 58J60
Received April 16, 2003

results and their proofs.

Throughout this paper all manifolds are assumed to be complete noncompact connected C^∞ Riemannian manifolds without boundary with dim ≥ 2.

2. Preliminaries

Throughout this paper, if M is a manifold and $q \in M$, M_q will denote the tangent space to M at q.

Definition 2.1 ([3]). Let M be an n-dimensional Riemannian manifold. A point $o \in M$ is a pole of M iff the exponential mapping $\exp_o$: $M_o \to M$ is a diffeomorphism.

Note that if M possesses a pole, it is complete. To be convenient, we refer to an ordered pair (M, o) as a manifold with a pole if o is a pole of the Riemannian manifold M. In this paper, for any $x \in (M, o)$, $r = r(x)$ will always denote the geodesic distance from o to x.

Definition 2.2. A manifold with a pole (M, o) is strongly symmetric around o iff every linear isometry $\phi : M_o \to M_o$ is realized as the differential of an isometry $\Phi : M \to M$, i.e., $\Phi(o) = o$ and $\Phi_*(o) = \phi$, where $\Phi_*(o)$ denotes the differential of Φ at o.

The Euclidean space $\mathbf{R}^n$ is strongly symmetric. There are some discussions about strongly symmetric manifold in [3] (where they use the term "model" instead of "strongly symmetric manifold").

From now on in this paper let M be strongly symmetric around o. We assume that $n = dim M \geq 2$. Let (M, φ, x) be a global normal coordinate neighborhood around o. That is, there is an orthonormal basis $\{e_j, j = 1, \ldots, n\}$ of M_o such that $\varphi : M \to \mathbf{R^n}$, $q \in M \mapsto x = (x^1, \ldots, x^n) \in \mathbf{R^n}$, $\exp_o^{-1}(q) = \sum_j x^j e_j$, where $\mathbf{R^n}$ denotes the n-dimensional Euclidean space. To be convenient, we frequently write $q = x$ and denote by x a point of M. Let $(r, \theta)(\theta = (\theta^1, \ldots, \theta^{n-1}))$ be the (geodesic) polar coordinates of x.

In the polar coordinates, by means of the well-known Gauss Lemma, the metric of M can be expressed by

$$ds^2 = dr^2 + \sum_{i,j} d_{ij} d\theta^i d\theta^j = dr^2 + h(r)^2 d\Theta^2 \tag{2.1}$$

on $M - \{o\}$ (see [3]). Here $d_{ij} = g(\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\theta^j})$ and $d\Theta^2$ denotes the canonical metric on the unit sphere of M_o. h depends only on r but not on θ since M is strongly symmetric around o. Let S_r be the geodesic sphere of M with center o and radius r. Then the Riemannian volume element of S_r can be written as

$$dS_r = \sqrt{D(r, \theta)} d\theta^1 \cdots d\theta^{n-1}, \tag{2.2}$$

where $D \equiv \det(d_{ij})$.

A function $u(x)$ on M is said to be radially symmetric around o iff $u(x) = u(\widetilde{x})$ provided that $r(x) = r(\widetilde{x})$ for any $x, \widetilde{x} \in M$. If $u(x)$ is radially symmetric, we frequently write $u(x)$ as $u(r(x))$. Since M is strongly symmetric around o, it is easy to show that Δr is radially symmetric on $M - \{o\}$. Furthermore, if $u \in C^2(M - \{o\})$ and u is radially symmetric around o, then Δu is also radially symmetric around o.

Let u be a C^2 radially symmetric function, it is easy to verify

$$\Delta u = \frac{1}{\sqrt{D}} \partial_r (\sqrt{D}\, \partial_r\, u\,). \tag{2.3}$$

Thus we obtain

$$\Delta r = \frac{1}{\sqrt{D}} \partial_r (\sqrt{D}) = \partial_r \log \sqrt{D}, \tag{2.4}$$

and

$$\Delta u = u'' + (\Delta r) u', \tag{2.5}$$

where $u' \equiv du/dr$, $u'' \equiv d^2u/dr^2$.

We can define a scalar product operation η as following:

$$\begin{aligned} \eta : \mathbf{R} \times M &\to M; \\ (t, (r, \theta)) &\mapsto (tr, \theta). \end{aligned} \tag{2.6}$$

We always write $\eta(r, x) = rx$.

In the statement of our main results we also need the following notation.

Definition 2.3 ([2]). A noncompact Riemannian manifold is strongly parabolic if it admits no nonconstant negative subharmonic function.

It is well-known that $\mathbf{R^n}$ (with the Euclidean metric) is strongly parabolic if $n = 2$, and is not if $n \geq 3$. An important result about strong parabolicity is the following Karp's theorem.

Theorem A (Karp [2]). *A complete noncompact Riemannian Manifold is strongly parabolic if it has moderate volume growth.*

In the above theorem, the term "to have moderate volume growth" is defined as following.

Definition 2.4 ([2]). A complete noncompact Riemannian manifold N has moderate volume growth if there is a positive nondecreasing function $F(r)$ such that $\limsup_{r \to \infty} \frac{1}{r^2 F(r)} \operatorname{vol} B(x_0, r) < \infty$ for some (and hence all) $x_0 \in N$ while $\int_1^\infty \frac{dr}{rF(r)} = +\infty$. Here $B(x_0, r)$ denotes the geodesic ball in N with center x_0 and radius r.

The following known result will be needed in the proofs of our main results.

Theorem B (Yau [4]). *Let f be a harmonic function on a complete Riemannian Manifold with nonnegative Ricci curvature. If f is bounded above, then it has to be constant.*

3. Results

Lemma 3.1. *If M is strongly symmetric around o, then*

$$\lim_{r\to 0} r\Delta r = n-1. \tag{3.1}$$

Proof. Here and below, let $V(r)$ be the volume of S_r. By (2.2) and (2.4) we get

$$V'(r) = \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} \frac{\partial_r\sqrt{D}}{\sqrt{D}}\sqrt{D}d\theta^1\cdots d\theta^{n-2}d\theta^{n-1} = V(r)\Delta r,$$

so that

$$\Delta r = \frac{V'(r)}{V(r)}. \tag{3.2}$$

On the other hand (see [7, p. 256])

$$V(r) = \int_{S(1)} r^{n-1}J(r,\theta)d\theta_0.$$

Here $S(1)$ denotes the unit sphere of M_o (consider M_0 as an inner product space with the inner product defined by the restriction of the metric of M at o), $d\theta_0$ its volume element and $J(r,\theta)$ the Jacobian of $\exp_o$ at (r,θ). But J depends only on r since M is radially symmetric around o, thus

$$V(r) = r^{n-1}J(r)\omega_n,$$

where ω_n denotes the volume of $S(1)$. Hence

$$\lim_{r\to 0} r\Delta r = \lim_{r\to 0}\frac{rV'(r)}{V(r)} = n-1.$$

□

Let (M,o) be a manifold with a pole. Here and below, let $B(r)$ be the geodesic ball of radius r and center o, and $d\mu$ be the volume element of M. To formulate our main results it is convenient to introduce the following notation:

$$\mathcal{H} \equiv \left\{ h\in C^0(M) \mid h\geq 0, \int_1^{\infty}\frac{1}{V(r)}\int_{B(r)} h\,d\mu dr < +\infty \right\}.$$

Theorem 3.1. *Let (M,o) be a manifold with a pole. Suppose that M is strongly symmetric around o and strongly parabolic. Let $u\in C^2(M)$ be a subharmonic function on M. If there is $h\in\mathcal{H}$ such that $\Delta u\leq h$, then u is harmonic on M.*

Proof. Define

$$\widetilde{u}(r) \equiv \frac{1}{V(r)} \int_{S_r} u dS_r = \frac{1}{V(1)} \int_{S_1} u(r\xi) dS_1, \qquad (\xi \in S_1). \tag{3.3}$$

The second equality is by (2.1). (On $\mathbf{R^n}$, Ni [8] has used $\widetilde{u}(r)$ to study the conformal scalar curvature equation). It is obvious from the definition that $\widetilde{u}(r(x)) \in C^2(M - \{o\})$. Moreover we will see in the following that $\widetilde{u}(r(x)) \in C^2(M)$.

Step 1. We first prove that $\widetilde{u}(r)$ is nondecreasing.
According to (3.3), for any $r > 0$, we have

$$\widetilde{u}'(r) = \frac{1}{V(1)} \int_{S_1} \frac{\partial u}{\partial r}(r\xi) dS_1 = \frac{1}{V(r)} \int_{S_r} \partial_r u dS_r.$$

Thus for $r > 0$,

$$\widetilde{u}'(r) V(r) = \int_{S_r} \partial_r u dS_r. \tag{3.4}$$

For $r > 0$, by use of the divergence theorem and (3.4), we obtain

$$\int_{B(r)} \Delta u d\mu = \int_{S_r} \partial_r u dS_r = \widetilde{u}'(r) V(r). \tag{3.5}$$

By the assumption of the theorem we have $\Delta u \geq 0$, thus $\widetilde{u}'(r) \geq 0$ and hence $\widetilde{u}(r)$ is nondecreasing.

Step 2. We now prove that $\widetilde{u}(r)$ satisfies the following equality:

$$\Delta \widetilde{u}(r) = \frac{1}{V(r)} \int_{S_r} \Delta u dS_r. \tag{3.6}$$

In fact, for $r > 0$,

$$\int_{B(r)} \Delta u d\mu = \int_0^r \int_{S_t} \Delta u dS_t dt. \tag{3.7}$$

By (2.5), (3.2), (3.5) and (3.7), we have

$$\begin{aligned} \int_{S_r} \Delta u dS_r &= [\widetilde{u}'(r) V(r)]' \\ &= V(r) \left\{ \widetilde{u}''(r) + \frac{V'(r)}{V(r)} \widetilde{u}'(r) \right\} \\ &= V(r) \{ \widetilde{u}''(r) + (\Delta r) \widetilde{u}'(r) \} \\ &= V(r) \Delta \widetilde{u}(r). \end{aligned} \tag{3.8}$$

Then, for $r > 0$, we obtain

$$\Delta\widetilde{u}(r) = \frac{1}{V(r)}\int_{S_r}\Delta u dS_r.$$

As for $r = 0$, a simple computation with a use of Lemma 3.1 shows that

$$\widetilde{u}'(0) = 0, \quad \widetilde{u}''(0) = \frac{1}{n}\Delta u(0), \quad \lim_{r\to 0}\Delta\widetilde{u}(r) = \Delta u(0). \tag{3.9}$$

Now it is not hard to prove that $\widetilde{u}(r) \in C^2(M)$ and that (3.6) still holds for $r = 0$ (in the sense of limit).

Step 3. We complete the proof of the theorem.

Now $0 \le \Delta u \le h$ on M, $h \in \mathcal{H}$. From (3.6) we know $\Delta\widetilde{u} \ge 0$, so that $\widetilde{u}$ is also subharmonic on M. On the other hand, from (3.5) we get

$$\widetilde{u}'(r)V(r) = \int_{B(r)}\Delta u d\mu \le \int_{B(r)} h d\mu.$$

So we have

$$\widetilde{u}'(r) \le \frac{1}{V(r)}\int_{B(r)} h d\mu.$$

Integrating both sides from 0 to R we obtain

$$\widetilde{u}(R) \le \int_0^R \frac{1}{V(r)}\int_{B(r)} h\, d\mu dr + \widetilde{u}(0) \le \int_0^\infty \frac{1}{V(r)}\int_{B(r)} h\, d\mu dr + \widetilde{u}(0) < \infty$$

for any $R \ge 0$. Thus $\widetilde{u} \equiv$ *constant* on M since M is strongly parabolic. Then by (3.5) we imply $\Delta u \equiv 0$ on M.

□

Corollary 3.1. *Let (M, o) be a manifold with a pole. Suppose that M is strongly symmetric around o and strongly parabolic. Let f be a nonnegative continuous function on M such that $f(x_0) \ne 0$ for some $x_0 \in M$. If there is $h \in \mathcal{H}$ such that $f \le h$, then the following equation*

$$\Delta u = f \tag{3.10}$$

has no C^2 solutions on M.

Proof. This is an immediate consequence of Theorem 3.1. □

Theorem 3.2 (A Liouville-type Theorem). *Let (M, o) be a manifold with a pole. Suppose that M is strongly symmetric around o and strongly parabolic. Let $u \in C^2(M)$ be a positive subharmonic function on M. If there are $\alpha > 1$ and $h \in \mathcal{H}$ such that $\Delta u^\alpha \le h$, then u is constant on M.*

Proof. From the definition of Laplacian we have

$$\Delta u^\alpha = \alpha u^{\alpha-1}\Delta u + \alpha(\alpha-1)u^{\alpha-2}|\nabla u|^2 \geq 0. \tag{3.11}$$

Now applying theorem 3.1 to u^α we get $\Delta u^\alpha \equiv 0$, and from (3.11) we obtain $|\nabla u| \equiv 0$ and hence $u \equiv$ *constant.* □

Given a manifold with a pole (M, o), the radial vector field ∂_M is the unit vector field defined on $M - \{o\}$ such that for any $x \in M - \{o\}$, ∂_M is the unit vector tangent to the unique geodesic joining o to x and pointing away from o.

Theorem 3.3 (A Comparison Theorem for Strong Parabolicity)**.** *Let (M, o) and $(\overline{M}, \overline{o})$ be manifolds with poles of the same dimension. Let M (resp. $\overline{M}$) be strongly symmetric around o (resp. around $\overline{o}$). Let r (resp. $\overline{r}$) be the distance functions of M (resp. $\overline{M}$) relative to o (resp. $\overline{o}$). Suppose that the Ricci curvature of M is nonnegative, and*

$$\mathrm{Ric}(\partial_M, \partial_M)(x) \geq \mathrm{Ric}(\partial_{\overline{M}}, \partial_{\overline{M}})(\overline{x}) \tag{3.12}$$

for every $x \in M - \{o\}$ and $\overline{x} \in \overline{M} - \{\overline{o}\}$ such that $r(x) = \overline{r}(\overline{x})$. Here Ric *denotes Ricci curvature. If $\overline{M}$ is strongly parabolic, then M is also strongly parabolic.*

Proof. Let u be a negative C^2 subharmonic function on M. Define $\widetilde{u}(r)$ by (3.3). Then $\widetilde{u}(r)$ is negative. We also know from the proof of Theorem 3.1 that $\widetilde{u}(r)$ is nondecreasing and subharmonic on M. Let Δ and $\overline{\Delta}$ be the Laplacians of M and $\overline{M}$ respectively. By means of Laplacian Comparison Theorem (see [3, p. 26]), for every $x \in M - \{o\}$ and $\overline{x} \in \overline{M} - \{\overline{o}\}$ such that $r(x) = \overline{r}(\overline{x})$, we have

$$\overline{\Delta}\widetilde{u}(\overline{r}(\overline{x})) \geq \Delta\widetilde{u}(r(x)) = \frac{1}{V(r)}\int_{S_r} \Delta u dS_r \geq 0. \tag{3.13}$$

So $\widetilde{u}(\overline{r})$ is a negative subharmonic function on $\overline{M}$, and hence it is constant by the strong parabolicity of $\overline{M}$. This implies that $\overline{\Delta}\widetilde{u} \equiv 0$ on $\overline{M}$. It follows by (3.13) that $\Delta u \equiv 0$ on M. According to Theorem B we conclude that u is constant and hence M is strongly parabolic. □

Corollary 3.2. *Let (M, o) be an n-dimensional manifold with a pole such that $n \geq 3$. Let M be strongly symmetric around o. If M is strongly parabolic, then there is $v_p \in M_p$ for some point $p \in M$ such that* $\mathrm{Ric}(v_p, v_p) > 0$.

Proof. We argue by contradiction. Assume that for all $p \in M$ and all $v_p \in M_p$ $\mathrm{Ric}(v_p, v_p) \leq 0$. From the assumption of the theorem we know that M is strongly parabolic. By applying Theorem 3.3 to $\mathbf{R^n}$ and M we conclude that $\mathbf{R^n}$ is strongly parabolic for $n \geq 3$. This is a contradiction. □

In the following we will give a different proof of Huber's Theorem (A complete two-dimensional Riemannian manifold with non-negative Gaussian curvature does not admit a nonconstant negative subharmonic function, cf. [1] and [2]) in the case of 2-dimensional strongly symmetric manifolds.

Corollary 3.3 (Huber). *Let (M^2, o) be a 2-dimensional manifold with a pole and M^2 be strongly symmetric around o. If M^2 has nonnegative Gaussian curvature, then it is strongly parabolic.*

Proof. We will make use of Theorem 3.3. We choose $(M, o) = (M^2, o)$ and $(\overline{M}, \overline{o}) = (\mathbf{R^2}, O)$ in Theorem 3.3, where O denotes the origin of $\mathbf{R^2}$. It is easy to verify that all conditions of Theorem 3.3 are satisfied. Since $\mathbf{R^2}$ is strongly parabolic, M^2 is also strongly parabolic. □

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